CHERN CURRENTS OF SINGULAR CONNECTIONS ASSOCIATED WITH A SECTION OF A COMPACTIFIED BUNDLE

John Zweck

A compactification of the Chern–Weil theory for bundle maps developed in [HL] is described. For each section ν of the compactification $\mathbb{P}(\underline{\mathbb{C}} \oplus F) \to X$ of a rank n complex vector bundle $F \to X$ with connection, and for each Ad–invariant polynomial ϕ on $\mathfrak{gl}_n$, there is an associated current formula

$$\phi(\Omega_F) + \nu^*(\mathrm{Res}_\infty(\phi))\ \mathrm{Div}_\infty(\nu) - \phi(\Omega_0) - \mathrm{Res}_0(\phi)\ \mathrm{Div}_0(\nu) = dT \qquad \text{on } X,$$

generalizing that of [HL]. Here $\mathrm{Div}_0(\nu)$ and $\mathrm{Div}_\infty(\nu)$ are integrally flat currents supported on the zero and pole sets of ν, $\mathrm{Res}_0(\phi)$ and $\mathrm{Res}_\infty(\phi)$ are smooth residue forms which can be calculated in terms of the curvature Ω_F of F, T is a canonical transgression form with coefficients in $L^1_{\mathrm{loc}}(X)$, and $\phi(\Omega_0)$ is an $L^1_{\mathrm{loc}}(X)$ form canonically defined in terms of a singular connection naturally associated to ν.

This result holds for C^∞–meromorphic sections ν which are atomic. The notion of an atomic section of a vector bundle was first introduced and studied in [HS]. The formulae obtained include a generalization of the Poincaré–Lelong formula to C^∞– meromorphic sections of a bundle of arbitrary rank.

Introduction.

The main result of this paper is a generalization of the following version of the argument principle for smooth maps from a manifold into complex projective space.

The argument principle.

Let H be a copy of $\mathbb{CP}^{n-1}$ in complex projective space $\mathbb{CP}^n$, and let q be a point in $\mathbb{CP}^n \sim H \cong \mathbb{C}^n$. Let X be a compact oriented real $2n$–dimensional smooth manifold with boundary ∂X, and let $f : X \to \mathbb{CP}^n$ be a smooth map. Suppose that $f^{-1}(q) = \{z_1, ..., z_N\}$ and (for the sake of simplicity) that $P = f^{-1}(H)$ is a smooth oriented connected $2(n-1)$–dimensional submanifold of X. Suppose that $(f^{-1}(q) \cup P) \cap \partial X = \varnothing$. Then

$$(\star) \qquad \sum_{i=1}^N \mathrm{Mult}(f, z_i) \ - \ \mathrm{Deg}(f\,\big|_P)\ \mathrm{Mult}(f, P) \ = \ \mathrm{Wind}(f, \partial X).$$

Here $\mathrm{Deg}(f\,\big|_P)$ *is the degree of the mapping $f : P \to H$. Let $\pi : \mathbb{C}^n \sim \{q\} \to S^{2n-1}$ be radial projection. Then* $\mathrm{Mult}(f, z_i) := \mathrm{Deg}(\pi \circ f : \partial B(z_i) \to S^{2n-1})$ *where $B(z_i)$ is a ball about z_i not containing any other points of $f^{-1}(q) \cup P$. To define* $\mathrm{Mult}(f, P)$ *note that for* *each $x \in P$ f defines a map from the normal plane N_xP to $N_{f(x)}H$. Then, for almost any $x \in P$,* $\mathrm{Mult}(f,P)$ *is the degree of the induced map between circles. Finally the winding number,* $\mathrm{Wind}(f,\partial X)$*, of f around ∂X is defined to be the sum of the degrees of the induced maps from the various connected components of ∂X to the unit sphere in $\mathbb{C}^n \cong \mathbb{CP}^n \sim H$.*

This well known result is an immediate consequence of a C^∞ generalization to arbitrary rank vector bundles of the Poincaré–Lelong equation of complex analysis, which is derived in this paper. The generalized Poincaré–Lelong equation is in turn a compactification of a Chern current formula derived by Harvey and Lawson in their beautiful paper " A theory of characteristic currents associated with a singular connection", [HL] (also see [HL–A]). Their theory associates canonical current representatives of the characteristic classes of a vector bundle to sections of that bundle. In this paper we generalize their work by computing the currents associated to "C^∞–meromorphic" sections of a vector bundle. The analytic underpinning for [HL], and for the results described here, is given in detail in Harvey and Semmes' paper "Zero divisors of atomic functions", [HS].

Let $F \to X$ be a rank n complex vector bundle over an oriented manifold, and let ϕ be an Ad–invariant polynomial on $\mathfrak{gl}(n,\mathbb{C})$. For each choice of connection on F the Chern–Weil theory constructs a d–closed differential form $\phi(\Omega_F)$ on X, called the ϕ–Chern form, which represents the ϕ–Chern class of F and is defined in terms of the curvature of the connection. Harvey and Lawson generalize this theory by associating to each sufficiently well behaved smooth section α of the vector bundle F with connection, a canonically defined d–closed current $\phi(\Omega_{F,\alpha})$, called the *ϕ–Chern current* of α, which also represents the ϕ–Chern class of F. This current can be expressed as the sum of an absolutely continuous part and a singular part which is supported on the zero set of the section α. They also construct a canonically defined transgression current $T = T(\phi,\alpha)$ satisfying the current equation

$$\phi(\Omega_{F,\alpha}) - \phi(\Omega_F) = dT \qquad \text{on } X.$$

The most interesting example is obtained by choosing the invariant polynomial to be the top Chern polynomial $c_n = (\frac{i}{2\pi})^n \det$. Then, if 0 is a regular value of α, the Chern current $c_n(\Omega_{F,\alpha})$ is equal to the current $[Z]$ of integration over the zero set Z of α, and we have the fundamental current equation

$$[Z] - c_n(\Omega_F) = dT \qquad \text{on } X.$$

In this paper we study the ϕ–Chern currents associated to "meromorphic" sections of a complex vector bundle F. By a meromorphic section of F we mean a C^∞–section ν of the bundle $\mathbb{P}(\underline{\mathbb{C}} \oplus F) \to X$ of projective spaces which compactifies the bundle $F \to X$. The pole set of such a section ν is the subset P of X defined by $P := \nu^{-1}(\mathbb{P}(\underline{\mathbb{C}} \oplus F) \sim F)$. In analogy with Harvey and Lawson's work we associate to ν a canonically defined current $\phi(\Omega_{F,\nu})$ which also represents the ϕ–Chern class of F and whose singular support is contained in the union of the zero and pole sets of ν. On the complement of the pole set the meromorphic section ν defines a smooth section α of F and the current $\phi(\Omega_{F,\nu})$ is equal to Harvey and Lawson's current $\phi(\Omega_{F,\alpha})$. The main aim of this paper is to study the current $\phi(\Omega_{F,\nu})$ in a neighbourhood of the pole set P. For instance, in the special case that the zero and pole sets of ν do not intersect the critical set of the section ν, the c_n–Chern current of ν is of the form

$$c_n(\Omega_{F,\nu}) = [Z] - \mathrm{Res}\,[P] \qquad \text{on } X$$

where $[P]$ denotes the current of integration over the pole set P and Res is a d–closed smooth form. Furthermore, just as in Harvey and Lawson's setting, we derive a current equation of the form

$$[Z] - \mathrm{Res}\,[P] - c_n(\Omega_F) = dT \qquad \text{on } X.$$

This equation is a special case of a current equation which generalizes the Poincaré–Lelong formula of complex analysis. Note that the classical Poincaré–Lelong formula only holds for meromorphic sections of holomorphic *line* bundles, whereas the generalizations described in this paper hold for C^∞–meromorphic sections of smooth complex vector bundles of *arbitrary* rank.

Choosing F to be the the trivial bundle corresponds to the study of maps $f : X \to \mathbb{CP}^n$. Suppose that f satisfies the conditions given in the statement of the argument principle above. Then we have the current equation

$$\sum_{i=1}^{N} \mathrm{Mult}(f, z_i)\,[z_i] \;-\; \mathrm{Mult}(f, P)\,[P]\,f^*(\mathrm{Vol}) \;=\; dT$$

where Vol is the volume form on H in the Fubini–Study metric. The argument principle is obtained by letting the terms in this equation act on the test function $[X]$ on X.

We begin by describing Harvey and Lawson's construction of the Chern currents associated to a bundle map. Let E and F be complex vector bundles of ranks 1 and n defined over a smooth oriented manifold X. Endow E and F with connections D_E and D_F, and choose hermitian metrics on E and F. Let $E \overset{\alpha}{\to} F$ be a smooth bundle map, and let $Z = \{x \in X \,:\, \alpha_x \equiv 0\}$ be the zero set of α. Note that when E is the trivial bundle $\underline{\mathbb{C}}$ a bundle map α defines a section μ of F by $\alpha(1) = \mu$. Harvey and Lawson use the bundle map α to pushforward the connection D_E on E to a smooth connection $\overrightarrow{D}$ defined on F over $X \sim Z$ by

$$\overrightarrow{D} := \alpha\, D_E\, \beta + D_F\,(1 - \alpha\,\beta) = D_F - (D_F\,\alpha - \alpha\, D_E)\,\beta,$$

where $F \overset{\beta}{\to} E$ is the bundle map over $X \sim Z$ given by orthogonal projection of F onto the rank 1 subbundle $\mathrm{Im}(\alpha)$ followed by the inverse of the map $E \overset{\alpha}{\to} \mathrm{Im}(\alpha) \subset F$. The connection $\overrightarrow{D}$ is called the *singular pushforward connection associated with* α. It provides a way to relate the curvature operators of the bundles E and F to the zero set of the map $E \overset{\alpha}{\to} F$. This relationship is made explicit by computing "Chern currents" for $\overrightarrow{D}$.

Let ϕ be an Ad–invariant polynomial on $\mathfrak{gl}(n, \mathbb{C})$. The *$\phi$–Chern current of $\overrightarrow{D}$ at time zero*, $\phi((\overrightarrow{D}_0))$, on X is defined as follows. First we define a smooth family of smooth connections $\overrightarrow{D}_s$, for $0 < s < \infty$, on F over all of X by

$$\overrightarrow{D}_s = D_F + (\alpha\, D_E - D_F\,\alpha)\,\alpha^*\,(\alpha\alpha^* + s^2)^{-1},$$

where $F \xrightarrow{\alpha^*} E$ denotes the hermitian adjoint of α. This family has been chosen so that as $s \to 0$, $\overrightarrow{D}_s \to \overrightarrow{D}$ on F over $X \sim Z$ and as $s \to \infty$, $\overrightarrow{D}_s \to D_F$ on F over X. Then the Chern current at time zero on X is defined to be the current limit

$$\phi((\overrightarrow{D}_0)) := \lim_{s \to 0} \phi(\overrightarrow{D}_s) \qquad \text{on } X,$$

whenever this limit exists. Actually there are many families $\overrightarrow{D}_s$ which have the required limits as $s \to 0$ and $s \to \infty$. We choose to work with the particular family $\overrightarrow{D}_s$ defined above because it arises naturally from a geometric construction to be described below.

In order that the Chern current $\phi((\overrightarrow{D}_0))$ exist it is necessary to impose some conditions on the behaviour of the map α near its zero set. To that end Harvey and Semmes defined the notion of an *atomic* bundle map α and defined the *zero divisor*, $\mathrm{Div}_0(\alpha)$, of such a map. The zero divisor is an integrally flat current supported on the zero set of α which encodes the multiplicity of vanishing of α. Atomicity is a weak condition which ensures the existence of a zero divisor. The class of atomic sections includes those smooth sections which vanish algebraically on sets of the proper codimension. In particular real analytic sections whose zero sets are not too big are atomic.

Harvey and Lawson proved that, if the bundle map α is atomic, the Chern current $\phi((\overrightarrow{D}_0))$ exists and is of the form

$$\phi((\overrightarrow{D}_0)) = \phi(\overrightarrow{D}_0) + \mathrm{Res}_\phi(\overrightarrow{D}_0)\ \mathrm{Div}_0(\alpha) \qquad \text{on } X,$$

where $\phi(\overrightarrow{D}_0)$ is a d–closed L^1_{loc} form on X which agrees with the the smooth form $\phi(\overrightarrow{D})$ on $X \sim Z$, and the *residue form*, $\mathrm{Res}_\phi(\overrightarrow{D}_0)$, is a smooth form on X of classical Chern–Weil type which can be computed as a polynomial in the curvatures of the connections D_E and D_F. They also proved that the classical transgression formula

$$\phi(D_F) - \phi(\overrightarrow{D}_s) = dT_s,$$

involving smooth differential forms on X, converges, in the sense of currents as $s \to 0$, to the current equation

$$\phi(D_F) - \phi((\overrightarrow{D}_0)) = dT \qquad \text{on } X,$$

where the transgression current $T := \lim_{s \to 0} T_s$ converges in $L^1_{\mathrm{loc}}(X)$.

Let $\mathrm{Hom}(E, F) \to X$ denote the vector bundle of all bundle maps from E to F. This bundle can be compactified in the vertical directions by embedding it in the bundle of projective spaces $\mathbb{P}(E \oplus F) \to X$. This embedding is defined by sending a bundle map to its graph. Recall that $\mathbb{P}(E \oplus F) = \mathrm{Hom}(E, F) \sqcup \mathbb{P}(F)$. Each section ν of $\mathbb{P}(E \oplus F) \to X$ induces a bundle map $E \xrightarrow{\alpha} F$ defined over the complement of the pole set $P := \nu^{-1}(\mathbb{P}(F))$ of ν in X. We can apply the work of Harvey and Lawson described above to the induced map α over $X \sim P$ to construct the family $(\overrightarrow{D}_s, F)$ over $X \sim P$ and compute the Chern current at time zero on $X \sim P$. The **main results** of this paper can be stated as follows. The Chern forms, $\phi(\overrightarrow{D}_s)$ for $0 < s < \infty$, extend smoothly across P to all of X. Provided that the section ν of $\mathbb{P}(E \oplus F) \to X$ satisfies a suitable **atomic** hypothesis the Chern forms $\phi(\overrightarrow{D}_s)$ converge, in the sense of currents as $s \to \infty$, to the **Chern current at time infinity**, $\phi(\!(\overrightarrow{D}_\infty)\!)$, on X. Furthermore the Chern current $\phi(\!(\overrightarrow{D}_\infty)\!)$ is of the form

$$\phi(\!(\overrightarrow{D}_\infty)\!) = \phi(D_F) + \nu^*(\mathrm{Res}_\phi(\overrightarrow{D}_\infty))\ \mathrm{Div}_\infty(\nu) \qquad \text{on } X$$

where the **pole divisor**, $\mathrm{Div}_\infty(\nu)$, of ν is an integrally flat current supported on the pole set P of ν and the **residue form at infinity**, $\mathrm{Res}_\phi(\overrightarrow{D}_\infty)$, is a d–closed smooth form on $\mathbb{P}(F)$ whose cohomology class can be computed in terms of the Chern classes of the bundles E, F and of the tautological bundle L over $\mathbb{P}(F)$. Further properties of the residue form at infinity will be discussed in the body of the paper.

In addition the smooth transgression forms $T_{r,s}$, which satisfy the transgression formula

$$\phi(\overrightarrow{D}_r) - \phi(\overrightarrow{D}_s) = dT_{r,s}$$

for $0 < s < r < \infty$ on $X \sim P$, extend smoothly to all of X. As $r \to \infty$ and $s \to 0$ the transgression forms $T_{r,s}$ converge in $L^1_{\mathrm{loc}}(X)$ to the L^1_{loc} **transgression current** T and the current equation

$$\phi(\!(\overrightarrow{D}_\infty)\!) - \phi(\!(\overrightarrow{D}_0)\!) = dT \qquad \text{on } X$$

is the limiting form of the smooth transgression formula.

The main ideas used to prove that the smooth Chern and transgression forms, $\phi(\overrightarrow{D}_s)$ and $T_{r,s}$ for $0 < s < r < \infty$, on $X \sim P$ extend smoothly to all of X can be outlined as follows. Let $\mathbf{E}, \mathbf{F}$ be the pullbacks to $\mathbb{P}(E \oplus F)$ of E, F, and let $U \subset \mathbf{E} \oplus \mathbf{F}$ be the tautological line bundle over $\mathbb{P}(E \oplus F)$. We begin by constructing a certain family of smooth connections $D_{Q,s}$ for $0 < s < \infty$ on the pullback to X via ν of the quotient bundle $Q = \mathbf{E} \oplus \mathbf{F}/U$ over $\mathbb{P}(E \oplus F)$. Then we show that the family $(\overrightarrow{D}_s, F)$ is gauge equivalent to the family $(D_{Q,s}, \nu^*(Q))$ over the subset $X \sim P$ of X. That is, the family $(\overrightarrow{D}_s, F)$, which is only defined over $X \sim P$, is compactified by the family $(D_{Q,s}, \nu^*(Q))$, defined over all of X. Consequently the Chern and transgression forms of the family $(\overrightarrow{D}_s, F)$ over $X \sim P$ extend smoothly to all of X as the Chern and transgression forms of the family $(D_{Q,s}, \nu^*(Q))$ over X. In addition we will show that as $s \to \infty$ the family $D_{Q,s}$ converges smoothly over the complement of the pole set P of ν to a connection $D_{Q,\infty}$ defined on $\nu^*(Q)$ over $X \sim P$. Similarly, as the parameter $s \to 0$ the family $D_{Q,s}$ converges smoothly over the complement of the zero set Z of ν in X to a connection $D_{Q,0}$ defined on $\nu^*(Q)$ over $X \sim Z$. This last fact implies that the Chern current at time zero on X associated with an atomic section ν of $\mathbb{P}(E \oplus F) \to X$ has exactly the same form as it did in the case of a bundle map α.

Note that many of the results concerning the smooth forms $\phi(\overrightarrow{D}_s)$ induced by ν on X were already well understood by Harvey and Lawson (see [HL I.8,9]). We conclude this introduction with some additional comments.

The definition of an atomic section ν of $\mathbb{P}(E\oplus F)\to X$ is the one forced upon us in order that the zero and pole divisors of ν are well defined flat currents on X. If $[t,u]$ denotes a local homogeneous expression for the section ν of $\mathbb{P}(E\oplus F)\to X$ we simply require that the local n–vector–valued function $\frac{u}{t}$ and the scalar–valued functions $\frac{t}{u_j}$ for $j\in\{1,...,n\}$ are atomic functions in the sense of [HS].

The main result outlined above is nicely illustrated by seeing what it says in the trivial case. Let $E=\underline{\mathbb{C}}$ and $F=\underline{\mathbb{C}}^n$ be the trivial rank 1 and n bundles over $X=\mathbb{P}(\mathbb{C}\oplus\mathbb{C}^n)$, endowed with the trivial connection d. Let 0 denote the distinguished point $\mathbb{C}\oplus 0\in\mathbb{P}(\mathbb{C}\oplus\mathbb{C}^n)$ and let $\mathbb{P}(\mathbb{C}^n)$ be the hyperplane at infinity in $\mathbb{P}(\mathbb{C}\oplus\mathbb{C}^n)$. The identity section ν of $\mathbb{P}(\underline{\mathbb{C}}\oplus\underline{\mathbb{C}}^n)\to\mathbb{P}(\mathbb{C}\oplus\mathbb{C}^n)$ is atomic with $\mathrm{Div}_0(\nu)=[0]$ and $\mathrm{Div}_\infty(\nu)=[\mathbb{P}(\mathbb{C}^n)]$, where $[M]$ denotes the current of integration over a submanifold M of X. Then the c_n–Chern currents at time zero and time infinity on $\mathbb{P}(\mathbb{C}\oplus\mathbb{C}^n)$ are given by

$$c_n((\overrightarrow{D}_0))=[0]\qquad\text{and}\qquad c_n((\overrightarrow{D}_\infty))=\big(-c_1(D_L)\big)^{n-1}[\mathbb{P}(\mathbb{C}^n)],$$

where D_L is the connection induced on the tautological line bundle $L\subset\underline{\mathbb{C}}\oplus\underline{\mathbb{C}}^n$ by the trivial connection d. Note that the residue at infinity, $\big(-c_1(D_L)\big)^{n-1}$, is just the volume form on $\mathbb{P}(\mathbb{C}^n)$ in the Fubini–Study metric. Finally, the smooth volume forms $c_n(D_{Q,s})$ on $\mathbb{P}(\mathbb{C}\oplus\mathbb{C}^n)$ can be regarded as a compactification of a natural family of approximate identities on $\mathbb{C}^n$. This last fact can be generalized by saying that in the universal case the smooth forms $c_n(D_{Q,s})$ on $\mathbb{P}(\underline{\mathbb{C}}\oplus F)$ compactify a smooth family of Thom forms on the bundle F.

Two other special cases are worthy of comment. The first is the case that rank $F=1$. In this case we have the current equation

$$dT=c_1(D_F)^m-c_1(D_E)^m+\big(\mathrm{Div}_\infty(\nu)-\mathrm{Div}_0(\nu)\big)\,\frac{c_1(D_F)^m-c_1(D_E)^m}{c_1(D_F)-c_1(D_E)}$$

for $m\geq 1$ on X. When $m=1$ we have the formula

$$dT=c_1(D_F)-c_1(D_E)+\mathrm{Div}_\infty(\nu)-\mathrm{Div}_0(\nu),$$

which is a generalization of the Poincaré–Lelong formula of complex analysis to the smooth category. Note that these formulae are immediate corollaries of [HL II].

The second special case is that of a section of the compactification $\mathbb{P}(\underline{\mathbb{C}}\oplus F)\to X$ of a holomorphic vector bundle $F\to X$. It seems more natural to compute Chern currents of a section of the compactified bundle $\mathbb{P}(\underline{\mathbb{C}}\oplus F)$, rather than those of a meromorphic section of the bundle $F\to X$. Note that when the base manifold X is a Riemann surface the concepts of a meromorphic section of $F\to X$ and a holomorphic section of $\mathbb{P}(\underline{\mathbb{C}}\oplus F)$ coincide.

There are analogues of these results in the real case (see [HL IV] and [Z2]) and in the case of C^∞–meromorphic maps between quaternionic line bundles (see [HL V] and Section 6).

Contents.

Note that Sections 4,5 and 6 can be read independently of each other.

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1. Divisors of atomic sections of a compactified bundle.

Let E and F be hermitian bundles of ranks 1 and n over an oriented manifold X. In this section we apply the work of Harvey and Semmes [HS] to define the notion of an atomic section of the compactification $\mathbb{P}(E \oplus F) \to X$ of the bundle $\mathrm{Hom}(E, F) \to X$, define the zero and pole divisors of such a section and give some geometric conditions which ensure that a section is atomic. Harvey and Semmes make the following definition.

Definition 1.1. [HS 1.1]. Let $y = (y_1, \ldots, y_m)$ denote coordinates on $\mathbb{R}^m$ and let M be a smooth manifold. A smooth function $f : M \to \mathbb{R}^m$ $(m > 1)$ is called **atomic** if for each form $\frac{dy^I}{|y|^p}$ on $\mathbb{R}^m$, with $p = |I| \leq m - 1$, the pullback $f^*\left(\frac{dy^I}{|y|^p}\right)$ to M has an L^1_{loc} extension across the zero set Z of f. Also assume that f does not vanish identically in any connected component of its domain M.

Note. Harvey and Semmes [HS 1.2] prove that if f is atomic then the zero set Z has Lebesgue measure zero in M. Therefore the L^1_{loc} extension of $f^*\left(\frac{dy^I}{|y|^p}\right)$ across Z is unique.

We generalize Definition 1.1 by replacing the trivial bundle $\mathbb{R}^m \to \{0\}$ by a vector bundle $V \to X$ and the zero set Z of f by the inverse image of X under f. Here we identify X with the zero section of V.

Definition 1.2. Let $V \to X$ be a smooth real vector bundle of rank m and let M be a smooth manifold. A smooth mapping $f : M \to V$ is called **atomic with respect to X** if for every choice of local fibre coordinate y on V the locally defined function $y \circ f : M \to \mathbb{R}^m$ is atomic.

Remark 1.3. Harvey and Semmes [HS 2.5] define the notion of an *atomic section* of a bundle $V \to X$. This definition is simply the one given above in the special case that the mapping f is a section of the bundle $V \to X$. Furthermore let $\mathbf{V}$ denote the pullback to M of the bundle $\pi : V \to X$ via the mapping $\pi \circ f : M \to X$. Note that the mapping $f : M \to V$ defines a section $\widetilde{f}$ of the bundle $\mathbf{V} \to M$. Then f is atomic with respect to X iff $\widetilde{f}$ is an atomic section of $\mathbf{V} \to M$.

Note. This observation combined with [HS 2.4] ensures that the atomicity with respect to X of a mapping f is a well defined notion.

Let

$$\theta := \sum_{k=1}^{m} (-1)^{k-1} \frac{y_k dy_1 \wedge \cdots \wedge \widehat{dy_k} \wedge \cdots \wedge dy_m}{|y|^m}$$

denote the **solid angle kernel** on $\mathbb{R}^m$, and let γ_m denote the volume of the unit sphere in $\mathbb{R}^m$. The fundamental current equation $d\,(\gamma_m^{-1}\theta) = [0]$ on $\mathbb{R}^m$, where $[0]$ denotes the point mass at the origin, motivates the definition of divisor. Note that if $f : M \to \mathbb{R}^m$ is atomic then $f^*(\theta)$ defines an L^1_{loc} current on M.

Definition 1.4. Let V be a smooth oriented real vector bundle of rank m over a smooth oriented manifold X. Let $f : M \to V$ be atomic with respect to X and let y be any positively oriented local fibre coordinate on V. The **divisor of f with respect to X**, denoted $\text{Div}_X(f)$, is defined by the current equation

$$\text{Div}_X(f) := d(\gamma_m^{-1}\,(y \circ f)^*\theta) \qquad \text{on } M.$$

Note. The divisor $\text{Div}_X(f)$ is a degree m current supported on the subset $f^{-1}(X)$ of M.

Remark 1.5. Note that the divisor of f with respect to X is simply the divisor of the section $\widetilde{f}$ of $\mathbf{V} \to M$. Therefore we can apply Theorem 2.11 of [HS] to show that the divisor of a mapping which is atomic with respect to X is well defined independent of choice of positively oriented local fibre coordinate on V.

Let $\mathbb{P}(E \oplus F) \to X$ denote the bundle of projective spaces whose fibre over a point $x \in X$ is the set of all one dimensional complex subspaces of $E_x \oplus F_x$. The ideas presented above can be applied to a section of the compactified bundle $\mathbb{P}(E \oplus F) \to X$ as follows. First we recall some of the basic structure of the bundle $\mathbb{P}(E \oplus F) \to X$. The natural embedding

$$\begin{aligned} \text{Hom}(E, F) &\hookrightarrow \mathbb{P}(E \oplus F) \\ \alpha_x \quad &\mapsto \text{Graph}(\alpha_x), \end{aligned} \tag{1.6}$$

where $\text{Graph}(\alpha_x) = \{\, [v, \alpha_x(v)] \in \mathbb{P}(E_x \oplus F_x) \,/\, v \in E_x \sim \{0\} \,\}$, provides a compactification of the bundle $\text{Hom}(E, F) \to X$ in the vertical directions. The total space of the projectivized bundle $\mathbb{P}(F) \to X$ can be regarded as a complex–codimension 1 submanifold of $\mathbb{P}(E \oplus F)$ by the inclusion

$$\begin{aligned} \mathbb{P}(F) &\hookrightarrow \mathbb{P}(E \oplus F) \\ [f] &\mapsto \quad [0, f]. \end{aligned}$$

Note that $\mathbb{P}(E \oplus F) = \mathrm{Hom}(E, F) \sqcup \mathbb{P}(F)$. Also recall that X embeds in $\mathbb{P}(E \oplus F)$ as the image of the zero section of $\mathrm{Hom}(E, F) \subset \mathbb{P}(E \oplus F)$ and that the projection

$$\begin{aligned} \pi : \mathbb{P}(E \oplus F) \sim X &\longrightarrow \mathbb{P}(F) \\ [e, f] \quad &\longmapsto [f] \end{aligned} \tag{1.7}$$

has the natural structure of a complex line bundle. Finally, let $Z = \nu^{-1}(X)$ denote the **zero set** and $P = \nu^{-1}(\mathbb{P}(F))$ the **pole set** of a section ν of $\mathbb{P}(E \oplus F) \to X$. Note that $Z \cap P = \varnothing$.

Definition 1.8. A smooth section $\nu : X \to \mathbb{P}(E \oplus F)$ is called **atomic** if the following two conditions hold.

(1) The induced section $\nu : X \sim P \to \mathrm{Hom}(E, F)$ is atomic.
(2) The induced mapping $\nu : X \sim Z \to \mathbb{P}(E \oplus F) \sim X$ is atomic with respect to $\mathbb{P}(F)$.

Thanks to [HS 1.2] the zero and pole sets of an atomic section of the bundle $\mathbb{P}(E \oplus F) \to X$ have Lebesgue measure zero.

Definition 1.9. Let ν be an atomic section of the bundle $\mathbb{P}(E \oplus F) \to X$. Then

(1) The **zero divisor** of ν, denoted $\mathrm{Div}_0(\nu)$, is defined to be the extension from $X \sim P$ to all of X of the divisor of the induced section $\nu : X \sim P \to \mathrm{Hom}(E, F)$.
(2) The **pole divisor** of ν, denoted $\mathrm{Div}_\infty(\nu)$, is defined to be the extension from $X \sim Z$ to all of X of the divisor with respect to $\mathbb{P}(F)$ of the induced section $\nu : X \sim Z \to \mathbb{P}(E \oplus F) \sim X$.

Note that the zero (resp. pole) divisor of an atomic section of $\mathbb{P}(E \oplus F) \to X$ is a codimension (or degree) $2n$ (resp. 2) d–closed flat current on X.

Remark 1.10. Harvey and Semmes prove that a large class of smooth sections of $\mathbb{P}(E \oplus F) \to X$ are atomic. More specifically, those sections of $\mathbb{P}(E \oplus F) \to X$ which approach their zero and pole sets algebraically and whose zero and pole sets are not too big in the sense of Minkowski are atomic. In addition they show that the zero and pole divisors have some very beautiful structure. The reader is encouraged to consult [HS] for more information.

2. The induced singular connections and their smooth approximations.

We begin this section by recalling the definition of the singular pushforward and pullback connections and their smooth approximating families. Then we define the Chern currents at time zero and time infinity and review Harvey and Lawson's results concerning the Chern current at time zero. Next we present the main result of this section, proving that the smooth approximating Chern and transgression forms defined by Harvey and Lawson on the complement of the pole set of a section ν of $\mathbb{P}(E \oplus F) \to X$ have a natural extension to all of X. Finally we derive a formula for the smooth approximating transgression form in a local homogeneous coordinate system on $\mathbb{P}(E \oplus F)$. This formula will be useful in Section 3 where we address the problem of computing the Chern current at time infinity.

Let E and F be rank 1 and rank n hermitian bundles endowed with connections D_E and D_F. The **Chern forms** of a connection D on a complex vector bundle V of rank m are defined as usual (c.f. [BC]). Let ω be a locally defined connection matrix or gauge for D and let $\Omega := d\omega - \omega \wedge \omega$ be the corresponding curvature matrix. The **total Chern form** c is defined by

$$c(D) = c(\Omega) := \det\left(I + \tfrac{i}{2\pi}\Omega\right)$$

and the **Chern character** is defined by

$$\mathrm{ch}(D) = \mathrm{ch}(\Omega) := \mathrm{tr}\left(e^{\frac{i}{2\pi}\Omega}\right).$$

More generally, for any polynomial ϕ on $\mathfrak{gl}(m,\mathbb{C})$ which is invariant under the adjoint action, the $\boldsymbol{\phi}$**–Chern form** is defined by

$$\phi(D) := \phi(\Omega).$$

Let D_s $(0 < s < \infty)$ be a family of connections on V. Set $\phi(A;B) := \frac{d}{dt}\big|_{t=0} \phi(B+tA)$. The **transgression form** $T_{r,s}$ on X which satisfies the standard transgression formula

$$\phi(D_r) - \phi(D_s) = dT_{r,s} \qquad \text{on } X \tag{2.1}$$

is defined (c.f. [BC]) by

$$T_{r,s} := \int_s^r \phi\left(\tfrac{\partial \omega_s}{\partial s}\,;\,\Omega_s\right) ds. \tag{2.2}$$

Following [HL] the connection D_E on E can be pushed forward via a section ν of $\mathbb{P}(E \oplus F) \to X$ to a singular connection $\overrightarrow{D}$ defined on F over $X \sim (Z \cup P)$ as follows.

Definition 2.3. Let $E \overset{\alpha}{\to} F$ over $X \sim P$ be the bundle map induced from a section ν of $\mathbb{P}(E \oplus F) \to X$ by the inclusion (1.6). The **singular pushforward connection associated with $\boldsymbol{\nu}$** is the smooth connection $\overrightarrow{D}$ on F over $X \sim (Z \cup P)$ defined by

$$\overrightarrow{D} := \alpha\, D_E\, \beta + D_F\,(1 - \alpha\,\beta) = D_F - (D_F\,\alpha - \alpha\, D_E)\,\beta,$$

where $F \overset{\beta}{\to} E$ is the bundle map over $X \sim (Z \cup P)$ given by orthogonal projection of F onto the subbundle $\mathrm{Im}\alpha$ followed by the inverse of the map $E \overset{\alpha}{\to} \mathrm{Im}\alpha \subset F$.

Similarly the connection D_F on F can be pulled back by ν to a connection $\overleftarrow{D}$ on E over $X \sim (Z \cup P)$.

Definition 2.4. The **singular pullback connection associated with $\boldsymbol{\nu}$** is the smooth connection $\overleftarrow{D}$ on E over $X \sim (Z \cup P)$ defined by

$$\overleftarrow{D} := \beta\, D_F\, \alpha + (1 - \beta\,\alpha)\, D_E = D_E + \beta\,(D_F\,\alpha - \alpha\, D_E).$$

Note. Let $F \overset{\alpha^*}{\to} E$ denote the adjoint of α. Then the bundle map β defined above is given by

$$\beta = (\alpha^* \alpha)^{-1} \alpha^*.$$

Harvey and Lawson considered the case of a smooth bundle map $E \overset{\alpha}{\to} F$ over X (i.e. $P = \varnothing$). One of their main aims was to compute "Chern currents" for the singular connection $\overrightarrow{D}$ over X. To do this they constructed a smooth family $\overrightarrow{D}_s$, for $0 < s < \infty$, of smooth connections on F over all of X so that as $s \to 0$, $\overrightarrow{D}_s \to \overrightarrow{D}$ as smooth connections on F over $X \sim Z$, and as $s \to \infty$, $\overrightarrow{D}_s \to D_F$ as smooth connections on F over X.

Definition 2.5.
(1) Such a family $\overrightarrow{D}_s$ is called a **smooth approximating family** to the singular connection $\overrightarrow{D}$.
(2) If as $s \to 0$ the corresponding Chern forms $\phi(\overrightarrow{D}_s)$ on X converge weakly as currents on X, then the limit will be denoted $\phi(\!(\overrightarrow{D}_0)\!)$ and referred to as the **ϕ–Chern current** of the singular connection $\overrightarrow{D}$.

Harvey and Lawson use an "approximate one" (c.f. [HL I.4.1]) to construct a smooth approximating family to the singular connection $\overrightarrow{D}$. Different choices of approximate one yield different approximating families. Because of its geometrical significance (see Theorems 2.20 and 2.22 below) we will restrict our attention to the **algebraic approximation mode** $\chi : [0, \infty] \to [0, 1]$ defined by $\chi(t) = \frac{t}{1+t}$.

Returning to the case of a section ν of $\mathbb{P}(E \oplus F) \to X$ we make the following definitions.

Definition 2.6. The singular pushforward connection $\overrightarrow{D}$ associated with ν is smoothed over $X \sim P$ by the **approximating family of smooth connections** $\overrightarrow{D}_s$ $(0 < s \leq \infty)$ on F over $X \sim P$ defined by

$$\overrightarrow{D}_s = D_F + (\alpha\, D_E - D_F\, \alpha)\, \alpha^*\, (\alpha\alpha^* + s^2)^{-1} = (s^2\, D_F + \alpha\, D_E\, \alpha^*)\, (\alpha\alpha^* + s^2)^{-1}.$$

Note that

(2.7) as $s \to 0$, $\overrightarrow{D}_s \to \overrightarrow{D}_0 = \overrightarrow{D}$ as smooth connections on F over $X \sim (Z \cup P)$, and

(2.8) as $s \to \infty$, $\overrightarrow{D}_s \to \overrightarrow{D}_F$ as smooth connections on F over $X \sim P$.

Similarly in the pullback case we define

(2.9) $$\overleftarrow{D}_s := D_E + (\alpha^*\alpha + s^2)^{-1}\, \alpha^*\, (D_F\alpha - \alpha D_E) = (\alpha^*\alpha + s^2)^{-1}\, (s^2 D_E + \alpha^* D_F \alpha).$$

Note that $\overleftarrow{D}_\infty = D_E$.

To simplify the discussion we will restrict our attention to the pushforward case. The discussion in the pullback case proceeds along similar lines. We will show that for $0 < s < \infty$

the Chern forms $\phi(\overrightarrow{D}_s)$ and the corresponding transgression forms $T_{r,s}$ defined initially over $X \sim P$ extend across the pole set P to smooth forms on all of X. Suppose now that ν is an atomic section of $\mathbb{P}(E \oplus F) \to X$. The main aim of this paper is to show that the **Chern current at time zero**,

$$\phi(\!(\overrightarrow{D}_0)\!) := \lim_{s\to 0} \phi(\overrightarrow{D}_s), \tag{2.10}$$

and the **Chern current at time infinity**,

$$\phi(\!(\overrightarrow{D}_\infty)\!) := \lim_{r\to \infty} \phi(\overrightarrow{D}_r), \tag{2.11}$$

exist on all of X, and to compute them. Harvey and Lawson have solved this problem in the case that the pole set P is empty, i.e. in the case of an atomic bundle map $E \overset{\alpha}{\to} F$ over X.

Theorem 2.12. *[HL: III] Let $E \overset{\alpha}{\to} F$ be an atomic bundle map. Then*

(1) *The transgression forms $T_{\infty,s}$ converge as L^1_{loc}–currents on X to the transgression current $T_{\infty,0} := \lim_{s\to 0} T_{\infty,s}$.*
(2) *Therefore the Chern current $\phi(\!(\overrightarrow{D}_0)\!) := \lim_{s\to 0} \phi(\overrightarrow{D}_s)$ exists on X and is given by*

$$\phi(\!(\overrightarrow{D}_0)\!) = \phi(D_F) + dT_{\infty,0}.$$

Note that $\phi(\!(\overrightarrow{D}_0)\!)$ is a d–closed flat current on X which represents the ϕ–characteristic class of F.
(3) *Let $\phi(\overrightarrow{D}_0)$ denote the smooth form $\phi(\overrightarrow{D})$ on $X \sim Z$. Then $\phi(\overrightarrow{D}_0)$ extends across Z to a d–closed L^1_{loc} form on all of X, also denoted by $\phi(\overrightarrow{D}_0)$. Note that $\phi(\overrightarrow{D}_0)$ is called the L^1_{loc} part of the Chern current $\phi(\!(\overrightarrow{D}_0)\!)$.*
(4) *The Chern current $\phi(\!(\overrightarrow{D}_0)\!)$ is given by*

$$\phi(\!(\overrightarrow{D}_0)\!) = \phi(\overrightarrow{D}_0) + \mathrm{Res}_\phi(\overrightarrow{D}_0)\ \mathrm{Div}_0(\alpha),$$

where the residue form $\mathrm{Res}_\phi(\overrightarrow{D}_0)$ is a d–closed smooth form on X. Note that the residue form is independent of the choice of bundle map α and that the divisor is independent of the choice of invariant polynomial ϕ.
(5) *The current equation*

$$\phi(D_F) - \phi(\overrightarrow{D}_0) - \mathrm{Res}_\phi(\overrightarrow{D}_0)\ \mathrm{Div}_0(\alpha) = dT_{\infty,0} \qquad \text{on } X$$

is the limit as $s \to 0$ of the smooth equation

$$\phi(D_F) - \phi(\overrightarrow{D}_s) = dT_{\infty,s} \qquad \text{on } X$$

(6) *In particular if $\phi = c_n$ is the top Chern polynomial and $E = \underline{\mathbb{C}}$ is the trivial bundle we have*

$$c_n(D_F) - \mathrm{Div}_0(\alpha) = dT_{\infty,0} \qquad \text{on } X.$$

Note. Analogous results hold in the pullback case (see [HL III.8]). Also note that the residue form $\mathrm{Res}_\phi(\overrightarrow{D}_0)$ has many beautiful properties (see [HL III.2,7]).

In the general case the families $(\overrightarrow{D}_s, F)$ and $(\overleftarrow{D}_s, E)$ for $0 < s < \infty$ do not extend from $X \sim P$ to all of X. However we will construct families $(D_{Q,s}, Q)$ and $(D_{U,s}, U)$ defined over *all* of X whose Chern and transgression forms are smooth extensions to all of X of the Chern and transgression forms of the families $(\overrightarrow{D}_s, F)$ and $(\overleftarrow{D}_s, E)$ on $X \sim P$. The construction of the families $(D_{Q,s}, Q)$ and $(D_{U,s}, U)$ is most easily described in the universal case.

Remark 2.13. The universal case. Let $\mathbf{E} = p^*E$ and $\mathbf{F} = p^*F$ be the pullbacks of E and F to $\mathbb{P}(E \oplus F)$ via the projection map $p : \mathbb{P}(E \oplus F) \to X$, and let $\mathbf{D_E}$ and $\mathbf{D_F}$ be the pullbacks to $\mathbf{E}$ and $\mathbf{F}$ of the connections D_E and D_F. The induced bundle $\mathbb{P}(\mathbf{E} \oplus \mathbf{F}) \to \mathbb{P}(E \oplus F)$ has a tautological or universal section $\boldsymbol{\nu}$ which sends a line $l \in \mathbb{P}(E \oplus F)$ to itself. The section $\boldsymbol{\nu}$ is atomic with zero divisor $\mathrm{Div}_0(\boldsymbol{\nu}) = [X]$ and pole divisor $\mathrm{Div}_\infty(\boldsymbol{\nu}) = [\mathbb{P}(F)]$. For any section ν of $p : \mathbb{P}(E \oplus F) \to X$ we have $\nu^*\mathbf{E} = E$ and $\nu^*\mathbf{F} = F$ as bundles with connections, and $\nu^*(\boldsymbol{\nu}) = \nu$. So to prove results of a smooth nature on or over X which involve a section ν it often suffices to prove them on or over $\mathbb{P}(E \oplus F)$ for the universal section $\boldsymbol{\nu}$ and then pullback the result to X via ν.

Definition 2.14. Let V, W be isomorphic vector bundles. A family of connections $D_{V,s}$ on V is said to be **equivalent** to a family of connections $D_{W,s}$ on W $(0 < s < \infty)$ if there is a bundle isomorphism $\varphi : V \to W$ (independent of s) so that

$$D_{W,s} = \varphi \circ D_{V,s} \circ \varphi^{-1} \tag{2.15}$$

for all $s \in (0, \infty)$.

This concept is important because of the following fact.

Lemma 2.16. *Equivalent families of connections have equal Chern and transgression forms.*

In the universal case the family $(D_{U,s}, U)$ is constructed over $\mathbb{P}(E \oplus F)$ as follows. Let U denote the tautological line bundle over $\mathbb{P}(E \oplus F)$ and let U_s $(0 < s < \infty)$ be the pullback of U over $\mathbb{P}(E \oplus F)$ by the flow

$$\begin{aligned} \Psi_s : \mathbb{P}(E \oplus F) &\longrightarrow \mathbb{P}(E \oplus F) \\ [e, f] &\longmapsto [se, f]. \end{aligned} \tag{2.17}$$

We can induce a connection D_{U_s} on U_s over $\mathbb{P}(E \oplus F)$ by regarding U_s as a subbundle of $\mathbf{E} \oplus \mathbf{F}$ and defining

$$D_{U_s} = \mathrm{pr}_s \circ \mathbf{D_{E \oplus F}} \qquad \text{operating on sections of } U_s, \tag{2.18}$$

where $\mathrm{pr}_s : \mathbf{E} \oplus \mathbf{F} \to U_s$ denotes orthogonal projection. Then we define the family of connections $D_{U,s}$ on U over $\mathbb{P}(E \oplus F)$ by

$$D_{U,s} := \Psi_s^{-1} \circ D_{U_s} \circ \Psi_s, \tag{2.19}$$

where we now regard Ψ_s as a bundle map from U to U_s.

The following result is a corollary of Theorem I.8.14 of [HL]. Note that the polynomials $c_1, c_1^2, c_1^3, \ldots$ generate the set of all Ad–invariant polynomials on $\mathfrak{gl}(1,\mathbb{C})$ and that $c^{-1} := (1+c_1)^{-1} = \sum_{j=0}^{\infty}(-1)^j c_1^j$.

Theorem 2.20.
(1) The family $(\overleftarrow{D}_s, \mathbf{E})$ *(*$0 < s < \infty$*) is equivalent over* $\mathrm{Hom}(E,F)$ *to the family* $(D_{U,s}, U)$*. More specifically*

$$D_{U,s} = \gamma \circ \overleftarrow{D}_s \circ \gamma^{-1} \qquad \textit{over } \mathrm{Hom}(E,F)$$

where $\gamma : \mathbf{E} \to U$ *is the graphing map,* $\gamma(e) := (e, \alpha(e))$*, for* $e \in \mathbf{E}_\alpha$*.*

(2) Therefore the Chern forms $c^{-1}(\overleftarrow{D}_s)$ *(*$0 < s < \infty$*) and the transgression forms* $T_{r,s}$ *(*$0 < s < r < \infty$*) of the family* $(\overleftarrow{D}_s, E)$ *on* $\mathrm{Hom}(E,F)$ *extend smoothly to all of* $\mathbb{P}(E \oplus F)$ *as the Chern and transgression forms of the family* $(D_{U,s}, U)$ *over* $\mathbb{P}(E \oplus F)$*.*

The corresponding result in the pushforward case can be described as follows. Let $G_n(F^* \oplus E^*)$ denote the Grassmannian of n-planes in $F^* \oplus E^*$, and let U_n be the tautological rank n bundle over $G_n(F^* \oplus E^*)$. Recall that there is a natural bundle isomorphism

$$\varphi : \mathbb{P}(E \oplus F) \to G_n(F^* \oplus E^*)$$

defined by sending a line to its annihilator, and that the pullback $\varphi^*(U_n)$ to $\mathbb{P}(E \oplus F)$ of the tautological bundle over $G_n(F^* \oplus E^*)$ is the dual Q^* of the quotient bundle $Q = \mathbf{E} \oplus \mathbf{F}/U$. Let $D_{U_n,s}$ denote the connection defined on $U_n \to G_n(F^* \oplus E^*)$ as in (2.19).

Definition 2.21. The family of connections $D_{Q,s}$ on Q over $\mathbb{P}(E \oplus F)$ is defined to be the family dual to the family of connections induced on $\varphi^*(U_n)$ by the family $D_{U_n,s}$ on U_n over $G_n(F^* \oplus E^*)$.

Theorem 2.22.
(1) The family $(\overrightarrow{D}_s, \mathbf{F})$ *(*$0 < s < \infty$*) is equivalent over* $\mathrm{Hom}(E,F)$ *to the family* $(D_{Q,s}, Q)$*. More specifically*

$$D_{Q,s} = \psi \circ \overrightarrow{D}_s \circ \psi^{-1} \qquad \textit{over } \mathrm{Hom}(E,F),$$

where the bundle isomorphism $\psi : \mathbf{F} \to Q$ *over* $\mathrm{Hom}(E,F)$ *is defined by*

$$\psi(f) := [0, f] \qquad \textit{for } f \in \mathbf{F}.$$

Here $[e,f]$ *denotes the image of* (e,f) *in* Q*.*

(2) Therefore the Chern forms $\phi(\overrightarrow{D}_s)$ *(*$0 < s < \infty$*) and the transgression forms* $T_{r,s}$ *(*$0 < s < r < \infty$*) of the family* $(\overrightarrow{D}_s, F)$ *on* $\mathrm{Hom}(E,F)$ *extend smoothly to all of* $\mathbb{P}(E \oplus F)$ *as the Chern and transgression forms of the family* $(D_{Q,s}, Q)$ *over* $\mathbb{P}(E \oplus F)$*.*

Note. As suggested in Remark 2.13, Theorems 2.20 and 2.22 also hold for a smooth section ν of $\mathbb{P}(E \oplus F) \to X$ provided that we replace all spaces, bundles, connections and bundle maps by their pullbacks to X via ν.

Proof. First note that the restriction of the mapping φ to the finite part $\mathrm{Hom}(E, F)$ of $\mathbb{P}(E \oplus F)$ is given by

$$\begin{array}{ccc} \mathrm{Hom}(E,F) & \xrightarrow{\varphi} & \mathrm{Hom}(F^*,E^*) \\ \alpha & \longmapsto & -\widetilde{\alpha} \end{array} \tag{2.23}$$

where the dual homomorphism $\widetilde{\alpha}$ is defined by $\widetilde{\alpha}(f^*)(e) := f^*(\alpha(e))$. Now by Theorem 2.20, the family $(\overleftarrow{D}_{s,F^*}, \mathbf{F}^*)$ is equivalent over $\mathrm{Hom}(F^*, E^*)$ to the family $(D_{U_n,s}, U_n)$. The isomorphism $\widetilde{\gamma} : \mathbf{F}^* \to U_n$ setting up this equivalence is given by $\widetilde{\gamma}(f^*) = (f^*, \widetilde{\alpha}(f^*))$. This data pulls back via the mapping $\varphi : \mathrm{Hom}(E,F) \to \mathrm{Hom}(F^*, E^*)$ of (2.23) to the isomorphism

$$(\mathbf{F}^*, \overleftarrow{D}_{s,F^*}) \xrightarrow{\varphi^*(\widetilde{\gamma})} (\varphi^*(U_n), \varphi^*(D_{U_n,s})) \qquad \text{over } \mathrm{Hom}(E,F).$$

Since $\overrightarrow{D}_{s,F} = \big(\overleftarrow{D}_{s,F^*}\big)^*$ on $\mathbf{F}$ (see [HL I.4.28]) and

$$(Q, D_{Q,s}) = \big(\varphi^*(U_n)^*, (\varphi^*(D_{U_n,s}))^*\big)$$

the dual isomorphism is given by

$$(Q, D_{Q,s}) \xrightarrow{\eta} (\mathbf{F}, \overrightarrow{D}_s) \qquad \text{over } \mathrm{Hom}(E,F),$$

where $\eta := \widetilde{\varphi^*(\widetilde{\gamma})}$.

To complete the proof we just need to identify the map η and show that its inverse is the mapping ψ. Let $f^* \in \mathbf{F}^*$ and $(e,f) \in \mathbf{E} \oplus \mathbf{F}$ be arbitrary. An element $[e,f] \in Q$ defines a linear functional $T_{[e,f]}$ on $\varphi^*(U_n)$ by $T_{[e,f]}(v^*, w^*) = v^*(f) - w^*(e)$for $(v^*, w^*) \in \varphi^*(U_n) \subset \mathbf{F}^* \oplus \mathbf{E}^*$. So by the definition of the dual map and using the identification of $\mathbf{F}$ with $\mathbf{F}^{**}$ we have $\eta([e,f])(f^*) = T_{[e,f]}\big(\widetilde{\gamma}(f^*)\big) = f^*(f) - \widetilde{\alpha}(f^*)(e) = (f - \alpha(e))(f^*)$ and so

$$\eta([e,f]) = f - \alpha(e).$$

Finally it is easy to check that $\eta^{-1}(f) = [0,f] = \psi(f)$. $\square$

Note. The extensions of the Chern forms $c^{-1}(\overleftarrow{D}_s)$ and $\phi(\overrightarrow{D}_s)$ and the transgression forms $T_{r,s}$ from $\mathrm{Hom}(E,F)$ to all of $\mathbb{P}(E \oplus F)$ will also be denoted by $c^{-1}(\overleftarrow{D}_s)$, $\phi(\overrightarrow{D}_s)$ and $T_{r,s}$.

Lemma 2.24. *[HL I.9.2]*
(1) Over $\mathbb{P}(E \oplus F) \sim X$ *the smooth family of subbundles* $U_s \subset \mathbf{E} \oplus \mathbf{F}$ *for* $0 < s < \infty$ *extends smoothly to* $s = 0$ *where* $U_0 = \pi^* L$ *is the pullback to* $\mathbb{P}(E \oplus F) \sim X$ *of the universal bundle* $L \to \mathbb{P}(F)$ *via the natural map* $\pi : \mathbb{P}(E \oplus F) \sim X \to \mathbb{P}(F)$ *defined by (1.7).*
(2) Over $\mathrm{Hom}(E,F) \subset \mathbb{P}(E \oplus F)$ *the smooth family of subbundles* $U_s \subset \mathbf{E} \oplus \mathbf{F}$ *for* $0 < s < \infty$ *extends smoothly to* $s = \infty$ *where* $U_\infty = \mathbf{E}$.

Note.
(1) The bundle U_0 on $\mathbb{P}(E \oplus F) \sim X$ does not extend smoothly across X.
(2) Even though the bundle $U_\infty = \mathbf{E}$ on $\mathrm{Hom}(E,F)$ extends smoothly across $\mathbb{P}(F)$ the smooth family U_s over $\mathrm{Hom}(E,F)$ for $0 < s \le \infty$ does not extend smoothly across $\mathbb{P}(F)$.

Corollary 2.25.
(1) The families $(D_{U,s}, U)$ and $(D_{Q,s}, Q)$ extend smoothly to $s = 0$ over $\mathbb{P}(E \oplus F) \sim X$.
(2) The families $(D_{U,s}, U)$ and $(D_{Q,s}, Q)$ extend smoothly to $s = \infty$ over $\mathbb{P}(E \oplus F) \sim \mathbb{P}(F)$.

Combining this corollary with Theorem 2.12 we have

Theorem 2.26. *[HL I.9.12,III.7.6]*
Let ν be an atomic section of $\mathbb{P}(E \oplus F) \to X$.

(1) *The smooth form $\phi(\overrightarrow{D}_0)$ on $X \sim Z$ extends across Z to a d–closed L^1_{loc} form on all of X, also denoted by $\phi(\overrightarrow{D}_0)$.*
(2) *The Chern current $\phi(\!(\overrightarrow{D}_0)\!)$ exists on X and equals*

$$\phi(\!(\overrightarrow{D}_0)\!) = \phi(\overrightarrow{D}_0) + \mathrm{Res}_\phi(\overrightarrow{D}_0)\,[X].$$

Corollary 2.25 also enables us to show that the Chern current $\phi(\!(\overrightarrow{D}_\infty)\!)$ exists on $X \sim P$ and equals $\phi(D_F)$. The next section is devoted to computing $\phi(\!(\overrightarrow{D}_\infty)\!)$ on all of X. In order to do this we need to find a local formula for the transgression form $T_{r,s}$ associated to a section ν of $\mathbb{P}(E \oplus F) \to X$. To describe this we work universally in a local homogeneous coordinate on $\mathbb{P}(E \oplus F)$ and pullback the resulting local formula to X via ν.

Harvey and Lawson derived the following local formula for the transgression form $T_{r,s}$. Let $\boldsymbol{\alpha} : \mathbf{E} \to \mathbf{F}$ be the tautological homomorphism over $\mathrm{Hom}(E,F)$. Let e, f be local frames for $\mathbf{E}, \mathbf{F}$ and define a local fibre variable $u = (u_1, ..., u_n)$ on $\mathrm{Hom}(E,F)$ by $\boldsymbol{\alpha} e = uf$. Let $\boldsymbol{\alpha}^* f = u^* e$ define u^*. Set $|u|^2 = uu^*$,

$$Du = du + u\omega_F - \omega_E u \qquad \text{and} \qquad Du^* = du^* + u^*\omega_E - \omega_F u^*.$$

Finally let $u\frac{\partial}{\partial u}$ and $u^*\frac{\partial}{\partial u^*}$ be the globally defined Euler and co-Euler vector fields on $\mathrm{Hom}(E,F)$ (c.f. [HL I.5.24]). Then in the pushforward case we have [HL III.3.17] that

$$T_{r,s} = \int_{\chi_r}^{\chi_s} \phi\left(\frac{u^* Du}{|u|^2}\,;\,\Lambda(x)\right)\,dx \tag{2.27}$$

$$= -\int_{\chi_r}^{\chi_s} u^* \frac{\partial}{\partial u^*} \llcorner\, \phi\left(\Lambda(x)\right)\,\frac{dx}{x} \tag{2.28}$$

where $\chi_s = \frac{|u|^2}{s^2 + |u|^2}$ and

$$\Lambda(x) = \Omega_F + x\left(\frac{u^*\,\Omega_E\,u}{|u|^2} - \frac{u^* u}{|u|^2}\,\Omega_F - \frac{Du^* Du}{|u|^2}\right). \tag{2.29}$$

The formula for the transgression form $T_{r,s}$ in a local homogeneous coordinate on $\mathbb{P}(E\oplus F)$ is obtained as follows. Let u be a local fibre variable on $\mathrm{Hom}(E,F)$ defined as above. Let (t,u) be the local fibre variables on $E\oplus F$ defined by $c=(te,uf)$ where $c: E\oplus F\to \mathbf{E}\oplus\mathbf{F}$ is the tautological section. We call $[t,u]$ the **local homogeneous coordinate** on $\mathbb{P}(E\oplus F)$ corresponding to the fibre variable u.

Let ω be a smooth form on $\mathbb{P}(E\oplus F)$. Suppose that over the subset $\mathrm{Hom}(E,F)$ ω is expressed in terms of a local fibre variable u. Let $\omega_{HC} := \pi^*\omega$ be the pullback of ω to $E\oplus F\sim X$ via the natural projection map $\pi: E\oplus F\sim X\to\mathbb{P}(E\oplus F)$. Then to obtain an expression for ω_{HC} in the corresponding local homogeneous coordinate (t,u) we simply replace u by $\frac{u}{t}$ in the expression for ω. The resulting expression for ω_{HC} will be called the **expression for** ω **in homogeneous coordinates**. The following general lemma will be of use in finding expressions for the transgression and Chern forms in homogeneous coordinates.

Lemma 2.30. *Let u be a local fibre coordinate on* $\mathrm{Hom}(E,F)$. *Let*

$$\mathcal{C}=\left\{\frac{u^*u}{|u|^2},\ \frac{u^*Du}{|u|^2},\ \frac{Du^*u}{|u|^2},\ \frac{Du^*Du}{|u|^2}\right\}\cup\Omega^*(X),$$

and set $\chi_s=\frac{|u|^2}{s^2+|u|^2}$. Let ω be a (possibly 1×1) matrix valued form on the total space of the bundle $\mathrm{Hom}(E,F)\to X$. Suppose that ω can be expressed as a polynomial in χ_r, χ_s and $\frac{\partial\chi_s}{\partial s}$ with coefficients which are themselves polynomial expressions in (the entries of) the forms belonging to the set $\mathcal{C}$. Then the expression for ω in homogeneous coordinates $[t,u]$ on $\mathbb{P}(E\oplus F)$ is

$$\omega_{HC}=\omega-\frac{dt}{t}\,(u\frac{\partial}{\partial u}\llcorner\,\omega)-\frac{d\bar t}{\bar t}\,(u^*\frac{\partial}{\partial u^*}\llcorner\,\omega)-\frac{dt\,d\bar t}{|t|^2}\,(u\frac{\partial}{\partial u}\llcorner\, u^*\frac{\partial}{\partial u^*}\llcorner\,\omega) \tag{2.31}$$

where any occurence of χ_s on the R.H.S. is replaced by $\frac{|u|^2}{s^2|t|^2+|u|^2}$.

Proof. First note that (2.31) holds for each $\omega\in\mathcal{C}$. For instance

$$\begin{aligned}\left(\frac{Du^*Du}{|u|^2}\right)_{HC}&=\frac{\left(Du^*-u^*\,\frac{d\bar t}{\bar t}\right)\left(Du-u\,\frac{dt}{t}\right)}{|u|^2}\\&=\frac{Du^*Du}{|u|^2}+\frac{dt}{t}\,\frac{Du^*u}{|u|^2}-\frac{d\bar t}{\bar t}\,\frac{u^*Du}{|u|^2}-\frac{dt\,d\bar t}{|t|^2}\,\frac{u^*u}{|u|^2},\end{aligned}$$

which has the required form. The proof is completed by checking that if ω and η satisfy (2.31) then so do $\omega+\eta$ and $\omega\wedge\eta$. $\square$

Corollary 2.32. *Let $T_{r,s}$ be the smooth transgression form on $\mathbb{P}(E\oplus F)$ associated to the family of Chern forms $\phi(\overrightarrow{D}_s)$. Then the expression for $T_{r,s}$ in a local homogeneous coordinate system $[t,u]$ on $\mathbb{P}(E\oplus F)$ is given by*

$$(T_{r,s})_{HC}=T_{r,s}-\frac{dt}{t}\,(u\frac{\partial}{\partial u}\llcorner\,T_{r,s}), \tag{2.33}$$

where $T_{r,s}$ is defined by replacing χ_s by $\frac{|u|^2}{s^2|t|^2+|u|^2}$ in (2.27) or (2.28).

3. The transgression and Chern currents.

In this section we compute the Chern currents at time infinity,

$$\phi((\overrightarrow{D}_\infty)) := \lim_{r\to\infty} \phi(\overrightarrow{D}_r) \qquad \text{and} \qquad c^{-1}((\overleftarrow{D}_\infty)) := \lim_{r\to\infty} c^{-1}(\overleftarrow{D}_r)$$

on X associated with an atomic section ν of the bundle $\mathbb{P}(E \oplus F) \to X$. We begin with a discussion of the **pushforward** case. Let $T_{r,s}$ denote the transgression form induced on X in the pushforward case by a section ν of $\mathbb{P}(E \oplus F) \to X$ and an invariant polynomial ϕ.

Proposition 3.1. *Let ν be an atomic section of the bundle $\mathbb{P}(E \oplus F) \to X$. Then the transgression current $T_{\infty,0} = \lim_{\substack{r\to\infty \\ s\to 0}} T_{r,s}$ converges in $L^1_{\mathrm{loc}}(X)$ and equals*

$$(T_{\infty,0})_{HC} = T - \frac{dt}{t}\,(u\frac{\partial}{\partial u} \llcorner T) \tag{3.2}$$

in local homogeneous coordinates $[t, u]$ where

$$T = \int_0^1 \phi\left(\frac{u^* Du}{|u|^2}\,;\,\Lambda(x)\right)\,dx. \tag{3.3}$$

Here $\Lambda(x)$ is given by (2.29).

Proof. Since ν is atomic, $Z \cup P$ has Lebesgue measure zero in X and so by Corollary 2.25 $T_{r,s} \to T_{\infty,0}$ a.e. on X as $r \to \infty$ and $s \to 0$. So to prove that $T_{r,s} \to T_{\infty,0}$ in $L^1_{\mathrm{loc}}(X)$ we just need to show that the Lebesgue Dominated Convergence Theorem applies. We work locally in the pullback via ν of the various charts on $\mathbb{P}(E \oplus F)$. Let $[t, u]$ be a local coordinate expression for ν, defined with respect to a local frame e, f for $E \oplus F$ by $\nu = [te, uf]$. Recall that the coordinate expressions for the transgression form $T_{r,s}$ in the charts $\nu^{-1}(\mathcal{H}) = \{t \neq 0\}$ and $\nu^{-1}(\mathcal{U}_j) = \{u_j \neq 0\}$ for $j \in \{1, \ldots, n\}$ are given by setting $t = 1$ and $u_j = 1$ for $j \in \{1, \ldots, n\}$ respectively in (2.33). Also note that since ν is atomic the local functions t on $\nu^{-1}(\mathcal{U}_j)$ and u on $\nu^{-1}(\mathcal{H})$ are atomic.

Using the atomicity of u Harvey and Lawson [HL III.2.8] have shown that $T_{r,s}$ converges to $T_{\infty,0}$ in $L^1_{\mathrm{loc}}(\nu^{-1}(\mathcal{H}))$. Consulting (2.33) we see that in each of the charts $\nu^{-1}(\mathcal{U}_j)$ the transgression $T_{r,s}$ is a polynomial in χ_r and χ_s with coefficients which are either smooth forms on $\nu^{-1}(\mathcal{U}_j)$ or products of a smooth form on $\nu^{-1}(\mathcal{U}_j)$ with the $L^1_{\mathrm{loc}}(\nu^{-1}(\mathcal{U}_j))$ form $\frac{dt}{t}$. So since χ_r and χ_s are bounded and converge to 0 and 1 a.e. as $r \to \infty$ and $s \to 0$ the atomicity of t together with the Lebesgue Dominated Convergence Theorem imply that $T_{r,s} \to T_{\infty,0}$ in $L^1_{\mathrm{loc}}(\nu^{-1}(\mathcal{U}_j))$. Patching these results together we conclude that $T_{r,s}$ converges to $T_{\infty,0}$ in $L^1_{\mathrm{loc}}(X)$. $\square$

Corollary 3.4. *Let ν be an atomic section of the bundle $\mathbb{P}(E \oplus F) \to X$. Then the Chern current at time infinity, $\phi((\overrightarrow{D}_\infty)) := \lim_{r\to\infty} \phi(\overrightarrow{D}_r)$, exists on X and is of the form*

$$\phi((\overrightarrow{D}_\infty)) = \phi(D_F) + S$$

where S is a flat current supported on the pole set P of ν. Furthermore the current equation

$$dT_{\infty,0} = \phi((\overrightarrow{D}_\infty)) - \phi((\overrightarrow{D}_0)) \qquad \text{on } X$$

is the current limit as $s \to 0$ and $r \to \infty$ of the smooth transgression formula

$$dT_{r,s} = \phi(\overrightarrow{D}_r) - \phi(\overrightarrow{D}_s) \qquad \text{on } X.$$

Proof. By Corollary 2.25(1) and Proposition 3.1 the Chern current

$$\phi((\overrightarrow{D}_\infty)) := \lim_{r\to\infty} \phi(\overrightarrow{D}_r) = \phi(\overrightarrow{D}_0) + dT_{\infty,0}$$

exists and is flat on $X \sim Z$, and hence on all of X by Corollary 2.25(2). Finally by (2.8) the difference $S := \phi((\overrightarrow{D}_\infty)) - \phi(D_F)$ is a flat current supported on P. $\square$

The current S is called the singular part of the Chern current, and has several nice properties which we now describe. First we define the residue form at infinity, $\mathrm{Res}_\phi(\overrightarrow{D}_\infty)$. We work universally on $\mathbb{P}(E \oplus F)$. Set $\mathcal{R} := \mathbb{P}(E \oplus F) \sim X$. Choose a hermitian metric on the line bundle $\pi : \mathcal{R} \to \mathbb{P}(F)$ and let $\rho : S_\epsilon(\mathcal{R}) \to \mathbb{P}(F)$ denote the ϵ–circle bundle in $\mathcal{R}$. Note that the restriction of the transgression current $T_{\infty,0}$ to $S_\epsilon(\mathcal{R})$ is a smooth form. Finally suppose that the invariant polynomial ϕ has pure degree m.

Definition 3.5. The **residue form at infinity** is the smooth $2(m-1)$–form on $\mathbb{P}(F)$ defined by

$$\mathrm{Res}_\phi(\overrightarrow{D}_\infty) := \int_{\rho^{-1}} T_{\infty,0} \; = \rho_*(T_{\infty,0}).$$

Note that the residue form pulls back via the projection map π to a smooth form on $\mathbb{P}(E \oplus F) \sim X$, also denoted by $\mathrm{Res}_\phi(\overrightarrow{D}_\infty)$.

Note. Similarly Harvey and Lawson define the residue form $\mathrm{Res}_\phi(\overrightarrow{D}_0)$ by integrating the transgression current $-T_{\infty,0}$ over any ϵ–sphere bundle about X in $\mathrm{Hom}(E, F) \to X$.

Lemma 3.6.

(1) *The expression for the residue at infinity in a local homogeneous coordinate u on $\mathbb{P}(F)$ is*

$$\mathrm{Res}_\phi(\overrightarrow{D}_\infty)_{HC} = \tfrac{2\pi}{i}\, u\frac{\partial}{\partial u} \llcorner T$$

where

$$T = \int_0^1 \phi\left(\frac{u^* Du}{|u|^2}\,;\,\Lambda(x)\right)\,dx.$$

Therefore the residue form is independent of the choice of ϵ-circle bundle and hermitian metric on $\mathcal{R}$.

(2) *The residue form* $\mathrm{Res}_\phi(\overrightarrow{D}_\infty)$ *is d–closed.*

Proof.

(1) Let $[t,u]$ be homogeneous local coordinates on $\mathbb{P}(E \oplus F)$. By (3.2)

$$(T_{\infty,0})_{HC} = T + \tfrac{1}{2\pi i}\frac{dt}{t}\,(\tfrac{2\pi}{i}\,u\frac{\partial}{\partial u} \llcorner T).$$

So

$$\mathrm{Res}_\phi(\overrightarrow{D}_\infty)_{HC} = \int_{\rho^{-1}} (T_{\infty,0})_{HC} = \left(\tfrac{1}{2\pi i}\int_{|t|=\epsilon}\frac{dt}{t}\right)\tfrac{2\pi}{i}\,u\frac{\partial}{\partial u}\llcorner T = \tfrac{2\pi}{i}\,u\frac{\partial}{\partial u}\llcorner T,$$

since t can be regarded as a local fibre variable on the bundle $\pi : \mathcal{R} \to \mathbb{P}(F)$.

(2) It suffices to show that $\rho_*(dT_{\infty,0}) = 0$. Since $S_\epsilon(\mathcal{R}) \subset \mathrm{Hom}(E,F) \sim X$ we can apply part (5) of Theorem 2.12 to show that $dT_{\infty,0} = \phi(D_F) - \phi(\overrightarrow{D}_0)$ on $S_\epsilon(\mathcal{R})$. Now by [HL III.6.4] $\phi(\overrightarrow{D}_0) = \phi(\overrightarrow{\Omega}_0)$ where

$$\overrightarrow{\Omega}_0 = \left(1 - \frac{u^*u}{|u|^2}\right)\left(\Omega_F - \frac{Du^*Du}{|u|^2}\right) + \frac{u^*\Omega_E u}{|u|^2} \qquad \text{on } \mathrm{Hom}(E,F).$$

Furthermore by [Z1 I.3.7],

$$u\frac{\partial}{\partial u}\llcorner\phi(\overrightarrow{D}_0) = 0 \qquad \text{and} \qquad u^*\frac{\partial}{\partial u^*}\llcorner\phi(\overrightarrow{D}_0) = 0.$$

So by Lemma 2.30 the expressions for $\phi(D_F)$ and $\phi(\overrightarrow{D}_0)$ in homogeneous coordinates are of degree zero in the fibre differentials $dt, d\bar{t}$ of the bundle $p : \mathcal{R} \to \mathbb{P}(F)$. This implies that $\rho_*(dT_{\infty,0}) = \int_{\rho^{-1}} \phi(D_F) - \int_{\rho^{-1}} \phi(\overrightarrow{D}_0) = 0.$ $\square$

Finally we come to the main result of the paper.

Theorem 3.7. *Let ν be an atomic section of $\mathbb{P}(E \oplus F) \to X$. Then the singular part of the Chern current $\phi((\overrightarrow{D}_\infty))$ is given by*

$$S = \nu^*(\mathrm{Res}_\phi(\overrightarrow{D}_\infty))\ \mathrm{Div}_\infty(\nu). \tag{3.8}$$

Therefore

$$\phi((\overrightarrow{D}_\infty)) = \phi(D_F) + \nu^*(\mathrm{Res}_\phi(\overrightarrow{D}_\infty))\ \mathrm{Div}_\infty(\nu) \tag{3.9}$$

and

$$dT_{\infty,0} = \phi(D_F) + \nu^*(\mathrm{Res}_\phi(\overrightarrow{D}_\infty))\ \mathrm{Div}_\infty(\nu) - \phi(\overrightarrow{D}_0) - \mathrm{Res}_\phi(\overrightarrow{D}_0)\ \mathrm{Div}_0(\nu) \tag{3.10}$$

on X.

Note. The singular part of the Chern current at time infinity decomposes into the product of two terms — the pole divisor, which is independent of the invariant polynomial ϕ, and the smooth form $\nu^*(\mathrm{Res}_\phi(\overrightarrow{D}_\infty))$ on $X \sim Z$ which depends on both ν and ϕ. This phenomenon should be compared and contrasted to the decomposition of the singular part of the Chern current at time zero, where the residue form $\mathrm{Res}_\phi(\overrightarrow{D}_0)$ on X is ν–independent.

Corollary 3.11.

$$\mathrm{Res}_{c_1}(\overrightarrow{D}_\infty) = 1$$

so that

$$c_1((\overrightarrow{D}_\infty)) = c_1(D_F) + \mathrm{Div}_\infty(\nu) \qquad \textit{on } X.$$

Remark 3.12. Case rank F = 1. In the case rank $F = 1$ the Chern current formulae are particularly simple. Let $\phi = c_1^m$. Then

$$dT_{\infty,0} = c_1(D_F)^m - c_1(D_E)^m + \big(\mathrm{Div}_\infty(\nu) - \mathrm{Div}_0(\nu)\big)\, \frac{c_1(D_F)^m - c_1(D_E)^m}{c_1(D_F) - c_1(D_E)} \tag{3.13}$$

on X. The case $m = 1$ gives a C^∞–generalization of the classical Poincaré – Lelong formula,

$$dT_{\infty,0} = c_1(D_F) - c_1(D_E) + \mathrm{Div}_\infty(\nu) - \mathrm{Div}_0(\nu). \tag{3.14}$$

Proof of Theorem 3.7. We compute S by calculating $dT_{\infty,0}$ in each of the charts $\nu^{-1}(\mathcal{U}_j)$ By (3.2) and Lemma 3.6,

$$T_{\infty,0} = T + \tfrac{1}{2\pi i}\frac{dt}{t}\, \nu^*(\mathrm{Res}_\phi(\overrightarrow{D}_\infty)) \qquad \text{on } \nu^{-1}(\mathcal{U}_j)$$

where T and $\mathrm{Res}_\phi(\overrightarrow{D}_\infty)$ are smooth forms on $\nu^{-1}(\mathcal{U}_j)$. So

$$dT_{\infty,0} = dT + \nu^*(\mathrm{Res}_\phi(\overrightarrow{D}_\infty))\ \mathrm{Div}_\infty(\nu) \qquad \text{on } \nu^{-1}(\mathcal{U}_j)$$

since $d\,\mathrm{Res}_\phi(\overrightarrow{D}_\infty) = 0$. On the other hand we know that

$$dT_{\infty,0} = \phi(D_F) + S - \phi(\overrightarrow{D}_0),$$

where $\phi(\overrightarrow{D}_0)$ and $\phi(D_F)$ are smooth forms on $\nu^{-1}(\mathcal{U}_j)$. So

$$S - \nu^*(\mathrm{Res}_\phi(\overrightarrow{D}_\infty))\ \mathrm{Div}_\infty(\nu) = \phi(\overrightarrow{D}_0) - \phi(D_F) + dT,$$

and since the L.H.S. is zero on the dense subset $\nu^{-1}(\mathcal{U}_j) \sim P$ and the R.H.S. is smooth on $\nu^{-1}(\mathcal{U}_j)$ we are forced to conclude that both sides of the equation are zero on $\nu^{-1}(\mathcal{U}_j)$. This proves that $S = \nu^*(\mathrm{Res}_\phi(\overrightarrow{D}_\infty))\ \mathrm{Div}_\infty(\nu)$ on $\nu^{-1}(\mathcal{U}_j)$ and hence on all of X. $\square$

Remark 3.15. The pullback case. Analogous results hold in the pullback case, with $\phi = c^{-1}$. In particular

$$(T_{r,s})_{HC} = T_{r,s} - \frac{dt}{t}\,(u\frac{\partial}{\partial u} \llcorner T_{r,s})$$

where

$$T_{r,s} := \tfrac{i}{2\pi}\tfrac{Duu^*}{|u|^2}\,\tfrac{\left(c(D_E)+\chi_r\left(c(\overleftarrow{D}_0)-c(D_E)\right)\right)^{-1}-\left(c(D_E)+\chi_s\left(c(\overleftarrow{D}_0)-c(D_E)\right)\right)^{-1}}{c(\overleftarrow{D}_0)-c(D_E)}$$

and $c(\overleftarrow{D}_0) = c(\overleftarrow{\Omega}_0)$ where

$$\overleftarrow{\Omega}_0 = \frac{u\Omega_F u^*}{|u|^2} - \frac{Du\left(1-\frac{u^*u}{|u|^2}\right)Du^*}{|u|^2}.$$

The transgression current $T_{\infty,0} = \lim\limits_{\substack{r\to\infty\\ s\to 0}} T_{r,s}$ converges in $L^1_{\mathrm{loc}}(X)$ and equals

$$(T_{\infty,0})_{HC} = T - \frac{dt}{t}\,(u\frac{\partial}{\partial u} \llcorner T),$$

where

$$T = -\tfrac{i}{2\pi}\frac{Duu^*}{|u|^2}\;\frac{c(D_E)^{-1} - c(\overleftarrow{D}_0)^{-1}}{c(D_E) - c(\overleftarrow{D}_0)}.$$

The Chern current at time infinity exists and is given by

$$c^{-1}(\!(\overleftarrow{D}_\infty)\!) = c(D_E)^{-1} + c(D_E)^{-1}\nu^*(c(D_L)^{-1})\ \mathrm{Div}_\infty(\nu),$$

where L is the tautological line bundle on $\mathbb{P}(F)$ with connection D_L induced from that on F. Furthermore

(3.16)

$$dT_{\infty,0} = c(D_E)^{-1} + c(D_E)^{-1}\nu^*(c(D_L)^{-1})\ \mathrm{Div}_\infty(\nu) - c^{-1}(\overleftarrow{D}_0) - c(D_E)^{-1}c(D_F)^{-1}\ \mathrm{Div}_0(\nu).$$

Note that

$$\mathrm{Res}_{c^{-1}}(\overleftarrow{D}_\infty) = c(D_E)^{-1}c(D_L)^{-1} \qquad \text{on } \mathbb{P}(F) \tag{3.17}$$

and

$$\mathrm{Res}_{c^{-1}}(\overleftarrow{D}_0) = c(D_E)^{-1}c(D_F)^{-1} \qquad \text{on } X. \tag{3.18}$$

Also note that in the case rank $F = 1$ equation (3.16) reduces to the negative of equation (3.13).

Note. These results are proved just as in the pushforward case. To prove (3.17) note that $(U_0, \overleftarrow{D}_0) = (L, D_L)$ on the projective space $\mathbb{P}(F)$ at infinity.

4. Residues at infinity.

In this section we aim to compute the residue form at infinity, $\mathrm{Res}_\phi(\overrightarrow{D}_\infty)$, on the projective bundle $\mathbb{P}(F)$ at infinity in $\mathbb{P}(E \oplus F)$. Except for the following fact our results are entirely analogous to those of Harvey and Lawson [HL III.7]. They were able to prove that the residue form $\mathrm{Res}_\phi(\overrightarrow{D}_0)$ on X is a polynomial $\psi(c_1(D_E), c_1(D_F), \ldots, c_n(D_F))$ in the Chern forms of the bundles (E, D_E) and (F, D_F). In contrast to this the residue form at infinity on $\mathbb{P}(F)$ is given by

$$\mathrm{Res}_\phi(\overrightarrow{D}_\infty) = \psi(c_1(D_L),\, c_1(D_E),\, c_1(D_F), \ldots, c_n(D_F)) + dS_\phi$$

where $L \to \mathbb{P}(F)$ is the tautological line bundle and ψ is a polynomial in the Chern forms of the bundles (E, D_E) , (F, D_F) and (L, D_L). The important point here is that the differential form S_ϕ is *not* closed in general. Therefore we often prefer to compute the residue class, $[\mathrm{Res}_\phi(\overrightarrow{D}_\infty)]$, rather than the residue form itself.

The residue class is computable in the sense that for any invariant polynomial ϕ there is an explicit formula for the polynomial ψ for which

$$[\mathrm{Res}_\phi(\overrightarrow{D}_\infty)] = \psi(c_1(L),\, c_1(E),\, c_1(F), \ldots, c_n(F)).$$

Here $c_j(V)$ denotes the jth Chern class of the bundle V. Actually for any given invariant polynomial ϕ it is also possible to explicitly calculate the form S_ϕ in a local homogeneous coordinate system on $\mathbb{P}(F)$. In particular we will prove that $S_{c_1^m}$ and S_{c_2} are both zero. However this behaviour is far from typical as is suggested by the example of S_{c_3} which is not even a closed form on $\mathbb{P}(F)$. The following analogue of [HL III.2.24] provides an alternate definition of the residue form at infinity.

Theorem 4.1. *Let $\pi : \mathbb{P}(E \oplus F) \sim X \to \mathbb{P}(F)$ be defined by (1.7). Then for any invariant polynomial ϕ,*

$$\mathrm{Res}_\phi(\overrightarrow{D}_\infty) = \pi_*(\phi(D_Q)) = \int_{\pi^{-1}} \phi(D_Q).$$

Proposition 4.2. *Consider the bundle map $\eta : \mathbb{P}(F) \to X$. Then*

$$\mathrm{Res}_\phi(\overrightarrow{D}_0) = \int_{\eta - 1} \mathrm{Res}_\phi(\overrightarrow{D}_\infty).$$

Proof. Recall [HL III.7.6] that the residue form on X is given by the fibre integral

$$\begin{aligned}\mathrm{Res}_\phi(\overrightarrow{D}_0) &= \int_{\mathbb{P}(E\oplus F)_x} \phi(D_Q) = \int_{\mathcal{R}_x} \phi(D_Q) = \int_{\mathbb{P}(F)_x} \int_{\pi^{-1}} \phi(D_Q) \\ &= \int_{\mathbb{P}(F)_x} \mathrm{Res}_\phi(\overrightarrow{D}_\infty) = \int_{\eta^{-1}} \mathrm{Res}_\phi(\overrightarrow{D}_\infty). \qquad \square\end{aligned}$$

The next result will be used to calculate the residue class at infinity. It is proved by combining the analogue of Theorem 4.1 in the pullback case with equation (3.17).

Proposition 4.3. *Let $\pi : \mathbb{P}(E \oplus F) \sim X \to \mathbb{P}(F)$ be as above. Then*

$$\pi_*(c(D_U)^{-1}) = \int\limits_{\pi^{-1}} c(D_U)^{-1} = c(D_E)^{-1} c(D_L)^{-1}.$$

In particular

$$\pi_*(c_1^k(D_U)) = -\frac{c_1^k(D_L) - c_1^k(D_E)}{c_1(D_L) - c_1(D_E)} = -\sum_{i+j=k-1} c_1^i(D_E) c_1^j(D_L).$$

We are now in a position to calculate the residue class $[\mathrm{Res}_\phi(\overrightarrow{D}_\infty)]$ for an arbitrary invariant polynomial ϕ. Recall that any $\mathfrak{gl}(n, \mathbb{C})$ invariant polynomial can be expressed as a polynomial in $c_1, \ldots, c_n$ and that the residue of a sum of polynomials is the sum of the residues. Consequently to compute the residue for a general ϕ it suffices to compute it for the polynomials

$$\phi = \prod_{k=1}^{n} c_k^{j_k}. \tag{4.4}$$

Since

$$0 \to U \to \mathbf{E} \oplus \mathbf{F} \to Q \to 0 \tag{4.5}$$

is exact over $\mathbb{P}(E \oplus F)$ we have the sequence of formulae

$$c(E \oplus F) = c(U \oplus Q) \tag{4.6}$$

$$c(E)\, c(F) = c(U)\, c(Q) \tag{4.7}$$

$$c(Q) = c(U)^{-1}\, c(E)\, c(F) \qquad \text{and} \tag{4.8}$$

$$c_k(Q) = \{c(U)^{-1}\, c(E)\, c(F)\}_k \tag{4.9}$$

where $\{T\}_k$ denotes the degree $2k$ part of T. Therefore for any ϕ given by (4.4) we have

$$\begin{aligned} \phi(Q) &= \prod_{k=1}^{n} \left(\{c(U)^{-1}\, c(E)\, c(F)\}_k\right)^{j_k} \\ &= \sum_{l=0}^{\infty} c_1^l(U)\, \psi_l(c_1(E),\, c_1(F), \ldots, c_n(F)) \end{aligned} \tag{4.10}$$

where the polynomials ψ_l are defined by this last equation. Applying Theorem 4.1 and Proposition 4.3 we have the following result.

Theorem 4.11. *Let* $\phi = \prod_{k=1}^{n} c_k^{j_k}$. *Then*

$$[\mathrm{Res}_\phi(\overrightarrow{D}_\infty)] = -\sum_{l=0}^{\infty} \frac{c_1^l(L) - c_1^l(E)}{c_1(L) - c_1(E)}\ \psi_l(c_1(E),\, c_1(F), \dots, c_n(F))$$

in $H^*(\mathbb{P}(F)\,;\,\mathbb{R})$ *where* ψ_l *is the polynomial defined by equation (4.9). In particular,*

1. $[\mathrm{Res}_c(\overrightarrow{D}_\infty)] = c(L)^{-1}\, c(F)$.
2. $\mathrm{Res}_{c_1}(\overrightarrow{D}_\infty) = 1$.
3. *Let* $T = -\,c_1(L)$. *Then for* $k \in \{2, \dots, n\}$,

$$[\mathrm{Res}_{c_k}(\overrightarrow{D}_\infty)] = T^{k-1} + T^{k-2}\, c_1(F) + \cdots + c_{k-1}(F).$$

(4)

$$[\mathrm{Res}_{\mathrm{ch}}(\overrightarrow{D}_\infty)] = \sum_{k=1}^{\infty} \tfrac{1}{k!}\, \frac{c_1^k(L) - c_1^k(E)}{c_1(L) - c_1(E)}.$$

(5)

$$[\mathrm{Res}_{b_k}(\overrightarrow{D}_\infty)] = \frac{c_1^k(L) - c_1^k(E)}{c_1(L) - c_1(E)}$$

where the k*th trace power* b_k *is defined by* $b_k(\Omega) := \mathrm{tr}((\tfrac{i}{2\pi}\Omega)^k)$.

6. $[\mathrm{Res}_{c^{-1}}(\overrightarrow{D}_\infty)] = -\,c(E)^{-1}\, c(F)^{-1}$.
7. *Let* $\mathbb{H}(c)$ *be a multiplicative series of Chern classes associated to the formal power series* $h(x) = 1 + a_1\, x + a_2\, x^2 + \cdots \in \mathbb{R}[x]$. *Then*

$$[\mathrm{Res}_{\mathbb{H}}(\overrightarrow{D}_\infty)] = \pi_!(h^{-1}(c_1(U)))\, \mathbb{H}(E)\, \mathbb{H}(F).$$

In particular if $h^{-1}(x) = 1 + b_1\, x + b_2\, x^2 + \dots$ *is the formal power series for the reciprical of* h, *then*

$$[\mathrm{Res}_{\mathbb{H}}(\overrightarrow{D}_\infty)] = -\sum_{k=1}^{\infty} b_k\, \frac{c_1^k(L) - c_1^k(E)}{c_1(L) - c_1(E)}\ \mathbb{H}(E)\, \mathbb{H}(F).$$

Remark 4.12. See [LM] for the definition of a multiplicative series of Chern classes. The main examples are given by the Todd series $\mathbb{H} = \mathrm{Td}$ in which case $h(x) = \frac{x}{1-e^{-x}}$ and the series $\mathbb{H} = e^{c_1}$ generated by $h(x) = e^x$.

Note. The seven formulae above can be proved directly using the exact sequence (4.5), Proposition 4.3 and the properties of the particular invariant polynomials under consideration.

Remark 4.13. Flat case. Let $T = -\, c_1(L)$. Then in the flat case we have

$$[\mathrm{Res}_c(\overrightarrow{D}_\infty)] = c(L)^{-1}$$

$$[\mathrm{Res}_{\mathrm{ch}}(\overrightarrow{D}_\infty)] = \mathrm{Td}^{-1}(T)$$

$$[\mathrm{Res}_{b_k}(\overrightarrow{D}_\infty)] = b_{k-1}(L)$$

$$[\mathrm{Res}_{c^{-1}}(\overrightarrow{D}_\infty)] = 0$$

$$[\mathrm{Res}_{\mathbb{H}}(\overrightarrow{D}_\infty)] = -\sum_{k=1}^{n} b_k\, c_1^{k-1}(L)$$

and for any degree m invariant polynomial ϕ which is a simple product of the Chern polynomials $c_1, \ldots, c_n$,

$$[\mathrm{Res}_\phi(\overrightarrow{D}_\infty)] = T^{m-1}.$$

In particular

$$[\mathrm{Res}_\phi(\overrightarrow{D}_\infty)] = 0 \qquad \text{for degree } \phi > n.$$

Actually the next proposition shows that these formulae hold on the **form** level.

Proposition 4.14. *Let* $\phi = \prod_{k=1}^{n} c_k^{j_k}$ *be an invariant polynomial of degree* m*. Then in the flat case*

$$\mathrm{Res}_\phi(\overrightarrow{D}_\infty) = (-\, c_1(D_L))^{m-1}\,.$$

Proof. Let u be a homogeneous coordinate on $\mathbb{P}(F)$. First note that

$$b_k(du^* du) = \mathrm{tr}((\tfrac{i}{2\pi} du^* du)^k) = -\,(\tfrac{i}{2\pi} du du^*)^k.$$

Then by the Newton identities,

$$c_k(-\, du^* du) = (\tfrac{i}{2\pi} du du^*)^k$$

and so

$$\phi(-\, du^* du) = (\tfrac{i}{2\pi} du du^*)^m.$$

So by Lemma 3.6

$$\begin{aligned}
\mathrm{Res}_\phi(\overrightarrow{D}_\infty) &= -\,\tfrac{1}{m}\,(\tfrac{i}{2\pi})^{m-1}\; u\frac{\partial}{\partial u} \llcorner\, u^* \frac{\partial}{\partial u^*} \llcorner\, \phi\left(\frac{-du^* du}{|u|^2}\right) \\
&= (\tfrac{i}{2\pi})^{m-1} \left(\frac{du du^*}{|u|^2} - \frac{du u^* u du^*}{|u|^4}\right)^{m-1} \\
&= (-\, c_1(D_L))^{m-1}. \qquad \square
\end{aligned}$$

Next we state a formula which relates $[\mathrm{Res}_{c_1\psi}(\overrightarrow{D}_\infty)]$ to $[\mathrm{Res}_\psi(\overrightarrow{D}_\infty)]$. Its proof is analogous to [HL III.4.16] which states that $\mathrm{Res}_{c_n\psi}(\overrightarrow{D}_0) = \psi(D_F)$ on X.

Proposition 4.15. *Let ψ be any invariant polynomial. Then*
$$[\mathrm{Res}_{c_1\psi}(\overrightarrow{D}_\infty)] = c_1(F)\,[\mathrm{Res}_\psi(\overrightarrow{D}_\infty)] + \psi(E\oplus L^\perp).$$

Finally we state some results concerning residue **forms** (rather than residue classes) on $\mathbb{P}(F)$.

Theorem 4.16.
$$\mathrm{Res}_{c_1^m}(\overrightarrow{D}_\infty) = \frac{(c_1(D_F)+c_1(D_E)-c_1(D_L))^m - c_1(D_F)^m}{c_1(D_E)-c_1(D_L)}$$
as differential forms on $\mathbb{P}(F)$.

Proof. Let u be a homogeneous coordinate on $\mathbb{P}(F)$. Either by direct calculation or using [HL III.7.33] we have
$$c_1(D_L) = \tfrac{i}{2\pi}\left(\frac{u\Omega_F u^*}{|u|^2} - \frac{Du(1-\frac{u^*u}{|u|^2})Du^*}{|u|^2}\right).$$
Then by Lemma 3.6
$$\begin{aligned}
&\mathrm{Res}_{c_1^m}(\overrightarrow{D}_\infty)\\
&= -(\tfrac{i}{2\pi})^{m-1} u\frac{\partial}{\partial u}\llcorner u^*\frac{\partial}{\partial u^*}\llcorner \int_0^1 \mathrm{tr}^m\left(\Omega_F - x\left(\tfrac{u^*u}{|u|^2}(\Omega_F-\Omega_E) + \tfrac{Du^*Du}{|u|^2}\right)\right)\frac{dx}{x}\\
&= -(\tfrac{i}{2\pi})^{m-1} u\frac{\partial}{\partial u}\llcorner u^*\frac{\partial}{\partial u^*}\llcorner \int_0^1 \left(\mathrm{tr}(\Omega_F) + x\Omega_E - x\tfrac{u\Omega_F u^*}{|u|^2} + x\tfrac{DuDu^*}{|u|^2}\right)^m\frac{dx}{x}\\
&= \int_0^1 m\,(c_1(D_F)+x\,(c_1(D_E)-c_1(D_L))^{m-1}\,dx\\
&= \frac{(c_1(D_F)+c_1(D_E)-c_1(D_L))^m - c_1(D_F)^m}{c_1(D_E)-c_1(D_L)}. \qquad \square
\end{aligned}$$

Theorem 4.17.

(4.18) $$\mathrm{Res}_{c_2}(\overrightarrow{D}_\infty) = c_1(D_F) - c_1(D_L)$$

(4.19) $$\mathrm{Res}_{b_2}(\overrightarrow{D}_\infty) = c_1(D_E) + c_1(D_L)$$

(4.20) $$\mathrm{Res}_{b_3}(\overrightarrow{D}_\infty) = c_1^2(D_E) + c_1(D_E)\,c_1(D_L) + c_1^2(D_L) + dS$$

(4.21) $$\mathrm{Res}_{c_3}(\overrightarrow{D}_\infty) = c_2(D_F) - c_1(D_L)\,c_1(D_F) + c_1^2(D_L) + \tfrac{1}{3}dS$$

where
$$S = -\tfrac{3}{8\pi}\,\tfrac{i}{2\pi}\,\frac{Du\,(1-\frac{u^*u}{|u|^2})\,\Omega_F u^*}{|u|^2}$$
in a homogeneous coordinate u on $\mathbb{P}(F)$, and
$$dS \not\equiv 0.$$

The proof of this theorem relies on the following theorem of Harvey and Lawson.

Theorem 4.22. *[HL III.7.25] Let ϕ be a $\mathfrak{gl}(n+1,\mathbb{C})$ invariant polynomial. Then*

$$\phi(D_E \oplus D_F) - \phi(D_U \oplus D_Q) = dS_\phi \tag{4.23}$$

on $\mathbb{P}(E \oplus F)$ where the transgression form S_ϕ is given as follows. Let $v = [t,u]$ be a local homogeneous coordinate on $\mathbb{P}(E \oplus F)$. Let $P := \frac{v^*v}{|v|^2}$ and $\Omega := \Omega_{E\oplus F}$. Then

(4.24)
$$\begin{aligned} S_\phi = \int_0^1 \phi\Bigg(\frac{v^*Dv}{|v|^2}\,(1-P)\,;\, \Omega - x\,P\Omega(1-P) - x\,\frac{v^*Dv}{|v|^2}\,(1-P)\,\frac{Dv^*v}{|v|^2} \\ - x\,(1-P)\,\frac{Dv^*Dv}{|v|^2}\,(1-P)\Bigg)\,dx. \end{aligned}$$

Proof of Theorem 4.17. By equation (4.23) we have

$$b_k(D_Q) = b_k(D_E) + b_k(D_F) - b_k(D_U) - dS_{b_k}$$

and so

$$\begin{aligned} \mathrm{Res}_{b_k}(D_Q) &= -\,\pi_*(b_k(D_U)) - d\pi_*(S_{b_k}) \\ &= -\,\pi_*(c_1^k(D_U)) - d\pi_*(S_{b_k}) \\ &= \frac{c_1^k(D_L) - c_1^k(D_E)}{c_1(D_L) - c_1(D_E)} \;-\; d\pi_*(S_{b_k}). \end{aligned}$$

So to prove (4.19) and (4.20) it suffices to show that

$$\pi_*(S_{b_2}) = 0 \qquad \text{and} \qquad \pi_*(S_{b_3}) = S.$$

Note that (4.18) follows from (4.19) and Proposition 4.16 using the Newton identity

$$c_2 = \tfrac{1}{2}\,(c_1^2 - b_2).$$

We will discuss the proof of (4.21) at the end.

To prove that $\pi_*(S_{b_2}) = 0$ we use equation (4.24) to derive the formula

$$\begin{aligned} S_{b_2} = -\,\tfrac{1}{2\pi^2}\,\Bigg(\frac{Du\Omega_F u^*}{|u|^2+|t|^2} - \frac{Duu^*}{|u|^2+|t|^2}\,\frac{|t|^2\,\Omega_E + u\Omega_F u^*}{|u|^2+|t|^2} \\ + \frac{\bar{t}dt\,(|u|^2\Omega_E - u\Omega_F u^*)}{(|u|^2+|t|^2)^2}\Bigg) \end{aligned}$$

in homogeneous coordinates $[t,u]$ on $\mathbb{P}(E\oplus F)$, and note that since S_{b_2} has no terms of degree 2 in $dt, d\bar{t}$,

$$\pi_*(S_{b_2}) = \int\limits_{\pi^{-1}} S_{b_2} = 0.$$

To prove that $\pi_*(S_{b_3}) = S$ we use (4.24) to calculate that the part of S_{b_3} of degree 2 in $dt, d\bar{t}$ is given by

$$-\,3\,(\tfrac{i}{2\pi})^3\,\frac{dt\,d\bar{t}}{|t|^2}\,u\frac{\partial}{\partial u}\llcorner\, u^*\frac{\partial}{\partial u^*}\llcorner\, B$$

where

$$B = \chi\,\frac{Du\Omega_F u^*}{|u|^2} - \chi^2\,\frac{Duu^*\,(\Omega_E^2 + u\Omega_F u^*)}{|u|^4} - \chi^2\,\frac{Du\Omega_F u^*}{|u|^2}\left(\frac{DuDu^*}{|u|^2} - \chi\frac{Duu^*uDu^*}{|u|^4}\right)$$
$$+\,\chi^3\,\frac{Duu^*\,(\Omega_E + u\Omega_F u^*)\,DuDu^*}{|u|^6}$$

and $\chi := \frac{|u|^2}{|u|^2+|t|^2}$. Consequently

$$\begin{aligned}\pi_*(S_{b_3}) &= -\,3\,(\tfrac{i}{2\pi})^3\int\limits_{\pi^{-1}}\chi^2(1-\chi)\left(\frac{u\Omega_F u^*\,Duu^*}{|u|^4} - \frac{Du\Omega_F u^*}{|u|^2}\right)\frac{dt\,d\bar{t}}{|t|^2}\\ &= 3\,(\tfrac{i}{2\pi})^3\left(\frac{Du\Omega_F u^*}{|u|^2} - \frac{Duu^*\,u\Omega_F u^*}{|u|^4}\right)\int\limits_{\pi^{-1}}\frac{|u|^4\,dtd\bar{t}}{(|t|^2+|u|^2)^3}\\ &= S.\end{aligned}$$

Finally to prove (4.21) we use the Newton identity

$$6c_3 = 2b_3 - 3b_1b_2 + b_1^3$$

and check that

$$\pi_*(S_{b_1b_2}) = b_1(\Omega_{E\oplus F})\,\pi_*(S_{b_2}) = 0. \quad\square$$

5. The relationship to Levine's construction.

In this section we examine the relationship between the connections $D_{U,s}$ and $D_{Q,s}$ and the families of approximating pullback and pushforward connections induced on U and Q by certain naturally defined bundle maps whose Chern currents were first studied by Levine [L]. Levine's construction was subsequently examined by Gillet and Soule [GS] and by Harvey and Lawson [HL II.10]. Our main result is that in the pullback case the Chern current $c_1^m((\overleftarrow{D}_{\alpha,\infty}))$ at time infinity of the tautological homomorphism α is equal to the Chern current $c_1^m((\overleftarrow{D}_{\pi,0}))$ at time zero of the natural projection $\pi : U \to E$ over $\mathbb{P}(E\oplus F)$. Note that the zero set of the map π is the pole set of α. We also investigate the extent to which the pushforward analogue of this result holds. The discussion splits into four cases corresponding to the families of connections $D_{U,r}$ and $D_{Q,r}$ at times $r=0$ and $r=\infty$. We assume that rank $E = 1$ unless otherwise stated.

Pullback at zero.

Let $\pi : E \oplus F \to F$ and consider the blow up

$$\widehat{\mathbb{P}}(E \oplus F) = \{ (A, B) \in \mathbb{P}(E \oplus F) \times_X \mathbb{P}(F) \ / \ \pi(A) \subset B \}.$$

Recall that the projection $p : \widehat{\mathbb{P}}(E \oplus F) \to \mathbb{P}(E \oplus F)$ onto the first factor is a diffeomorphism over $\mathbb{P}(E \oplus F) \sim X$ and that $p^{-1}(X) = \mathbb{P}(F)_0$ is a copy of $\mathbb{P}(F)$ over spatial zero. Let $\mathbf{U}$ and $\mathbf{L}$ be the pullbacks of the tautological line bundles $U \to \mathbb{P}(E \oplus F)$ and $L \to \mathbb{P}(F)$. The blow up $\widehat{\mathbb{P}}(E \oplus F)$ has been chosen so that the induced map $\pi : U \to L$ over $\mathbb{P}(E \oplus F) \to X$ desingularises to define an atomic bundle map $\pi : \mathbf{U} \to \mathbf{L}$ over $\widehat{\mathbb{P}}(E \oplus F)$, with $\mathrm{Div}(\pi) = [\mathbb{P}(F)_0]$. Let $\overleftarrow{D}_{\pi,s}$ be the pullback family of connections on $\mathbf{U}$ induced by π using the algebraic approximation mode and let $D_{\mathbf{U},r}$ be the pullback to $\mathbf{U}$ of $D_{U,r}$ on U.

Theorem 5.1. *Let $0 < r < 1$ and $0 < s < \infty$ be related by $r = \frac{s}{1+s}$. Then*

$$D_{\mathbf{U},r} = \overleftarrow{D}_{\pi,s} \qquad \text{on } \mathbf{U} \text{ over } \widehat{\mathbb{P}}(E \oplus F).$$

Remark 5.2. Let $1 \leq p = \mathrm{rank}\, E \leq \mathrm{rank}\, F$ and consider the blow up $\widehat{G}_p(E \oplus F)$ defined as above. Let $U \to G_p(E \oplus F)$ and $L \to G_p(F)$ be the tautological rank p bundles and define $\pi : \mathbf{U} \to \mathbf{L}$ as before. Then $D_{\mathbf{U},r} = \overleftarrow{D}_{\pi,s}$ on $\mathbf{U}$.

Corollary 5.3. *The current equation*

$$c_1^m(D_{\mathbf{U}}) - c_1^m(D_{\mathbf{L}}) - \mathrm{Res}_m(\overleftarrow{D}_{\pi,0})\,[\mathbb{P}(F)_0] = dT_{\infty,0} \qquad \text{on } \widehat{\mathbb{P}}(E \oplus F)$$

pushes down via p to the current equation

$$c_1^m(\overleftarrow{D}_{\alpha,1}) - c_1^m(\overleftarrow{D}_{\alpha,0}) - \mathrm{Res}_m(\overleftarrow{D}_{\alpha,0})\,[X] = dT_{1,0} \qquad \text{on } \mathbb{P}(E \oplus F).$$

In particular the residue $\mathrm{Res}_m(\overleftarrow{D}_{\alpha,0})$ of a map into a bundle of arbitrary rank is the fibre integral

$$\mathrm{Res}_m(\overleftarrow{D}_{\alpha,0}) = \int_{\mathbb{P}(F)_0} \mathrm{Res}_m(\overleftarrow{D}_{\pi,0}) \tag{5.4}$$

of the residue $\mathrm{Res}_m(\overleftarrow{D}_{\pi,0})$ of a map between line bundles.

Note. The fibre integral (5.4) is computed using Bott's Theorem (see [B] or [HL III.7.28]).

Proof of Theorem 5.1. Let $\gamma_s : E \to U_s$ be the graphing map defined over $\mathrm{Hom}(E, F)$ by $\gamma_s(e) = (se, \alpha(e))$ and let $\gamma := \gamma_1$. It suffices to prove that $\gamma^{-1} \circ \overleftarrow{D}_{\pi,s} \circ \gamma = \overleftarrow{D}_{\alpha,r}$ on E over $\mathrm{Hom}(E, F)$. Since $\gamma^{-1}\pi^*\pi\gamma = (1 + \alpha^*\alpha)^{-1}\alpha^*\alpha$, $\pi\gamma = \gamma_0$ and $\gamma_s^{-1} \circ D_{U_s} \circ \gamma_s = \overleftarrow{D}_{\alpha,s}$ we have that

$$\begin{aligned}
\gamma^{-1} \circ \overleftarrow{D}_{\pi,s} \circ \gamma &= \gamma^{-1} (\pi^*\pi + s)^{-1} (sD_U + \pi^* D_{U_0} \pi)\, \gamma \\
&= \left((1 + \alpha^*\alpha)^{-1}\alpha^*\alpha + s\right)^{-1} (1 + \alpha^*\alpha)^{-1} \left(s(D_E + \alpha^* D_F \alpha) + \alpha^* D_F \alpha\right) \\
&= \left(\tfrac{s}{1+s} + \alpha^*\alpha\right)^{-1} \left(\tfrac{s}{1+s} D_E + \alpha^* D_F \alpha\right) = \overleftarrow{D}_{\alpha,r} \qquad \square
\end{aligned}$$

Pullback at infinity.

Swapping the roles of E and F in the previous case, let $\pi : E \oplus F \to E$ and consider the induced map $\pi : U \to E$ over $\mathbb{P}(E \oplus F)$. Note that the mapping π is atomic with $\mathrm{Div}(\pi) = [\mathbb{P}(F)]$ since $U \subset F$ on $\mathbb{P}(F)$.

Theorem 5.5. *Let $0 < s < \infty$ and $\infty > r > 1$ be related by $r = \frac{1+s}{s}$. Then*

$$D_{U,r} = \overleftarrow{D}_{\pi,s} \qquad \text{on } U \text{ over } \mathbb{P}(E \oplus F).$$

Remark 5.6. If $1 \le p = \operatorname{rank} E \le \operatorname{rank} F$ then $\overleftarrow{D}_{\pi,s} = D_{U,r}$ on U over $G_p(E \oplus F)$.

Corollary 5.7. *The Chern current $c_1^m((\overleftarrow{D}_{\alpha,\infty}))$ at time infinity of the tautological homomorphhism α exists and equals the Chern current $c_1^m((\overleftarrow{D}_{\pi,0}))$ at time zero of the map $\pi : U \to E$ of line bundles.*

Proof of Theorem 5.5. Observe that $\gamma^{-1}\pi^*\pi\gamma = (1 + \alpha^*\alpha)^{-1}$ and that $\pi\gamma = 1$. The proof now follows along the same lines as that of Theorem 5.1 $\square$

Having noted the relationship between the pullback families of connections arising from Levine's construction and the family $D_{U,r}$ we search for natural bundle maps whose families of pushforward connections are related to the family $D_{Q,r}$ on Q.

Pushforward at zero.

Let $\pi : E \to Q$ be the atomic bundle map over $\mathbb{P}(E \oplus F)$ defined by $\pi(e) = [e, 0]$ where $[e, f]$ denotes the image of (e, f) in Q. Note that $\mathrm{Div}(\pi) = [X]$ since $U = E$ on X.

Theorem 5.8. *Let $0 < r < 1$ and $0 < s < \infty$ be related by $r = \frac{s}{1+s}$. Then*

$$D_{Q,r} = \overrightarrow{D}_{\pi,s} \qquad \text{on } Q \text{ over } \mathbb{P}(E \oplus F),$$

where $D_{Q,r}$ is defined by (2.21).

Corollary 5.9.

$$\phi((\overrightarrow{D}_{\pi,0})) = \phi((\overrightarrow{D}_{\alpha,0})) \qquad \text{on } \mathbb{P}(E \oplus F).$$

Proof of Theorem 5.8. Let $\psi : F \to Q$ be defined by $\psi(f) = [0, f]$. We need to show that $\psi^{-1} \circ \overrightarrow{D}_{\pi,s} \circ \psi = \overrightarrow{D}_{\alpha,r}$ on F over $\mathrm{Hom}(E, F)$. Now $\pi = -\psi\alpha$ and $\psi^* = \left(1 - \frac{\alpha\alpha^*}{1+|\alpha|^2}\right) \psi^{-1}$. So $\pi^* = -\frac{\alpha^*\psi^{-1}}{1+|\alpha|^2}$. Now follow your nose. $\square$

Pushforward at infinity.

In analogy with the previous three cases we consider the map $\pi : \mathbf{L} \to \mathbf{Q}$ over $\widehat{\mathbb{P}}(E \oplus F)$ given by $\mathbf{L} \hookrightarrow \mathbf{F} \xrightarrow{\psi} \mathbf{Q}$. Since $\mathbf{L} = \mathbf{U}$ on $\mathbb{P}(F)$ π vanishes on the codimension 2 submanifold $\mathbb{P}(F)$ and so is *not* atomic unless rank $F = 1$. Furthermore $\overrightarrow{D}_{\pi,s} \neq D_{\mathbf{Q},r}$ unless rank $F = 1$. Locally however π is of the form $(t, 0, ..., 0)$ where t is an atomic $\mathbb{C}$–valued function and so there is still some hope that the Chern current $\phi((\overrightarrow{D}_{\pi,0}))$ of the degenerately vanishing map π exists.

Theorem 5.10. *Let $0 < s < \infty$ and $\infty > r > 1$ be related by $r = \frac{1+s}{s}$, and let $\omega(s)$ be the global* $\mathrm{Hom}(\mathbf{Q},\mathbf{Q})$*–valued 1–form defined over $\widehat{\mathbb{P}}(E\oplus F)$ by $\omega(s) := \overrightarrow{D}_{\pi,s} - D_{\mathbf{Q},r}$. Then in a local fibre variable u on* $\mathrm{Hom}(E,F)$*,*

$$\omega(s) = -\,\widetilde{\chi}_s\,\frac{u^*Du}{|u|^2}\,(1-P) \qquad \text{on } \mathrm{Hom}(E,F)\sim X, \tag{5.11}$$

where $\widetilde{\chi}_s := \frac{1}{1+s+s|u|^2}$ and $P := \frac{u^*u}{|u|^2}$. In particular when rank $F=1$,

$$\overrightarrow{D}_{\pi,s} = D_{\mathbf{Q},r} \qquad \text{on } \mathbf{Q} \text{ over } \widehat{\mathbb{P}}(E\oplus F).$$

Proof. Let e, f be local frames for E, F. Then $l = \alpha(e)$ and $q = [0,f]$ are local frames for L and Q over $\mathrm{Hom}(E,F)$. Define u and w by $\alpha(e) = uf$ and $\pi(l) = wq$. Let $\overrightarrow{\omega}_{\alpha,r}$ and $\overrightarrow{\omega}_{\pi,s}$ be the corresponding local gauges for $\overrightarrow{D}_{\alpha,r}$ and $\overrightarrow{D}_{\pi,s}$. Recall that

$$\omega_Q = \overrightarrow{\omega}_{\alpha,1} = \omega_F - \frac{u^*Du}{|u|^2+1} \qquad \text{and} \qquad \omega_L = \overleftarrow{\omega}_{\alpha,0} = \omega_E + \frac{Duu^*}{|u|^2}.$$

Then

$$\omega(s) = \overrightarrow{\omega}_{\pi,s} - \overrightarrow{\omega}_{\alpha,r} = \omega_Q - \frac{w^*Dw}{|w|^2+s} - \omega_F + \frac{u^*Du}{|u|^2+r}$$

$$= -\frac{u^*Du}{|u|^2+1} + \frac{u^*Du}{|u|^2+r} - \frac{w^*Dw}{|w|^2+s}. \tag{5.12}$$

Now $w = u$, $w^* = \frac{u^*}{|u|^2(|u|^2+1)}$ and $\frac{|w|^2}{|w|^2+s} = \widetilde{\chi}_s$. Substituting

$$\frac{w^*Dw}{|w|^2+s} = \frac{u^*Du\,\big((1-P) - |u|^2\,P\big)}{|u|^2(|u|^2+1)(1+s+s|u|^2)}$$

into (5.12) yields (5.11). $\square$

Set $D_s := \psi^{-1}\circ\overrightarrow{D}_{\pi,s}\circ\psi$ on F and define

$$D_\sigma := \sigma D_s + (1-\sigma)\,\overrightarrow{D}_{\alpha,r} \qquad \text{for } 0<\sigma<1 \text{ on } F.$$

Let $\Omega_{\alpha,r}$, Ω_s, Ω_σ be the local curvature matricies of the connections $\overrightarrow{D}_{\alpha,r}$, D_s, D_σ.

Proposition 5.13. *Let $\chi_r = \frac{|u|^2}{|u|^2+r} = \frac{s|u|^2}{1+s+s|u|^2}$ and $\Lambda = -\frac{Du^*Du}{|u|^2}$. Then on* $\mathrm{Hom}(E,F)\sim X$*,*

$$\Omega_\sigma = \sigma\,(\Omega_s - \Omega_{\alpha,r}) + \Omega_{\alpha,r} \tag{5.14}$$

where

$$\Omega_{\alpha,r} = (1-\chi_r P)(\Omega_F + \chi_r\Lambda) + \chi_r P\Omega_E \tag{5.15}$$

and

$$\Omega_s = \Omega_{\alpha,r} + \widetilde{\chi}_s\big(\Omega_0 - \Omega_F - \chi_r\,P\Lambda(1-P)\big). \tag{5.16}$$

Here

$$\Omega_0 = \Omega_F + (1-P)\Lambda(1-P) + P(\beta-\lambda) - P\Omega_F(1-P). \tag{5.17}$$

where $\lambda = \mathrm{Tr}(\Lambda) = \frac{DuDu^*}{|u|^2}$ and $\beta = \frac{Duu^*uDu^*}{|u|^2}$.

Corollary 5.18.
$$\mathrm{Tr}(\Omega_s) = \mathrm{Tr}(\Omega_{\alpha,r}).$$
Consequently the Chern current $c_1^m((\overrightarrow{D}_{\pi,0}))$ *exists and equals*
$$c_1^m((\overrightarrow{D}_{\pi,0})) = c_1^m((\overrightarrow{D}_{\alpha,\infty})) \qquad \textit{on } \widehat{\mathbb{P}}(E \oplus F).$$

More generally let

$$T_s(\phi) := \int_0^1 \phi\left(\omega(s); \Omega_\sigma\right) \, d\sigma \tag{5.19}$$

Then

$$\phi(\overrightarrow{D}_{\pi,s}) - \phi(\overrightarrow{D}_{\alpha,r}) = dT_s(\phi) \qquad \text{on } \widehat{\mathbb{P}}(E \oplus F). \tag{5.20}$$

Proposition 5.21. $T_s(c_2) \to T_0(c_2)$ *in* $L^1_{\mathrm{loc}}(\widehat{\mathbb{P}}(E \oplus F))$ *as* $s \to 0$. *In local homogeneous coordinates* $[t, u]$ *for* $\mathbb{P}(E \oplus F)$
$$T_0(c_2) = -\frac{Du\Omega_F u^*}{|u|^2} + \frac{u\Omega_F u^*}{|u|^2}\,\frac{Duu^*}{|u|^2} \in C^\infty\left(\mathbb{P}(E \oplus F) \sim X\right),$$
and so the Chern current $c_2((\overrightarrow{D}_{\pi,0}))$ *exists and equals*
$$c_2((\overrightarrow{D}_{\pi,0})) = c_2((\overrightarrow{D}_{\alpha,\infty})) + dT_0(c_2).$$
In particular the singular parts of the two Chern currents are equal.

Proposition 5.22. *Suppose that* D_E *and* D_F *are flat connections. Then for any invariant polynomial* ϕ,
$$T_s(\phi) \to 0 \qquad \textit{in } L^1_{\mathrm{loc}}(\widehat{\mathbb{P}}(E \oplus F)) \textit{ as } s \to 0.$$
Consequently if ϕ *is of pure degree* N *then*
$$\phi((\overrightarrow{D}_{\pi,0})) = \phi((\overrightarrow{D}_{\alpha,\infty})) = \left(-c_1(D_L)\right)^{N-1}\,[\mathbb{P}(F)_\infty] \qquad \textit{on } \widehat{\mathbb{P}}(E \oplus F).$$

Remark 5.23. In the nonflat case $T_s(c_3)$ does not converge in L^1_{loc} to $T_0(c_3)$ (see Example 5.26 below). Most likely it converges to a current of the form $T = \omega + \eta\,[\mathbb{P}(F)_\infty]$ where $\omega \in C^\infty(\widehat{\mathbb{P}}(E \oplus F) \sim \mathbb{P}(F)_\infty) \cap L^1_{\mathrm{loc}}(\widehat{\mathbb{P}}(E \oplus F))$ and η is a degree 3 L^1_{loc} form on $\mathbb{P}(F)_\infty$. Therefore the Chern current $c_3((\overrightarrow{D}_{\pi,0}))$ will still exist though it is impossible to predict the precise form of its singular part without much calculation!

Proof of Proposition 5.13. Simplifying [HL III.2.3] in the case that χ is the algebraic approximation mode gives
$$\Omega_{\alpha,r} = (1 - \chi_r P)(\Omega_F + \chi_r \Lambda) + \chi_r P \Omega_E$$

which can be rewritten as

$$\Omega_{\alpha,r} = \chi_r \Omega_{\alpha,0} + (1-\chi_r)(\Omega_F + \chi_r P\Lambda) \tag{5.24}$$

where

$$\Omega_{\alpha,0} = (1-P)(\Omega_F + \Lambda) + P\Omega_E. \tag{5.25}$$

Replacing $(\alpha, u, \Omega_F, \Omega_E)$ by $(\pi, w, \Omega_Q, \Omega_L)$ in (5.24) and (5.25) and simplifying yields (5.16) and (5.17). $\square$

Proof of Proposition 5.21. Recall that $c_2 = \frac{1}{2}(c_1^2 - b_2)$. Let $\alpha = \frac{Duu^*}{|u|^2}$ and note that $\alpha^2 = 0$. Plugging (5.11) and (5.14) into (5.19) gives

$$T_s(b_2) = \widetilde{\chi}_s \chi_r \, \alpha\lambda + \widetilde{\chi}_s \left(\frac{u\Omega_F u^*}{|u|^2} \, \alpha - \frac{Du\Omega_F u^*}{|u|^2} \right).$$

Since

$$u\frac{\partial}{\partial u} \llcorner T_s(b_2) = \widetilde{\chi}_s \chi_r \, (\lambda - \beta) \qquad \text{and} \qquad u^* \frac{\partial}{\partial u^*} \llcorner T_s(b_2) = 0$$

the expression for $T_s(b_2)$ in homogeneous coordinates is

$$T_s(b_2)_{HC} = T_s(b_2) - \widetilde{\chi}_s \chi_r \, (\lambda - \beta) \, \frac{dt}{t}.$$

Since $\widetilde{\chi}_s \chi_r$ is bounded by 1 and converges to zero a.e. as $s \to 0$ and $(\lambda - \beta) \frac{dt}{t} \in L^1_{\text{loc}}(\widehat{\mathbb{P}}(E \oplus F))$ the Lebesgue Dominated Convergence Theorem implies that $T_s(b_2) \to T_0(b_2)$ in $L^1_{\text{loc}}(\widehat{\mathbb{P}}(E \oplus F))$ as $s \to 0$. $\square$

Example 5.26. The expression for $T_s(b_3)$ in homogeneous coordinates includes the term

$$\frac{s|u|^2 \, (1+s)|t|^2}{((1+s)|t|^2 + s|u|^2)^3} \; \frac{Du(1-P)\Omega_F u^*}{|u|^2} \, dt d\bar{t}$$

which convergees to zero a.e. but not in L^1_{loc}.

Proof of Proposition 5.22. We can assume that $\phi = \prod_{j=1}^{m} b_{k_j}$. First use the fact that $\alpha^2 = 0$ to obtain the recursion formula

$$\text{Tr}(\omega(s)\Omega_\sigma^{k-1}) = -\sigma\widetilde{\chi}_s \lambda \; \text{Tr}(\omega(s)\Omega_\sigma^{k-2}) + (-1)^k \widetilde{\chi}_s \chi_r (\sigma\widetilde{\chi}_s + \chi_r)^{k-2} \, \alpha\lambda^{k-1}.$$

Therefore $b_k(\omega(s); \Omega_\sigma) = \chi_r C_1(s, \sigma, k) \, \alpha\lambda^{k-1}$ where $C_1(s, \sigma, k) \le C$. Furthermore since $b_k(\Omega_\sigma) = C_2(s, \sigma, k)\lambda^k$ mod α where $C_2(s, \sigma, k) \le C$ and the typical term of $\phi(\omega(s); \Omega_\sigma)$ is $b_{k_1}(\omega(s); \Omega_\sigma) \prod_{j=2}^{m} b_{k_j}(\Omega_\sigma)$ we have that

$$T_s(\phi) = C_3(s, k) \, \alpha\lambda^{N-1},$$

where $N = \deg \phi$ and $C_3(s, k) \le C$ with $C_3(s, k) \to 0$ as $s \to 0$. So since $u^* \frac{\partial}{\partial u^*} \llcorner T_s(\phi) = 0$ we can conclude as before that $T_s(\phi) \to 0$ as $s \to 0$ in $L^1_{\text{loc}}(\widehat{\mathbb{P}}(E \oplus F))$. $\square$

6. Meromorphic quaternionic line bundle homomorphisms.

The theory of Chern currents associated with endomorphisms of smooth quaternionic line bundle maps has strong analogies with that of maps between complex line bundles (see [HL II,V]). In this section we study the Chern currents associated with meromorphic maps between quaternionic line bundles. As we shall see the analytic aspects of this case are much more delicate than in the complex case.

First we review Harvey and Lawson's results in the **complex case**. A *meromorphic* section ν of a complex line bundle $F \to X$ is defined to be one which can be expressed in each local frame f for F as $\nu = \frac{a}{b} f$ where a, b are smooth complex valued functions on X. Such a section is called *atomic* if each of the local functions a, b are atomic. The *divisor* of ν is defined by $\mathrm{Div}(\nu) = \mathrm{Div}(a) - \mathrm{Div}(b)$. These notions are well defined since if $g, h : X \to \mathbb{C}$ are atomic then gh is atomic and $\mathrm{Div}(gh) = \mathrm{Div}(g) + \mathrm{Div}(h)$. Fix a connection D_F on F. The 1–form $\tau := \frac{da}{a} - \frac{db}{b} + \omega_F$ is a globally defined L^1_{loc} form on X and we have the current equation

$$\tfrac{1}{2\pi i}\, d\tau = \mathrm{Div}(\nu) - c_1(D_F) \qquad \text{on } X.$$

Now we turn to the **quaternionic case**. Let E, F be quaternionic line bundles over a manifold X.

Definition 6.1. A **meromorphic quaternion–linear bundle map** ν from E to F is a map which can be expressed in each pair of local frames e for E and f for F as

$$\nu\, e = ab^{-1}\, f \tag{6.2}$$

where a and b are smooth quaternion valued local functions on X. Such a mapping ν is called **atomic** if each of the local $\mathbb{R}^4$–valued functions a and b are atomic.

We would like to define the **divisor** of such an atomic map ν locally by the current equation

$$\mathrm{Div}(\nu) = \mathrm{Div}(a) - \mathrm{Div}(b). \tag{6.3}$$

The only difficulty in proving the quaternionic analogues of the results described above is to show that the divisor $\mathrm{Div}(\nu)$ is a well defined global current on X. We will discuss this problem in more generality later, but first we restrict our attention to a simple class of atomic meromorphic quaternionic maps from E to F for which it is easy to show that the divisor is well defined. This class consists of all maps ν for which the union Z of the zero sets of all the local functions a defined by (6.2) is disjoint from the union P of the zero sets of the local functions b. If this is the case we say that ν is a **determined** meromorphic map and call Z and P the zero and pole sets of ν. Note that this class of meromorphic maps includes all those maps which can be defined by a section of the bundle $\mathbb{P}_{\mathbb{H}}(E \oplus F) \to X$ of quaternionic projective spaces.

Proposition 6.4. *Let ν be a determined meromorphic quaternion linear bundle map between quaternionic line bundles E and F. Suppose that ν is atomic. Then the divisor,* $\operatorname{Div}(\nu)$, *of ν defined by (6.3) is a well defined current on X.*

Next we state a generalization of Harvey and Lawson's main theorem on homomorphisms between quaternionic line bundles, [HL V.2.26,36]. Endow E and F with metrics with respect to which scalar multiplication by unit quaternions is a pointwise isometry. Choose metric compatible connections D_E and D_F on E and F with respect to which scalar multiplication by any quaternion is parallel. The **instanton form**, $u(D_F)$, is defined by

$$u(D_F) := \tfrac{1}{16\pi^2}\operatorname{tr}(\Omega_F^2).$$

We use the notation $e := u(D_E)$ and $f := u(D_F)$.

Theorem 6.5. *Let ν be an atomic determined meromorphic quaternionic bundle map from E to F. Then there is a canonical L^1_{loc} form T on X called the* **instanton transgression current** *such that*

$$f - e - \operatorname{Div}(\nu) = dT$$

as currents on X where

$$16\pi^2 T = -\operatorname{tr}(\tau R_E) - \operatorname{tr}(\tau' R_F) + \tfrac{1}{3}\operatorname{tr}(\tau^3). \tag{6.6}$$

Here $\tau = \nu^{-1}\circ D\nu = \nu^{-1}\circ D_F\circ\nu - D_E$ and $\tau' = D\nu\circ\nu^{-1} = D_F - \nu\circ D_E\circ\nu^{-1}$ are L^1_{loc} forms on X with values in $\operatorname{End}_{\mathbb{H}}(E)$ and $\operatorname{End}_{\mathbb{H}}(F)$ respectively, and R_E and R_F are the curvature operators on E and F.

Remark 6.7. Suppose that ν is a section of the bundle $\mathbb{P}_{\mathbb{H}}(E\oplus F)$ of quaternionic projective spaces and let $E \overset{\alpha}{\to} F$ be the bundle homomorphism induced by ν over $X \sim P$. Just as in the complex case the pullback family of connections on E defined by $\overleftarrow{D}_s := D_E + \chi_s\, \alpha\circ D\alpha$ using the algebraic approximation mode $\chi(t) = \frac{t}{1+t}$ is compactified by the family of connections $\nu^*(D_{U,s})$ on the pullback $\nu^*(U)$ of the tautological quaternionic line bundle $U \to \mathbb{P}_{\mathbb{H}}(E\oplus F)$. Furthermore the current limit of the corresponding smooth transgression formula

$$u(\overleftarrow{D}_r) - u(\overleftarrow{D}_s) = dT_{r,s} \qquad \text{on } X$$

is the current equation

$$e - f + \operatorname{Div}(\nu) = -dT$$

of Theorem 6.5.

The proof of Proposition 6.4 relies on the following fundamental lemma.

Lemma 6.8. *Let $a, b : X \to \mathbb{H}$ be atomic. Suppose that the zero sets A, B of a, b are disjoint. Then*

(1) *ab is atomic.*

(2) *Let* $\tau(b) = db\, b^{-1}$, $\tau'(a) = a^{-1}\, da$ *and set*

$$\Omega = \tfrac{1}{2}\operatorname{Re}(\tau'(a)\tau(b)) = \tfrac{1}{2}\operatorname{Re}\left(\frac{\overline{a}da\, db\overline{b}}{|a|^2\, |b|^2}\right).$$

Then the smooth forms Ω *and* $d\Omega$ *on* $X \sim (A \cup B)$ *have (unique)* L^1_{loc} *extensions* $\widetilde{\Omega}$ *and* $\widetilde{d\Omega}$ *to all of* X*. Furthermore*

$$d\widetilde{\Omega} = \widetilde{d\Omega} \qquad \text{in } L^1_{\text{loc}}(X).$$

(3) *Let* $\theta(f) := f^*\theta \in L^1_{\text{loc}}(X)$ *denote the pullback of the solid angle kernel* θ *on* $\mathbb{H}$ *to* X *via an atomic function* f*. Then*

$$\theta(ab) = \theta(a) + \theta(b) + d\widetilde{\Omega} \qquad \text{in } L^1_{\text{loc}}(X).$$

(4) $\operatorname{Div}(ab) = \operatorname{Div}(a) + \operatorname{Div}(b)$ *on* X*.*
(5) *(2) holds with* b *replaced by* b^{-1}*.*
(6) *The smooth form* $\theta(ab^{-1})$ *on* $X \sim (A \cup B)$ *has an* L^1_{loc} *extension to all of* X *and*

$$\theta(ab^{-1}) = \theta(a) - \theta(b) + d\widetilde{\Omega} \qquad \text{in } L^1_{\text{loc}}(X)$$

where $\Omega = \frac{1}{2}\operatorname{Re}(\tau'(a)\tau'(b))$*. In particular,*

$$\theta(b^{-1}) = -\,\theta(b).$$

Proof of Lemma 6.8.

(1) We need to show that $\frac{d(ab)^I}{|ab|^p} \in L^1_{\text{loc}}(X)$ for all I such that $p = |I| \le 3$. Now each such form is dominated by sums of terms of the form $\frac{da^J}{|a|^j}\,\frac{db^K}{|b|^k}$ where $j = |J| \le 3$, $k = |K| \le 3$. But all of these forms belong to $L^1_{\text{loc}}(X)$ since the zero sets of a and b are disjoint and, in a neighbourhood of the zero set of a, $\frac{db^K}{|b|^k}$ is bounded while $\frac{da^J}{|a|^j} \in L^1_{\text{loc}}(X)$.

(2) Similarly Ω and $d\Omega$ have unique L^1_{loc} extensions $\widetilde{\Omega}$ and $\widetilde{d\Omega}$ to all of X. Let $\chi_s = \frac{|a|^2}{s^2+|a|^2}\,\frac{|b|^2}{s^2+|b|^2}$. Using the atomicity of a and b to dominate $d\chi_s\Omega$ by an s–independent L^1_{loc} form and applying the Lebesgue Dominated Convergence Theorem three times gives

$$d\widetilde{\Omega} = d(\lim_{s\to 0} \chi_s\, \Omega) = \lim_{s\to 0} d\chi_s\, \Omega + \lim_{s\to 0} \chi_s\, d\Omega = 0 + \widetilde{d\Omega}.$$

(3) Let x be a coordinate on $\mathbb{H}$ and set $\tau = dx\, x^{-1}$ and $\tau' = x^{-1}dx$. Then (see [HL V.2.34]) the solid angle kernel θ on $\mathbb{H}$ is given by

$$\theta = -\,\tfrac{1}{6}\operatorname{Re}(\tau^3).$$

Let $\tau(f) := f^*\tau$ and $\tau'(f) := f^*\tau'$ for any function $f : X \to \mathbb{H}$, and note that

$$\tau(ab) = \tau(a) + a\tau(b)a^{-1}.$$

Then

$$\theta(ab) = -\tfrac{1}{6}\operatorname{Re}(\tau^3(ab)) = \theta(a) + \theta(b) + \tfrac{1}{2}\operatorname{Re}\big(d\tau'(a)\tau(b) - \tau'(a)\, d\tau(b)\big),$$

since $\tau^2 = d\tau$ and $(\tau')^2 = -\, d\tau'$. So we have the equation of smooth forms

$$\theta(ab) = \theta(a) + \theta(b) + d\Omega \qquad \text{on } X \sim (A \cup B).$$

Therefore

$$\widetilde{d\Omega} = \theta(ab) - \theta(a) - \theta(b) \qquad \text{in } L^1_{\text{loc}}(X),$$

since it is true almost everywhere. (3) now follows from (2) and (4) from (3). $\square$

Proof of Proposition 6.4. Suppose that $\nu e = ab^{-1} f$ and $\nu e' = cd^{-1} f'$ where $e' = ge$ and $f' = hf$. Since $gab^{-1} = cd^{-1}h$ with g and h nonzero, Lemma 6.8(6) implies that

$$\theta(ga) - \theta(b) + d\widetilde{\Omega}(ga, b^{-1}) = \theta(c) - \theta(h^{-1}d) + d\widetilde{\Omega}(c, h^{-1}d).$$

Combining this with Lemma 6.8(4) gives $\operatorname{Div}(a) - \operatorname{Div}(b) = \operatorname{Div}(ga) - \operatorname{Div}(b) = \operatorname{Div}(c) - \operatorname{Div}(h^{-1}d) = \operatorname{Div}(c) - \operatorname{Div}(d)$ as required. $\square$

Proof of Theorem 6.5. We check that the smooth form T defined on $X \sim (Z \cup P)$ by (6.6) has an L^1_{loc} extension $\widetilde{T}$ to all of X and compute $d\widetilde{T}$. Write $\nu e = ab^{-1} f$. Then T is a sum of terms, each of which is the product of an L^∞_{loc} form on X with one of the forms

$$\frac{da^I\, db^J}{|a|^p\, |b|^q} \qquad \text{where } p = |I|,\ \ q = |J| \text{ and } p + q \le 3.$$

But each of these forms is in $L^1_{\text{loc}}(X)$ since a and b are atomic and have disjoint zero sets. Therefore T has an L^1_{loc} extension $\widetilde{T}$ to all of X. Next note that the smooth form dT on $X \sim (Z \cup P)$ has a C^∞ extension $\widetilde{dT}$ to all of X since by [HL V.2.10] $dT = u(D_F) - u(D_E)$ on $X \sim (Z \cup P)$. Define $S := d\widetilde{T} - \widetilde{dT}$ on X. Since $\widetilde{dT} = u(D_F) - u(D_E)$ we just need to check that $S = -\operatorname{Div}(\nu)$. Let T_{top} be that part of $\widetilde{T}$ which is of top degree 3 in the differentials da_i, db_j and let $T' := \widetilde{T} - T_{\text{top}}$. Then, since trace equals four times the real part,

$$T_{\text{top}} = \tfrac{1}{48\pi^2}\operatorname{tr}(\tau^3(ab^{-1})) = -\tfrac{1}{2\pi^2}\,\theta(ab^{-1}),$$

and so, by Lemma 6.8(6), $dT_{\text{top}} = -\,(\operatorname{Div}(a) - \operatorname{Div}(b)) = -\operatorname{Div}(\nu)$. Finally since dT' is L^1_{loc} and $\widetilde{dT} + S = dT' - \operatorname{Div}(\nu)$ we are forced to conclude that $S = -\operatorname{Div}(\nu)$, as required. $\square$

Let $a, b : X \to \mathbb{H}$ be atomic. Ideally we would like to prove that ab is atomic and that $\operatorname{Div}(ab) = \operatorname{Div}(a) + \operatorname{Div}(b)$. This has not been possible. Neither have we found any counterexamples. Instead we describe some extra assumptions which can be placed on the pair a, b to ensure that the desired conclusions hold. Proofs are left to the reader. The first result is motivated by [HS 3.2].

Definition 6.9. **[HS 3.2].** Let u be a smooth $\mathbb{R}^n$–valued function on $X^{\text{open}} \subset \mathbb{R}^m$. We say that u *vanishes algebraically* if, for each compact set $K \subset X$, there exist positive constants c and N such that

$$|u(x)| \geq c\,\mathrm{dist}(x,\,\mathrm{Zero}(u))^N \qquad \text{for all } x \in K.$$

The zero set of u is said to have *strong codimension greater than* $n-1$ if for each compact $K \subset X$ there exists an $\epsilon > 0$ such that the upper Minkowski content of $\mathrm{Zero}(u) \cap K$ in dimension $m - n + 1 - \epsilon$ is finite.

Note. Harvey and Semmes proved that if u vanishes algebraically and if its zero set has strong codimension greater than $n-1$ then u is atomic.

Proposition 6.10. *Let $a, b : X \to \mathbb{H}$ be smooth.*

(1) *If a and b vanish algebraically then so does ab.*
(2) *If the zero sets of a and b have strong codimension greater than 3 then so does that of ab.*
(3) *Therefore if the hypotheses of (1) and (2) hold then a, b and ab are all atomic.*

Corollary 6.11. *Let $a, b : X \to \mathbb{H}$ be real analytic and suppose that each irreducible component of the zero sets of a and b has codimension ≥ 4. Then a, b and ab are all atomic.*

Let $\mathcal{H}^k$ denote k-dimensional Hausdorff measure. The following result follows from the Federer Support Theorem for flat currents (see [F]).

Proposition 6.12. *Let $a, b : X \to \mathbb{H}$ be atomic. Let A, B denote the zero sets of a, b and let $n :=\ \dim X$. Suppose that ab is atomic and that either*
(1) $\mathcal{H}^{n-3}(A \cap B) = 0$ and $\Omega = \frac{1}{2}\,\mathrm{Re}\big(\frac{\overline{a}da\, db\overline{b}}{|a|^2\,|b|^2}\big)$ has an L^1_{loc} extension $\widetilde{\Omega}$ to all of X, or
(2) $\mathcal{H}^{n-4}(A \cap B) = 0$.
Then

$$\mathrm{Div}(ab) = \mathrm{Div}(a) + \mathrm{Div}(b) \qquad \textit{on } X.$$

Note. In contrast to this note that if $a : X \to \mathbb{H}$ is atomic then a^n is atomic and $\mathrm{Div}(a^n) = n\ \mathrm{Div}(a)$ on X.

We conclude this section with an example of a pair of polynomial functions a and b which do not satisfy the hypotheses of Lemma 6.8 or Proposition 6.12. Nevertheless we can compute directly that $\mathrm{Div}(ab) = \mathrm{Div}(a) + \mathrm{Div}(b)$. One obvious way to try to prove that $\mathrm{Div}(ab) = \mathrm{Div}(a) + \mathrm{Div}(b)$ for *any* pair of atomic functions a and b for which ab is also atomic would be to find a natural current extension $\widetilde{\Omega}$ to all of X of the smooth form Ω on $X \sim (A \cup B)$ and to show that $d\widetilde{\Omega} = \theta(ab) - \theta(a) - \theta(b)$. The example below shows that we cannot hope to extend Ω to an L^1_{loc} form on all of X. We have not been able to find a natural current extension of Ω. Neither have we been able to find a pair of functions a, b for which $\mathrm{Div}(ab) \neq \mathrm{Div}(a) + \mathrm{Div}(b)$.

Example 6.13. *Let $a, b : \mathbb{H} \to \mathbb{H}$ be defined by*

$$a(x_0, x_1, x_2, x_3) = (x_0, x_1^3, x_2^3, x_3^3), \qquad \text{and} \qquad b(x_0, x_1, x_2, x_3) = (x_0, x_1, x_2, x_3).$$

Then

(1) *a, b and ab are all atomic,*
(2) *$\mathcal{H}^0(A \cap B) = 1 > 0$,*
(3) *$\Omega \notin L^1_{\text{loc}}(\mathbb{H})$, but*
(4) *$\text{Div}(a) = [0]$, $\text{Div}(b) = [0]$ and $\text{Div}(ab) = 2[0]$, so that*

$$\text{Div}(ab) = \text{Div}(a) + \text{Div}(b) \qquad \text{on } \mathbb{H}.$$

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Rice University, Houston, Texas.