THE PRODUCT FORMULA FOR THE TOTAL CHERN AND PONTRYJAGIN SPARKS

F. Reese Harvey, H. Blaine Lawson and John Zweck

June 21, 1999

Abstract

This paper represents a systematic study of certain secondary classes associated to a map $\alpha : E \to F$ between bundles with connection over a manifold X. For each integer $k < \min\{\text{rank}(E), \text{rank}(F)\}$, there is a canonical and functorial L^1_{loc}-form $S_k(\alpha)$ on X such that

$$dS_k(\alpha) \ = \ TP_k(E, F) - \mathbb{D}_k(\alpha)$$

where $TP_k(E, F)$ is the Chern-Weil representative of the k^{th} Thom-Porteous class, and $\mathbb{D}_k(\alpha)$ is the k^{th} degeneracy current, which correctly represents the dropping of α to rank $\leq k$. The form $S_k(\alpha)$, called the k^{th}-Thom-Porteous spark, can be thought of as a representative of Chern-Weil type for the differential character $\widehat{TP}_k(E, F)$.

We introduce the space of secondary de Rham classes $\widehat{H}^*(X)$, defined by the L^1_{loc}-forms on X modulo the equivalence relation $\sim$ where $\phi \sim \psi$ iff $\phi - \psi = R + d\eta$ where R is a rectifiable current and η is L is an L^1_{loc}-form. The class of $S_k(\alpha)$ in $\widehat{H}^*(X)$ is shown to be independent of α, and is called the secondary k^{th} **Thom-Porteous class** of (E, F).

Given two maps $\alpha : E \to F$ and $\alpha' : E' \to F'$, a global spark product $S_k(\alpha) * S_k(\alpha')$. Two fundamental examples are the total Chern spark of a complex bundle E, constructed from bundle maps $\mathbf{C}^k \to E$, and the total Ronga-Pontrjagin spark of a real bundle E, which is constructed similarly. One of the main theorems establishes formulas for the behavior of these under the Whitney sum of bundles.

The relation of characteristic sparks to differential characters is examined. The spark product is shown to carry over to the Cheeger product on differential characters. A large number of specific examples arising in geometry are then analysed.

Contents

1. Introduction.

2. The Top Chern Spark.

In this section we summarize the results of [HS],[HL2],[HZ2] concerning the divisor and top Chern spark of an atomic section of a complex vector bundle with connection.

Let $F \to X$ be a complex vector bundle of rank n and let α be a smooth section of F. In [HS] the authors defined the notion of the divisor or zero current, $\mathrm{Div}(\alpha)$, of an atomic section α. The divisor is a d-closed current of degree $2n$ on X which is supported on the zero set and encodes the integer multiplicity of vanishing of the section. Atomicity is a weak measure-theoretic condition which ensures the existence of the divisor of a section.

The divisor of an atomic section α is defined, locally on X, as follows. Let $u : X \to \mathbf{C}^n$ be a local coordinate expression for the section. The **solid angle kernel**, θ, on $\mathbf{C}^n$ is the locally Lebesgue integrable (abbr. $L^1_{\rm loc}$) form on $\mathbf{C}^n$ defined by pulling back the normalized volume form on the unit sphere to $\mathbf{C}^n \sim \{0\}$ via the radial projection map. It satisfies the equation $d\theta = [0]$ which motivates the definition of the divisor. The atomicity of the section α ensures that the smooth form $u^*(\theta)$ on the complement of the zero set in X has an $L^1_{\rm loc}$ extension to all of X and so defines a current. Then the **divisor** of α is locally defined by the current equation

$$\mathrm{Div}(\alpha) \ := \ d(u^*(\theta)).$$

Finally a result of [HS] (c.f. [HZ2]) ensures that the divisor is globally defined on X.

If zero is a regular value of the section α, then α is atomic and the divisor is the current of integration over the zero submanifold of α. However, the set of atomic sections is much larger than the set of sections which have zero as a regular value. For example [HS], all real analytic sections of a real rank m bundle whose zero sets have codimension at least m are atomic. More generally, any smooth section which vanishes algebraically and whose zero set has Minkowski codimension strictly greater than $m - 1$ is atomic.

The top Chern spark of an atomic section α of F is defined as follows. Endow the vector bundle $\pi : F \to X$ with a connection D_F and a metric. The **spherical Chern potential**

σ is an $L^1_{\rm loc}$ form of degree $n-1$ on F which satisfies the current equation

$$d\sigma \ = \ \pi^* c_n(\Omega_F) \ - \ [X] \qquad \text{on } F, \tag{2.1}$$

where $\pi^* c_n(\Omega_F)$ is the pullback to F of the top Chern form $c_n(\Omega_F)$ of the connection D_F, and where $[X]$ is the current of integration over the zero section of $F \to X$. The potential σ was discovered by Bott and Chern [BCh1]. An explicit formula for σ is also due to Chern [BCh2] (c.f. [HL2,III.4.22]). In particular the restriction of σ to each fibre of F is the solid angle kernel θ of the fibre. The **top Chern spark**, $\sigma(\alpha)$, of an atomic section α of F (relative to the connection and metric) is the $L^1_{\rm loc}$ form on X defined, on the complement of the zero set of α in X, by the equation $\sigma(\alpha) := \alpha^*(\sigma)$. (The atomicity of the section α ensures that the smooth form $\alpha^*(\sigma)$ on $X \sim \text{Zero}(\alpha)$ has a unique $L^1_{\rm loc}$ extension to all of X.)

There are three fundamental current equations which are central to the theory. The first is the **local Gauss-Bonnet-Chern formula** [HL2] which states that

$$d\sigma(\alpha) \ = \ c_n(\Omega_F) \ - \ \text{Div}(\alpha) \qquad \text{on } X. \tag{2.2}$$

This equation can be regarded as the pullback to X via α of the current equation (2.1). The classical Gauss-Bonnet-Chern formula [BCh1] is recovered from equation (2.2) by letting each of the currents in (2.2) act on the test function 1 on a compact $2n$-dimensional manifold X. The proof of the classical Gauss-Bonnet-Chern formula just outlined is in the same spirit as Chern's proof of the formula in [BCh1].

The second equation is the **comparison formula for divisors** [HS] which states that, if α and α' are two atomic sections of F, then there is a locally rectifiable current R so that

$$\text{Div}(\alpha) \ - \ \text{Div}(\alpha') \ = \ -dR \qquad \text{on } X. \tag{2.3}$$

This equation implies that the divisor of an atomic section is a d- closed locally integrally flat current (i.e. it is of the form $R + dS$ where R, S are locally rectifiable). Consequently a theorem of Federer [F] implies that the divisor of an atomic section determines an integer cohomology class. Furthermore equation (2.3) states that this cohomology class – the top Chern class – is independent of the choice of atomic section of F.

The third equation is the **comparison formula for top Chern sparks** [HZ2] which states that, if α and α' are two atomic sections of F, then there is a locally rectifiable current R and an $L^1_{\rm loc}$ current L so that

$$\sigma(\alpha) \ - \ \sigma(\alpha') \ = \ R \ + \ dL \qquad \text{on } X. \tag{2.4}$$

Note that equation (2.3) can be obtained by taking the exterior derivative of equation (2.4).

Because of equation (2.4) it is natural to define an equivalence relation $\sim$ on $L^1_{\rm loc}$ currents by

$$\omega \sim \eta \ \text{ if and only if } \ \omega - \eta = R + dL, \ \text{ where } R \text{ is locally rectifiable and } L \text{ is } L^1_{\rm loc}.$$

Let $\widetilde{H}^*(X) = L^1_{\mathrm{loc}}(X)/\sim$ denote the space of equivalence classes $[\omega]$. The equivalence classes $[\omega]$ will be called **secondary de Rham cohomology classes**. Because of the comparison formula (2.4) we can define the **secondary top Chern class**, $\widetilde{c}_n(D_F)$, **of a vector bundle** F **with connection** D_F to be the secondary de Rham cohomology class

$$\widetilde{c}_n(D_F) \;:=\; [\sigma(\alpha)] \in \widetilde{H}^*(X),$$

where α is any atomic section of F.

The intersection of two divisors and the product of two top Chern sparks is defined as follows. Let F_1 and F_2 be complex vector bundles over X with connections D_{F_1} and D_{F_2} and metrics. Endow the direct sum $F_1 \oplus F_2$ with the direct sum connection $D_{F_1 \oplus F_2}$ and metric. Let α_1 and α_2 be atomic sections of F_1 and F_2 and suppose that the divisors of α_1 and α_2 **intersect properly** in that the section (α_1, α_2) of $F_1 \oplus F_2 \to X$ is atomic. Then the intersection, $\mathrm{Div}(\alpha_1) \wedge \mathrm{Div}(\alpha_2)$, of the divisors of α_1 and α_2 is defined to be the divisor of the section (α_1, α_2),

$$\mathrm{Div}(\alpha_1) \wedge \mathrm{Div}(\alpha_2) \;:=\; \mathrm{Div}(\alpha_1, \alpha_2). \tag{2.5}$$

Similarly, the **product of the top Chern sparks** of α_1 and α_2 is defined to be the top Chern spark of the section (α_1, α_2),

$$\sigma(\alpha_1) * \sigma(\alpha_2) \;:=\; \sigma(\alpha_1, \alpha_2). \tag{2.6}$$

¿From the properties of the determinant we know that for the direct sum connection $D_{F_1 \oplus F_2} = D_{F_1} \oplus D_{F_2}$ on $F_1 \oplus F_2$, we have

$$c_n\left(D_{F_1}\right) \wedge c_m\left(D_{F_2}\right) \;=\; c_{n+m}\left(D_{F_1 \oplus F_2}\right) \tag{2.7}$$

where $n = \mathrm{rank}(F_1)$ and $m = \mathrm{rank}(F_2)$.

With these definitions the three products – the top Chern spark product, the usual product of top Chern forms and intersection of zero divisors – are related by the equation

$$d(\sigma(\alpha_1) * \sigma(\alpha_2)) \;=\; c_n(\Omega_{F_1}) \wedge c_m(\Omega_{F_2}) \;-\; \mathrm{Div}(\alpha_1) \wedge \mathrm{Div}(\alpha_2). \tag{2.8}$$

Note that the intersection of divisors and the product of total Chern sparks are both **associative** and **commutative**. For instance, the associativity of the spark product is automatic, i.e.

$$\{\sigma(\alpha_1) * \sigma(\alpha_2)\} * \sigma(\alpha_3) \;=\; \sigma(\alpha_1) * \{\sigma(\alpha_2) * \sigma(\alpha_3)\},$$

whenever both sides are defined. (In this case both sides equal $\sigma(\alpha_1, \alpha_2, \alpha_3)$.) The commutativity,

$$\sigma(\alpha_1) * \sigma(\alpha_2) \;=\; (-1)^{nm} \sigma(\alpha_2) * \sigma(\alpha_1),$$

of the spark product is also straightforward.

Suppose now that (α_1, α_2) and (α_1', α_2') are sections of $F_1 \oplus F_2$ each of which intersect properly as above. Then from 2.3 and 2.5 we see that

$$\operatorname{Div}(\alpha_1) \wedge \operatorname{Div}(\alpha_2) - \operatorname{Div}(\alpha_1') \wedge \operatorname{Div}(\alpha_2') \;=\; dR \tag{2.9}$$

for a rectifiable current R. Furthermore from 2.4 and 2.6 we have

$$\sigma(\alpha_1) * \sigma(\alpha_2) - \sigma(\alpha_1') * \sigma(\alpha_2') \;=\; R + dL \tag{2.10}$$

where R is rectifiable and L is L^1_{loc}. It follows that our spark product induces a product of secondary classes:

$$\widetilde{c}_n(D_{F_1}) * \widetilde{c}_m(D_{F_2}) := \widetilde{c}_{n+m}(D_{F_1 \oplus F_2}). \tag{2.11}$$

which is independent of the choice of the sections α_1 and α_2.

3. Degeneracy Currents and Thom-Porteous Sparks.

In this section we shall introduce sparks and their products for the general degeneracy currents defined in [HL3]. We work in the following set-up. Fix a smooth bundle map

$$\begin{array}{ccc} E & \xrightarrow{\alpha} & F \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

between smooth complex vector bundles with unitary connections. Let $m = \operatorname{rank}(E)$, $n = \operatorname{rank}(F)$, and fix an integer k with $0 \le k < \min\{m, n\}$. To this data [GRT] associates an equation

$$TP_k(\Omega_E, \Omega_F) \;-\; \mathbb{D}_k(\alpha) \;=\; dS_k(\alpha) \qquad \text{on } X. \tag{3.1}$$

expressing the difference of a kth Thom-Porteous characteristic form of $E^* \otimes F$ and the kth degeneracy current of α as the exterior derivative of a canonical L^1_{loc}-form on X.

Equation (3.1) is derived from results of the previous section as follows. Set $r = m - k$ and let

$$\pi : G_r(E) \to X$$

be the Grassmann bundle of r-planes in E. Denote by $U \to G_r(E)$ the tautological r-plane bundle, and note the splitting $\pi^* E = U \oplus U^\perp$. Lifting α over π and restricting gives a bundle map $U \to \pi^* F$ which we will consider as a *section* $\widehat{\alpha}$ of $H \equiv \operatorname{Hom}(U, \pi^* F)$ over

$G_r(E)$. The map α is called **k-atomic** if the section $\widehat{\alpha}$ is atomic. Assuming this, we apply equation (2.2) to obtain

$$d\sigma(\widehat{\alpha}) \;=\; c_{nr}(U^* \oplus \pi^* F) - \mathrm{Div}(\widehat{\alpha}) \qquad \text{on } \; G_r(E) \tag{3.2}$$

where $\sigma(\widehat{\alpha})$ is a canonical L^1_{loc}-form on $G_r(E)$.

To reexpress the middle term in (3.2) we recall the Shur polynomials which are defined as follows. Let $\eta = 1 + \eta_1 + \eta_2 + \dots$ be a differential form, where each η_k is of pure degree $2k$, and let a, b be non-negative integers. The **Shur polynomial** of type (a, b) in η is

$$\Delta_a^{(b)}(\eta) \;=\; \det(\eta_{a-i+j})_{1\leq i,j\leq b}.$$

and one has that

$$c_{nr}(U^* \oplus \pi^* F) \;=\; \Delta_a^{(b)}\{\pi^* c(\Omega_F)\, c(\Omega_U)^{-1}\}.$$

Now there exists a canonical smooth form s *which depends only on the connections D_E and D_F and not on α*, such that

$$ds \;=\; \Delta_a^{(b)}\left\{\pi^* c((\Omega_F)\, c(\Omega_E)^{-1})\, c(\Omega_{U^\perp})\right\} - \Delta_a^{(b)}\{\pi^* c(\Omega_F)\, c(\Omega_U)^{-1}\}$$

Combining the equations above gives

$$d\{\sigma(\widehat{\alpha}) + s\} \;=\; \Delta_a^{(b)}\left\{\pi^* c((\Omega_F)\, c(\Omega_E)^{-1})\, c(\Omega_{U^\perp})\right\} - \mathrm{Div}(\widehat{\alpha}) \qquad \text{on } \; G_r(E). \tag{3.3}$$

Integrating this over the fibre of π gives equation (3.1) where by definition

$$TP_k(\Omega_E, \Omega_F) \;:=\; \Delta_{n-k}^{(m-k)}(c(\Omega_F)\, c(\Omega_E)^{-1}) \tag{3.4}$$

$$\mathbb{D}_k(\alpha) := \pi_* \,\mathrm{Div}(\widehat{\alpha}) \qquad \text{and} \qquad S_k(\alpha) := \pi_*(\sigma(\widehat{\alpha}) + s). \tag{3.5}$$

We recall that the kth degeneracy current $\mathbb{D}_k(\alpha)$ is the current which properly represents the collapsing of α to rank $\leq k$. (See [HL3].) In many interesting cases we have $\pi_* s = 0$.

Definition 3.6. The **kth Thom-Porteous spark** of α is the canonical L^1_{loc}-form $S_k(\alpha)$ on X.

Suppose now that $\alpha : E \to F$ and $\alpha' : E \to F$ are two k-atomic bundle maps. Applying (2.3) and (2.4) to $\widehat{\alpha}$ and $\widehat{\alpha}'$ and pushing down by π gives the comparison formulae

$$\begin{aligned} \mathbb{D}_k(\alpha) - \mathbb{D}_k(\alpha') \;&=\; -dR \qquad \text{and} \\ S_k(\alpha) \;-\; S_k(\alpha') \;&=\; R + dL \end{aligned} \tag{3.7}$$

where R is a rectifiable current and L is an L^1_{loc}-form on X. Note that the contribution of the smooth form s is the same for $S_k(\alpha)$ and $S_k(\alpha')$ and cancels when taking the difference.

We now consider products. Let $\alpha_1 : E_1 \to F_1$ and $\alpha_2 : E_2 \to F_2$ be bundle maps over X which are k_1- and k_2-atomic respectively. Assume that $\mathrm{Div}(\widehat{\alpha}_1)$ and $\mathrm{Div}(\widehat{\alpha}_2)$ intersect properly on $G_{r_1}(E_1) \times_X G_{r_2}(E_2)$ and define

$$\mathbb{D}_{k_1}(\alpha_1) \wedge \mathbb{D}_{k_2}(\alpha_2) \ := \ \pi_* \left\{ \mathrm{Div}(\widehat{\alpha}_1) \wedge \mathrm{Div}(\widehat{\alpha}_2) \right\}$$

where $\pi : G_{r_1}(E_1) \times_X G_{r_2}(E_2) \to X$ is the projection.

For each $k = 1, 2$ we write equation (3.3) on $G_{r_k}(E_k)$ in condensed form as

$$d\{\sigma(\widehat{\alpha}_k) + s_k\} \ = \ \widehat{\Delta}_k - \mathrm{Div}(\widehat{\alpha}_k)$$

¿From (2.8) we have that

$$d\left(\sigma(\widehat{\alpha}_1) * \sigma(\widehat{\alpha}_2)\right) \ = \ (\widehat{\Delta}_1 - ds_1) \wedge (\widehat{\Delta}_2 - ds_2) - \mathrm{Div}(\widehat{\alpha}_1) \wedge \mathrm{Div}(\widehat{\alpha}_1) \tag{3.8}$$

on the fibre product $G_{r_1}(E_1) \times_X G_{r_2}(E_2)$.

Definition 3.9. By the **spark product** of $S_{k_1}(\alpha_1)$ and $S_{k_2}(\alpha_2)$ we mean the L^1_{loc}-form

$$S_{k_1}(\alpha_1) * S_{k_2}(\alpha_2) \ := \ \pi_* \left\{ \sigma(\widehat{\alpha}_1) * \sigma(\widehat{\alpha}_2) + s_1 \wedge \widehat{\Delta}_2 + \widehat{\Delta}_1 \wedge s_2 - s_1 \wedge ds_2 \right\} \tag{3.10}$$

Let $\Delta_k = \pi_* \widehat{\Delta}_k$ be the form appearing in formula (3.1). Then pushing forward equation (3.8) by π gives

$$d\left\{ S_{k_1}(\alpha_1) * S_{k_2}(\alpha_2) \right\} \ = \ \Delta_1 \wedge \Delta_2 - \mathbb{D}_{k_1}(\alpha_1) \wedge \mathbb{D}_{k_2}(\alpha_2) \tag{3.11}$$

Suppose now that we consider new bundle maps $\alpha'_k : E_k \to F_k$ so that the divisors of α_k and α'_k intersect properly. We want to compare the product of the degeneracy currents and the spark products for the two pairs α_1, α_2 and α'_1, α'_2. Note first that the last three terms in (3.10) are independent of the choice of sections α_k. Therefore applying equations (2.9) and (2.10) upstairs and pushing forward by π give the comparison formulae

$$\begin{aligned} \mathbb{D}_{k_1}(\alpha_1) \wedge \mathbb{D}_{k_2}(\alpha_2) - \mathbb{D}_{k_1}(\alpha'_1) \wedge \mathbb{D}_{k_2}(\alpha'_2) \ &= \ -dR \qquad \text{and} \\ S_{k_1}(\alpha_1) * S_{k_2}(\alpha_2) - S_{k_1}(\alpha'_1) * S_{k_2}(\alpha'_2) \ &= \ R + dL \end{aligned} \tag{3.12}$$

where R is a rectifable current and L is an L^1_{loc}-form on X.

Definition 3.13. To each α and k as above we associate the **secondary Thom-Porteous class** of $H = \mathrm{Hom}(E, F)$ to be the image

$$\widetilde{\Delta}_k(\Omega_H) = \widetilde{\Delta}^{(n-k)}_{m-k}(c(\Omega_F)c(\Omega_E)^{-1}) \ := \ \left[S_k(\alpha)\right] \ \in \ \widetilde{H}^*(X)$$

in the secondary deRham classes of X. By (3.7), $\widetilde{\Delta}_k(\Omega_H)$ is independent of the section α.

Definition 3.14. To each pair α_1, k_1 and α_2, k_2 as above we define the product of secondary classes

$$\widetilde{\Delta}_{k_1}(\Omega_{H_1}) * \widetilde{\Delta}_{k_2}(\Omega_{H_2}) \ := \ [S_{k_1}(\alpha_1) * S_{k_2}(\alpha_2)] \ \in \ \widetilde{H}^*(X).$$

By (3.12), this product is independent of the sections α_1 and α_2.

All the definitions and results above carry over to real bundles (cf. [HL3]). The straightforward details are omitted.

4. Functoriality.

The sparks and their associated secondary classes are functorial in the following sense. Let

$$
(4.1) \qquad
\begin{array}{ccc}
E & \xrightarrow{\alpha} & F \\
& \searrow \quad \swarrow & \\
& X &
\end{array}
$$

be a smooth bundle morphism of bundles with unitatry connections and let k be an integer as in §2. Given a manifold Y and a smooth mapping $f : Y \to X$, there are induced bundles with induced connections and an induced bundle morphism

$$
(4.2) \qquad
\begin{array}{ccc}
f^*E & \xrightarrow{f^*\alpha} & f^*F \\
& \searrow \quad \swarrow & \\
& Y &
\end{array}
$$

together with a morphism from diagram (4.1) to (4.2). The Chern- Weil constructions for (4.1) pull-back to give the corresponding Chern- Weil constructions for (4.2).

Definition 4.3. The mapping $f : Y \to X$ is (α, k)-regular if the induced section $f^*\alpha$ is k-atomic.

Lemma 4.4. *If $f : Y \to X$ is (α, k)-regular, then*

$$
f^* S_k(\alpha) \;=\; S_k(f^*\alpha)
$$

as smooth forms in $Y - \mathbb{D}_k(f^*\alpha)$ with an L^1_{loc}-extension across the degeneracy set.

Proof. See I.10 and III.3.26 of [HL2].

We now observe that there is a universal object in this category. Let $G = Gr_m(\mathbf{C}^N) \times Gr_n(\mathbf{C}^N)$ be the product of the Grassmannians of m-planes and n-planes respectively in $\mathbf{C}^N$, and let

$$
\pi : H \longrightarrow \text{Hom}(\xi_m, \xi_n)
$$

where ξ_m and ξ_n are the pull-back to the product of the tautological bundles over these Grassmannians. These bundles carry natural unitarily-invariant connections. Now the pull-back $\pi^* H \to H$ has a tautological cross-section which gives a diagram

$$
(4.5) \qquad
\begin{array}{ccc}
\pi^*\xi_m & \xrightarrow{\alpha_H} & \pi^*\xi_n \\
& \searrow \quad \swarrow & \\
& H &
\end{array}
$$

Theorem 4.6. *Every diagram (4.1) is induced from the diagram (4.5) by a smooth mapping $f : X \to H$ for N sufficiently large. Furthermore, if α is k-atomic, then f is (α, k)-regular.*

Proof. The existence of $f_0 : X \to G$ classifying E and F with their connections follows from [NR]. The given map α determines a unique lifting of f_0 to $f : X \to H$. If α is k-atomic, then f is (α, k)-regular by definition. $\square$

Now for each k the tautological section α_H over H is k- atomic and its kth degeneracy current is just the manifold $\mathbb{D}_k(\alpha_H) = [\Sigma_k]$ where

$$\Sigma_k \ = \ \{A \in H : \mathrm{rank}(A) = k\}$$

Hence there exist **universal degeneracy sparks** $S_k(\alpha_H)$ which satisfy the equation

$$dS_k(\alpha_H) \ = \ \Delta_{m-k}^{(n-k)} \left\{c(\Omega_{\xi_m})c(\Omega_{\xi_m})^{-1}\right\} - [\Sigma_k] \qquad \text{on} \ \ H.$$

Corollary 4.7. *Given a k-atomic bundle map (4.1) with degeneracy spark $S_k(\alpha)$ and a classifying map $f : X \to H$ as in Theorem 4.6, one has*

$$f^* S_k(\alpha_H) \ = \ S_k(\alpha)$$

as smooth forms in $X - supp(\mathbb{D}_k(\alpha))$ with L^1_{loc}- extensions to all of X.

5. Chern Sparks.

Each Chern form of a complex vector bundle with connection can be related to the linear dependency set of a certain collection of sections of the bundle. This was analysed in [HL3] and represents a special case of the discussion in §2. That discussion yields a Chern spark and its associated secondary Chern class. In this section these classes and their products are examined in detail.

Let $\alpha = (\alpha^1, ..., \alpha^k)$ be a collection of k sections of a rank n complex vector bundle $F \to X$ with unitary connection D_F, and let $p := n - k + 1$. This corresponds to a bundle map $\alpha : \underline{\mathbf{C}}^k \to F$, where $\underline{\mathbf{C}}^k$ denotes the trivial bundle of rank k. The family is called $\mathbb{LD}$-atomic if the map α is $(k-1)$-atomic in the sense of §2. In this case the **linear dependency current** $\mathbb{LD}(\alpha) := \mathbb{D}_{k-1}(\alpha)$ of α is defined. It is a current of degree $2p$ on X which is supported on the linear dependency set of the collection α.

Furthermore, with the trivial connection on $\underline{\mathbf{C}}^k$, the discussion of §2 gives a canonical L^1_{loc} $(2p-1)$-form $S_p(\alpha)$ on X, called the **pth Chern spark** of α and D_F, which satisfies the equation

$$dS_p(\alpha) \ = \ c_p(\Omega_E) \ - \ \mathbb{LD}(\alpha) \qquad \text{on } X \tag{5.1}$$

where $c_p(\Omega_F)$ is the pth Chern form of D_F.

We shall examine the derivation of this equation in detail for use in the next section. Let $\pi : \mathbf{P}(\underline{\mathbf{C}}^k) \to X$ denote the denote the projectivization of the bundle $\underline{\mathbf{C}}^k$, and let $U \subset \pi^* \underline{\mathbf{C}}^k$ denote the tautological line bundle over $\mathbf{P}(\underline{\mathbf{C}}^k)$. Consider the **induced section**

$$\widehat{\alpha} \;=\; \pi^* \alpha\big|_U$$

of the bundle $\mathrm{Hom}(U, \pi^* F)$ over $\mathbf{P}(\underline{\mathbf{C}}^k)$. Then, by definition, α is $\mathbb{LD}$-atomic if the induced section $\widehat{\alpha}$ is atomic, and when this holds the linear dependency current, is defined by

$$\mathbb{LD}(\alpha) \;:=\; \pi_*(\mathrm{Div}(\widehat{\alpha})) \qquad \text{on } X. \tag{5.2}$$

Give $\mathrm{Hom}(U, \pi^* F) = U^* \otimes \pi^* F$ the tensor product connection. Then the pth Chern spark is given by the current pushforward (or fibre integral)

$$S_p(\alpha) \;:=\; \pi_*(\sigma(\widehat{\alpha})) \;=\; \int_{\mathbf{P}(\mathbf{C}^k)} \sigma(\widehat{\alpha}) \qquad \text{on } X, \tag{5.3}$$

of the top Chern spark $\sigma(\widehat{\alpha})$. By [HL3,4.4] and [HL2,III.7.25], we have

$$\pi_* c_n(\Omega_{\mathrm{Hom}(U,\pi^* E)}) = c_p(\Omega_E). \tag{5.4}$$

Equation (5.1) follows immediately from equation (2.2) for the induced section $\widehat{\alpha}$.

The linear dependency current and pth Chern spark inherit most of the properties of the divisor and top Chern spark. In particular the following comparison formulae are immediate consequences of the comparison formulae (2.3) and (2.4). Let α_1 and α_2 be two $\mathbb{LD}$-atomic collections of sections of $F \to X$. Then there is a locally rectifiable current R and an L^1_{loc} current L so that

$$\mathbb{LD}(\alpha_1) \;-\; \mathbb{LD}(\alpha_2) \;=\; -dR, \tag{5.5}$$

$$S_p(\alpha_1) \;-\; S_p(\alpha_2) \;=\; R \;+ dL. \tag{5.6}$$

These comparison formulae allow us to make the following definition.

Definition 5.7. The **secondary pth Chern class** of the vector bundle F with connection D_F is the equivalence class

$$\widetilde{c}_p(D_F) \;:=\; [S_p(\alpha)] \in \widetilde{H}^*(X),$$

of the pth Chern spark of any $\mathbb{LD}$-atomic collection α of $k = n - p + 1$ sections of F.

The intersection of linear dependency currents and the product of Chern sparks are defined as follows. Let E and F be vector bundles of ranks m and n with unitary connections, and let α and β be $\mathbb{LD}$-atomic collections of k sections of E and l sections of F respectively. Let $U_E \to \mathbf{P}(\underline{\mathbf{C}}^k)$ and $U_F \to \mathbf{P}(\underline{\mathbf{C}}^l)$ denote the tautological line bundles. Suppose that the linear dependency currents of α and β intersect properly in that the divisors of the pullbacks of $\widehat{\alpha}$ and $\widehat{\beta}$ to the fibre product $\mathbf{P}(\underline{\mathbf{C}}^k) \times_X \mathbf{P}(\underline{\mathbf{C}}^l)$ intersect properly, i.e., the induced section $(\widehat{\alpha}, \widehat{\beta})$ of the bundle $\mathrm{Hom}(U_E, E) \oplus \mathrm{Hom}(U_F, F) \to \mathbf{P}(\underline{\mathbf{C}}^k) \times_X \mathbf{P}(\underline{\mathbf{C}}^l)$ is atomic. Let $\pi : \mathbf{P}(\underline{\mathbf{C}}^k) \times_X \mathbf{P}(\underline{\mathbf{C}}^l) \to X$ be the natural projection. Then the intersection of the linear dependency currents of α and β is defined by

$$\mathbb{LD}(\alpha) \wedge \mathbb{LD}(\beta) \ := \ \pi_*(\mathrm{Div}(\widehat{\alpha}) \wedge \mathrm{Div}(\widehat{\beta})) \qquad \text{on } X. \tag{5.8}$$

This current is supported on the intersection of the linear dependency sets of the collections α and β.

Definition 5.9. Let $\alpha = (\alpha^1, ..., \alpha^k)$ and $\beta = (\beta^1, ..., \beta^l)$ be $\mathbb{LD}$-atomic collections of sections of E and F respectively, and let $p = m - k + 1$ and $q = n - l + 1$. Suppose that the linear dependency currents of α and β intersect properly. Then the **spark product**, $S_p(\alpha) * S_q(\beta)$, of the Chern sparks of α and β is the L^1_{loc} form of degree $2(p+q)-1$ on X defined by

$$S_p(\alpha) * S_q(\beta) \ := \ \pi_*(\sigma(\widehat{\alpha}) * \sigma(\widehat{\beta})), \tag{5.10}$$

where $\pi : \mathbf{P}(\underline{\mathbf{C}}^k) \times_X \mathbf{P}(\underline{\mathbf{C}}^l) \to X$ denotes projection.

Note that the singular support of the L^1_{loc} form $S_p(\alpha) * S_q(\beta)$ is contained in the intersection of the linear dependency sets of the collections α and β.

The product of Chern sparks is related to the usual product of Chern forms and intersection of linear dependency currents by the equation

$$d(S_p(\alpha) * S_q(\beta)) \ = \ c_p(\Omega_E) \wedge c_q(\Omega_F) \ - \ \mathbb{LD}(\alpha) \wedge \mathbb{LD}(\beta) \qquad \text{on } X. \tag{5.11}$$

Furthermore the comparison formulae

$$\mathbb{LD}(\alpha_1) \wedge \mathbb{LD}(\beta_1) \ - \ \mathbb{LD}(\alpha_2) \wedge \mathbb{LD}(\beta_2) \ = \ -dR \tag{5.12}$$

$$S_p(\alpha_1) * S_q(\beta_1) \ - \ S_p(\alpha_2) * S_q(\beta_2) \ = \ R \ + \ dL \tag{5.13}$$

hold.

The comparison formula (5.13) ensures that the product $S_p(\alpha) * S_q(\beta)$ of Chern sparks induces a well defined product, $\widetilde{c}_p(D_E) * \widetilde{c}_q(D_F)$, of secondary Chern classes by the formula

$$\widetilde{c}_p(D_E) * \widetilde{c}_q(D_F) \ := \ [S_p(\alpha) * S_q(\beta)] \in \widetilde{H}^*(X).$$

Multiple intersections of linear dependency currents and products of Chern sparks can also be defined. Furthermore, as in the case of the divisor and top Chern spark of a section, these intersections and products are associative and commutative (up to sign).

6. The Product Theorem for the Total Chern Spark.

In this section we present the product theorem for the total Chern spark. This result is a generalization of the classical product formula $c(\Omega_{E\oplus F}) = c(\Omega_E)\wedge c(\Omega_F)$ for the total Chern form $c = 1 + c_1 + \cdots + c_n$, where $c(\Omega_{E\oplus F})$ denotes the total Chern form of the direct sum connection $D_{E\oplus F} := D_E \oplus D_F$ on $E\oplus F$. We begin with a definition.

Definition 6.1. Let $E \to X$ be a rank n complex vector bundle and $\alpha = (\alpha^1, ..., \alpha^N)$ a collection let of N sections of E where $N \geq n$.
(1) α is called $\mathbb{TLD}$**-atomic** if $(\alpha^1, ..., \alpha^k)$ is $\mathbb{LD}$-atomic for $k = 1, ..., n+1$.
(2) The **total linear dependency current**, $\mathbb{TLD}(\alpha)$, of a $\mathbb{TLD}$-atomic collection α is defined by

$$\mathbb{TLD}(\alpha) \ := \ 1 + \mathbb{LD}(\alpha^1, ..., \alpha^n) + \mathbb{LD}((\alpha^1, ..., \alpha^{n-1}) + \cdots + \mathbb{LD}(\alpha^1, \alpha^2) + \mathrm{Div}(\alpha^1).$$

(3) Similarly, the **total Chern spark**, $TS(\alpha)$, of α is defined by

$$TS(\alpha) \ = \ 1 + S_1(\alpha^1, ..., \alpha^n) + S_2(\alpha^1, ..., \alpha^{n-1}) + \cdots + S_{n-1}(\alpha^1, \alpha^2) + S_n(\alpha^1),$$

and the total secondary Chern class, $\widetilde{c}(D_E) \in \widetilde{L}(X)$, by $\widetilde{c}(D_E) := 1 + \widetilde{c}_1(D_E) + \cdots + \widetilde{c}_n(D_E)$.

Note. Note that $\mathbb{LD}(\alpha^1, ..., \alpha^{n+1}) = 1$, by [HL3, 2.9-10], and that $\mathbb{LD}(\alpha^1, ..., \alpha^k) := 0$ for $k > n+1$.

Let α and β be $\mathbb{TLD}$-atomic collections of $n+m+1$ sections of E and F respectively, and suppose that the collection (α, β) of $n+m+1$ sections of $E\oplus F \to X$ is also $\mathbb{TLD}$-atomic. Then, by equation (5.1) ,

$$dTS(\alpha,\beta) \ = \ c(\Omega_{E\oplus F}) \ - \ \mathbb{TLD}(\alpha,\beta) \qquad \text{on } X. \tag{6.2}$$

Suppose also that the **total linear dependency currents** $\mathbb{TLD}(\alpha)$ and $\mathbb{TLD}(\beta)$ **intersect properly** in that the linear dependency currents $\mathbb{LD}(\alpha^1, ..., \alpha^p)$ and $\mathbb{LD}(\beta^1, ..., \beta^q)$ intersect properly for all p, q with $1 \leq p \leq n+1$ and $1 \leq q \leq m+1$. Summing equation (5.11) over all possible p, q gives the fundamental current equation

$$d(TS(\alpha) * TS(\beta)) \ = \ c(\Omega_E)\wedge c(\Omega_F) \ - \ \mathbb{TLD}(\alpha)\wedge \mathbb{TLD}(\beta) \qquad \text{on } X. \tag{6.3}$$

The main result of this section is the following product theorem for total Chern sparks and total linear dependency currents which relates the currents in equations (6.2) and (6.3).

Product Theorem 6.4. *Let α and β be $\mathbb{TLD}$-atomic collections of $n+m+1$ sections of E and F. Suppose that the collection (α,β) of sections of $E\oplus F$ is also $\mathbb{TLD}$-atomic and that the total linear dependency currents of α and β intersect properly.*

Then there is a locally rectifiable current R and a locally integrable current L on X so that

$$\mathbb{TLD}(\alpha,\beta) \;=\; \mathbb{TLD}(\alpha)\wedge\mathbb{TLD}(\beta) \;-\; dR, \tag{6.5}$$

and

$$TS(\alpha,\beta) \;=\; TS(\alpha)*TS(\beta) \;+\; R \;+\; dL. \tag{6.6}$$

Consequently

$$[\mathbb{TLD}(\alpha,\beta)] \;=\; [\mathbb{TLD}(\alpha)]\cdot[\mathbb{TLD}(\beta)] \qquad \text{in } H^*(X,\mathbf{Z}), \tag{6.7}$$

and

$$\widetilde{c}(D_{E\oplus F}) \;=\; \widetilde{c}(D_E)*\widetilde{c}(D_F) \qquad \text{in } \widetilde{H}^*(X). \tag{6.8}$$

Note. An explicit formula for the locally rectifiable current R is given in equation (6.22) below.

Note. Equation (6.5) implies that, for any k with $1\le k\le n+m+1$,

$$\mathbb{LD}((\alpha^1,\beta^1),...,(\alpha^k,\beta^k)) \;=\; \sum_{i=\max(1,k-m)}^{\min(k,n+1)} \mathbb{LD}(\alpha^1,...,\alpha^i)\wedge\mathbb{LD}(\beta^1,...,\beta^{k+1-i}) \;+\; dR_k,$$

where R_k is locally rectifiable. Similarly equation (6.6) implies that for any p with $1\le p\le n+m$,

$$\begin{aligned} &S_p((\alpha^1,\beta^1),...,(\alpha^{n+m-p+1},\beta^{n+m-p+1}))\\ &\quad=\sum_{j=\max(0,p-m)}^{\min(p,n)} S_j(\alpha^1,...,\alpha^{n-j+1})*S_{p-j}(\beta^1,...,\beta^{m-p+j+1}) \;+\; R_{2p-1}+dL_{2p-2}\end{aligned}$$

where R_{2p-1} is locally rectifiable and L_{2p-2} is L^1_{loc}.

Outline of the Proof of the Product Theorem. First notice that it is sufficient to prove the product formula (6.6) for the total Chern spark. This is because the product formula (6.5) for the total linear dependency current can be proved by taking the exterior derivative of the spark product formula (6.6) and then applying equations (6.2,6.3) and the product formula for the total primary Chern form. However the direct proof of equation (6.5) given below is also instructive, partly because it motivates the proof of equation (6.6).

The proof of the product formula (6.6) for the total Chern spark has three main steps, which can be outlined as follows.

The first step is to observe that the total Chern spark $TS(\alpha,\beta)$ of the collection (α,β) of $n+m+1$ sections of the bundle $E\oplus F$ is equal to the pushforward to the base X of a *single* current, denoted by S_1, on the product manifold $\mathbf{P}^{n+m}\times X$, where $\mathbf{P}^{n+m} := \mathbf{P}(\underline{\mathbf{C}}^{n+m+1})$. Similarly the product $TS(\alpha) * TS(\beta)$ of the total Chern sparks of α and β is equal to the pushforward to the base X of a *single* current S_2 on the product manifold $\mathbf{P}^{n+m}\times\mathbf{P}^{n+m}\times X$. The currents S_1 and S_2 are defined in equations (6.12) and (6.15) below.

Let $\Delta : \mathbf{P}^{n+m} \hookrightarrow \mathbf{P}^{n+m}\times\mathbf{P}^{n+m}$ denote the **diagonal embedding**. The second step of the proof is to derive a current equation which relates the current S_2 on $\mathbf{P}^{n+m}\times\mathbf{P}^{n+m}\times X$ to the pushforward $\Delta_*(S_1)$. This current equation (see (6.21) below) is an application of the approach to Morse theory developed in [HL4].

In the final step of the proof the spark product formula (6.6) on X is obtained by pushing the current equation of step two on $\mathbf{P}^{n+m}\times\mathbf{P}^{n+m}\times X$ down to the base X by the natural projection map.

Remark 6.9. In the proof of the product theorem it is necessary to multiply currents by divisors and by top Chern sparks of atomic sections. These products are defined by a smoothing process which will be described in Remark 10.13 of the Appendix.

Step One of the proof of the Product Theorem. We show that the total linear dependency current and total Chern spark of a collection γ of $N+1$ sections of a vector bundle with connection can be computed by pushing forward a *single* current on $\mathbf{P}(\mathbf{C}^{N+1})\times X$ to the base X. For this we define a flag current $\mathbf{F}$ on $\mathbf{P}(\mathbf{C}^{N+1})\times X$ as follows. Consider the flag $\mathbf{C}\subset\mathbf{C}^2\subset ...\subset\mathbf{C}^{N+1}$, where the inclusion $\mathbf{C}^k\subset\mathbf{C}^{k+1}$ is given by $z\mapsto(z,0)$, and its induced flag of projective spaces $\mathbf{P}^0\subset\mathbf{P}^1\subset ...\subset\mathbf{P}^N$, where $\mathbf{P}^k := \mathbf{P}(\mathbf{C}^{k+1})$. Then the **flag current**, $\mathbf{F}$, is defined by

$$\mathbf{F} \;:=\; [\mathbf{P}^N\times X] + \cdots + [\mathbf{P}^1\times X] + [\mathbf{P}^0\times X] \qquad \text{on } \mathbf{P}^N\times X. \tag{6.10}$$

Proposition 6.11. *Let γ be a $\mathbb{TLD}$-atomic collection of $N+1$ sections of a complex vector bundle $V\to X$ with connection, and let $\widehat{\gamma}$ denote the induced section of $\mathrm{Hom}(U,V)\to\mathbf{P}^N\times X$, where U is the tautological bundle over $\mathbf{P}^N$. Then the total linear dependency current and total Chern spark of the collection γ are given by the formulae*

$$\mathbb{TLD}(\gamma) \;=\; \pi_*(\mathbf{F}\wedge\mathrm{Div}(\widehat{\gamma})) \qquad \textit{and} \qquad TS(\gamma) \;=\; \pi_*(\mathbf{F}\wedge\sigma(\widehat{\gamma})),$$

where $\pi : \mathbf{P}^N\times X\to X$ denotes projection.

Note. The assumption that γ is $\mathbb{TLD}$-atomic ensures that the products $\mathbf{F} \wedge \mathrm{Div}(\widehat{\gamma})$ and $\mathbf{F} \wedge \sigma(\widehat{\gamma})$ are defined.

Proof. The proposition follows immediately from the fact that the tautological line bundle U and the section $\widehat{\gamma}$ over $\mathbf{P}^N \times X$ have the appropriate restrictions to each of the subprojective spaces $\mathbf{P}^k \times X$. $\square$

Let α and β be collections of $n+m+1$ sections of E and F which satisfy the hypotheses of the Product Theorem. By Proposition 6.11 the total linear dependency current and Chern spark of the collection (α, β) of sections of $E \oplus F$ are given by

$$(6.12) \qquad \mathbb{TLD}(\alpha,\beta) \ = \ \pi_*(\mathbf{F} \wedge \mathrm{Div}\,\widehat{(\alpha,\beta)}) \qquad \text{and} \qquad TS(\alpha,\beta) \ = \ \pi_*(\mathbf{F} \wedge \sigma\widehat{(\alpha,\beta)}).$$

where $\mathbf{F}$ is the flag current (6.10) on $\mathbf{P}^{n+m} \times X$ and $\pi : \mathbf{P}^{n+m} \times X \to X$ is projection.

Similarly, the intersection of the total linear dependency currents of α and β and the product of their total Chern sparks can be calculated using a product of flags in $\mathbf{P}^{n+m}_E \times \mathbf{P}^{n+m}_F \times X$. Here and below the subscripts E, F serve to identify the different roles of the two factors $\mathbf{P}^{n+m}$. More precisely, let $\mathbf{F}_E$ and $\mathbf{F}_F$ be the flag currents in $\mathbf{P}^{n+m}_E$ and $\mathbf{P}^{n+m}_F$ given by equation (6.10) and define

$$(6.13) \qquad \mathbf{G} \ := \ \sum_{k=0}^{n+m} \mathbf{G}_k \qquad \text{in } \mathbf{P}^{n+m}_E \times \mathbf{P}^{n+m}_F \times X,$$

where $\mathbf{G}_k$ is the complex fibre-dimension k part of the fibre product $\mathbf{F}_E \times_X \mathbf{F}_F$. Let

$$\pi_{EF} : \mathbf{P}^{n+m}_E \times \mathbf{P}^{n+m}_F \times X \to X \qquad \text{and} \qquad p_E : \mathbf{P}^{n+m}_E \times \mathbf{P}^{n+m}_F \times X \to \mathbf{P}^{n+m}_E \times X$$

denote the natural projections. Since the total linear dependency currents of α and β intersect properly the products $\mathbf{G} \wedge \mathrm{Div}(p_E^*\widehat{\alpha}) \wedge \mathrm{Div}(p_F^*\widehat{\beta})$ and $\mathbf{G} \wedge (\sigma(p_E^*\widehat{\alpha}) * \sigma(p_F^*\widehat{\beta}))$ exist. Then equations (5.8) and (5.10) imply that

$$(6.14) \qquad \mathbb{TLD}(\alpha) \wedge \mathbb{TLD}(\beta) \ = \ (\pi_{EF})_*\{\mathbf{G} \wedge \mathrm{Div}(p_E^*\widehat{\alpha}) \wedge \mathrm{Div}(p_F^*\widehat{\beta})\},$$

and

$$(6.15) \qquad TS(\alpha) * TS(\beta) \ = \ (\pi_{EF})_*\{\mathbf{G} \wedge (\sigma(p_E^*\widehat{\alpha}) * \sigma(p_F^*\widehat{\beta}))\}.$$

¿From here on we will write $\widehat{\alpha}$ and $\widehat{\beta}$ for $p_E^*\widehat{\alpha}$ and $p_F^*\widehat{\beta}$.

Step Two in the proof of the Product Theorem. The second step of the proof is to relate the currents on the right hand sides of equations (6.12) and (6.14-6.15). This is achieved by using the **diagonal embedding** $\Delta : \mathbf{P}^{n+m} \hookrightarrow \mathbf{P}_E^{n+m} \times \mathbf{P}_F^{n+m}$ to relate the flag current $\mathbf{F}$ in $\mathbf{P}^{n+m} \times X$ to the product of flag currents $\mathbf{G}$ in $\mathbf{P}_E^{n+m} \times \mathbf{P}_F^{n+m} \times X$.

The following proposition is an application of the approach to Morse Theory developed in [HL4]. Its proof is given in the Appendix. Recall that a current $\mathbf{I}$ is called *locally integral* if $\mathbf{I}$ and $d\mathbf{I}$ are locally rectifiable.

Proposition 6.16. *Let $\mathbf{F}$ and $\mathbf{G}$ be the flag currents defined above on $\mathbf{P}^{n+m} \times X$ and $\mathbf{P}_E^{n+m} \times \mathbf{P}_F^{n+m} \times X$. Then there is a canonically defined locally integral current $\mathbf{I}$ of odd degree so that*

$$d\mathbf{I} \;=\; \Delta_*(\mathbf{F}) \;-\; \mathbf{G} \qquad \text{on } \mathbf{P}_E^{n+m} \times \mathbf{P}_F^{n+m} \times X, \tag{6.17}$$

where $\Delta : \mathbf{P}^{n+m} \times X \hookrightarrow \mathbf{P}_E^{n+m} \times \mathbf{P}_F^{n+m} \times X$ is the diagonal embedding.

Let α, β be collections of sections which satisfy the hypotheses of the Product Theorem. In Lemma 10.19 of the Appendix we prove that the product of the diagonal flag current $\Delta_*(\mathbf{F})$ with the divisor $\mathrm{Div}\,(\widehat{\alpha}, \widehat{\beta})$ and with the top Chern spark $\sigma(\widehat{\alpha}, \widehat{\beta})$ exist and equal

$$\Delta_*(\mathbf{F}) \wedge \mathrm{Div}\,(\widehat{\alpha}, \widehat{\beta}) \;=\; \Delta_*(\mathbf{F} \wedge \mathrm{Div}\,\widehat{(\alpha, \beta)}) \tag{6.18}$$

$$\Delta_*(\mathbf{F}) \wedge \sigma(\widehat{\alpha}, \widehat{\beta}) \;=\; \Delta_*(\mathbf{F} \wedge \sigma\widehat{(\alpha, \beta)}) \tag{6.19}$$

on $\mathbf{P}_E^{n+m} \times \mathbf{P}_F^{n+m} \times X$.

Next suppose that the intersection, $\mathbf{I} \wedge \mathrm{Div}(\widehat{\alpha}, \widehat{\beta})$ exists. Then, by Proposition 6.16 one would expect that

$$d(\mathbf{I} \wedge \mathrm{Div}(\widehat{\alpha}, \widehat{\beta})) \;=\; \Delta_*(\mathbf{F}) \wedge \mathrm{Div}(\widehat{\alpha}, \widehat{\beta}) \;-\; \mathbf{G} \wedge \mathrm{Div}(\widehat{\alpha}, \widehat{\beta}) \tag{6.20}$$

holds on $\mathbf{P}_E^{n+m} \times \mathbf{P}_F^{n+m} \times X$.

Furthermore, suppose that the product $\mathbf{I} \wedge \sigma(\widehat{\alpha}, \widehat{\beta})$ of $\mathbf{I}$ with the top Chern spark $\sigma(\widehat{\alpha}, \widehat{\beta})$ exists and is locally flat. Let $\sigma = \sigma(\widehat{\alpha}, \widehat{\beta})$, $D = \mathrm{Div}(\widehat{\alpha}, \widehat{\beta})$, $c_n = p_E^* c_n(\Omega_{\mathrm{Hom}(U_E, E)})$ and $c_m = p_F^* c_m(\Omega_{\mathrm{Hom}(U_F, F)})$. Applying Proposition 6.16 and the local Gauss-Bonnet-Chern Theorem (2.2) one would expect that there is an L^1_{loc} current K so that

$$dK \;=\; \Delta_*(\mathbf{F}) \wedge \sigma \;-\; \mathbf{G} \wedge \sigma \;-\; \mathbf{I} \wedge c_n \wedge c_m \;+\; \mathbf{I} \wedge D \tag{6.21}$$

on $\mathbf{P}_E^{n+m} \times \mathbf{P}_F^{n+m} \times X$.

In Lemma 10.26 of the Appendix we will prove that for "almost all" pairs α, β of collections of sections, equations (6.20) and (6.21) hold with the potential $\mathbf{I} \wedge \mathrm{Div}(\widehat{\alpha}, \widehat{\beta})$ being locally rectifiable and the potential K being L^1_{loc}.

Step Three in the proof of the Product Theorem. The final step of the proof is to pushforward the current equations (6.20) and (6.21) on $\mathbf{P}_E^{n+m} \times \mathbf{P}_F^{n+m} \times X$ down to the base X.

Proof of Equation (6.5). Suppose that the product $\mathbf{I} \wedge \mathrm{Div}\,(\widehat{\alpha}, \widehat{\beta})$ exists, that it is locally rectifiable and that equation (6.20) holds. Then

$$\begin{aligned}
\mathbb{TLD}(\alpha,\beta) &= \pi_*(\mathbf{F} \wedge \mathrm{Div}\,\widehat{(\alpha,\beta)}) \qquad \text{by (6.12)},\\
&= (\pi_{EF})_*\,(\Delta_*(\mathbf{F}) \wedge \mathrm{Div}(\widehat{\alpha},\widehat{\beta})) \qquad \text{as } \pi = \pi_{EF} \circ \Delta \text{ and by (6.18)},\\
&= (\pi_{EF})_*(\mathbf{G} \wedge \mathrm{Div}(\widehat{\alpha},\widehat{\beta})) \;+\; (\pi_{EF})_* d(\mathbf{I} \wedge \mathrm{Div}(\widehat{\alpha},\widehat{\beta})) \qquad \text{by (6.20)},\\
&= \mathbb{TLD}(\alpha) \wedge \mathbb{TLD}(\beta) \;-\; dR \qquad \text{by (6.14)},
\end{aligned}$$

where

$$R := \,-\,(\pi_{EF})_*(\mathbf{I} \wedge \mathrm{Div}(\widehat{\alpha},\widehat{\beta})) \tag{6.22}$$

is locally rectifiable.

Consequently, by Lemma 10.26 of the Appendix, equation (6.5) holds for "almost all" pairs of collections α, β. To establish equation (6.5) for *any* pair α, β of collections of sections which satisfy the hypotheses of the Product Theorem we simply choose a pair μ, ν for which equation (6.5) does hold, apply the comparison formulae (5.5) and (5.12) for linear dependency currents, and conclude that equation (6.5) also holds for the pair of collections α, β, (though not with the canonical R of equation (6.22)). $\square$

Proof of Equation (6.6). By the comparison formulae (5.6) and (5.13) for Chern sparks and by the preceding discussion we may assume that the current $\mathbf{I} \wedge \mathrm{Div}\,(\widehat{\alpha}, \widehat{\beta})$ is defined and locally rectifiable and that there is an L^1_{loc} current K so that equation 6.21 holds. Then, by equations (6.12) and (6.19),

$$TS(\alpha,\beta) \;=\; \pi_*(\mathbf{F} \wedge \sigma(\widehat{\alpha},\widehat{\beta})) \;=\; (\pi_{EF})_*(\Delta_*(\mathbf{F}) \wedge \sigma),$$

where here and below we employ the notation $\sigma = \sigma(\widehat{\alpha},\widehat{\beta})$, etc, of Step Two of the proof. Therefore, by equations (6.21) and (6.15),

$$\begin{aligned}
TS(\alpha,\beta) &= (\pi_{EF})_*\,(\mathbf{G} \wedge \sigma - \mathbf{I} \wedge D + \mathbf{I} \wedge c_n \wedge c_m + dK)\\
&= TS(\alpha) * TS(\beta) \;+ R + dL,
\end{aligned}$$

where R is the locally rectifiable current defined by equation (6.22) and where $L = (\pi_{EF})_*(K)$ is L^1_{loc}.

The second equality above follows from the fact that $(\pi_{EF})_*(\mathbf{I} \wedge c_n \wedge c_m) = 0$. This fibre integral is zero since $\mathbf{I}$ is a current of *odd* degree in the fibre variables and, by Theorem A.22 of [HL3], $c_n(\Omega_{\mathrm{Hom}(U_E,E)}) = \left[c(\Omega_E) \wedge c(\Omega_U)^{-1}\right]_n$, and hence $c_n \wedge c_m$, are of *even* degree in the fibre variables. This completes the proof of equation (6.6). $\square$

7. The Product Theorem for the Total Ronga-Pontrjagin Spark.

In this section we derive product formulae for the total Ronga-Pontrjagin spark and the total 2-dependency current of a collection of sections of a real vector bundle with metric compatible connection. Because it has additional terms which are torsion of order two, the product formula for the total 2-dependency current of a collection of sections of a real bundle is more complicated than the product formula for total linear dependency currents in the case of a complex bundle. These torsion terms appear in the product formula because the product formula for the primary total integer Pontrjagin class $p \in H^*(X, \mathbf{Z})$, is only valid modulo two (see [M], [T]),

$$p(E \oplus F) \;=\; p(E)\, p(F) \;+\; \text{elements of order two} \qquad \text{in } H^*(X, \mathbf{Z}),$$

and also because the integer cohomology class of a 2-dependency current differs from the associated Pontrjagin class by a 2-torsion term, (see [R]).

The 2-dependency current and Ronga-Pontrjagin spark were introduced in [HL3] and further studied in [HZ1]. We begin with a brief summary. Let $E \to X$ be a real rank n vector bundle and let $\alpha \;=\; (\alpha^1, ..., \alpha^k)$ be a collection of k sections of E, where $k \equiv n$ mod 2 and $k \leq n+2$. Define q by $k = n+2-2q$. The 2-dependency current, $\mathbb{LD}_2(\alpha)$, of α is a current of degree $4q$ which is supported on the subset of X where the rank of the bundle map $\alpha : \underline{\mathbf{R}}^k \to E$ is less than or equal to $k-2$.

The current $\mathbb{LD}_2(\alpha)$ is a locally integrally flat d-closed current and so represents an integer cohomology class $[\mathbb{LD}_2(\alpha)] \in H^{4q}(X, \mathbf{Z})$. The comparison formula (5.5) implies that $[\mathbb{LD}_2(\alpha)]$ depends only on the vector bundle $E \to X$ and not on α. Ronga [R] showed that

$$[\mathbb{LD}_2(\alpha)] \;=\; p_q(E) \;+\; W_{2q-1}(E)\, W_{2q+1}(E) \in H^{4q}(X, \mathbf{Z}), \tag{7.1}$$

where p_q denotes the qth integral Pontrjagin class and W_{2j+1} is the $(2j+1)$st integer Stiefel-Whitney class defined by $W_{2j+1} := \delta_*(w_{2j})$ where w_{2j} is the $2j$th mod 2 Stiefel-Whitney class and δ_* is the Bockstein coboundary associated with the exact sequence $0 \to \mathbf{Z} \xrightarrow{2} \mathbf{Z} \to \mathbf{Z}_2 \to 0$ (c.f. [Bo],[W]). In particular note that W_{2j+1} is a 2-torsion class. The integer class $p_q + W_{2q-1} W_{2q+1}$ of the 2-dependency current will be referred to as the qth **Ronga-Pontrjagin class**.

Let D_E be a metric connection on E and let $p_q(\Omega_E)$ denote the qth Pontrjagin form of D_E. Then there is the fundamental current equation

$$dS_q(\alpha) \;=\; p_q(\Omega_E) \;-\; \mathbb{LD}_2(\alpha) \qquad \text{on } X, \tag{7.2}$$

where the qth Ronga-Pontrjagin spark, $S_q(\alpha)$, of α is a canonically defined L^1_{loc} form.

The 2-dependency current and Ronga-Pontrjagin spark are defined as follows. Let $\pi : G_2(\underline{\mathbf{R}}^k) \to X$ be the trivial Grassmann bundle of *oriented* two dimensional subspaces of the trivial bundle $\underline{\mathbf{R}}^k$ and let $U \to G_2(\underline{\mathbf{R}}^k)$ be the tautological rank 2 bundle. Note that the fibre $G_2(\mathbf{R}^k)$ and the tautological bundle U are both canonically oriented. Suppose that the collection α is $\mathbb{LD}_2$**-atomic** in that the induced section $\widehat{\alpha}$ of $\mathrm{Hom}(U, \pi^*E) \to G_2(\underline{\mathbf{R}}^k)$ is atomic. Let $\chi(\Omega_{\mathrm{Hom}(U,\pi^*E)})$ be the Chern-Euler form of the induced metric connection on the even rank bundle $\mathrm{Hom}(U, \pi^*E)$. Arguing as in the proof of [HL3, 5.2] and using the fact that $\underline{\mathbf{R}}^k$ is trivial it follows that

$$p_q(\Omega_E) \;=\; \tfrac{1}{2}\pi_*(\chi(\Omega_{\mathrm{Hom}(U,\pi^*E)})).$$

The Euler spark $\sigma(\widehat{\alpha}) := \widehat{\alpha}^*(\sigma)$ of the section $\widehat{\alpha}$ is the L^1_{loc} form on $G_2(\underline{\mathbf{R}}^k)$ obtained by pulling back the spherical Chern potential σ of the bundle $\mathrm{Hom}(U, \pi^*E)$ by the section $\widehat{\alpha}$, (see [Ch1], [HL2, IV.1.35]). Then the **2-dependency current** and qth **Ronga-Pontrjagin spark** of α are defined by

$$(7.3) \qquad \mathbb{LD}_2(\alpha) \;:=\; \tfrac{1}{2}\pi_*(\mathrm{Div}(\widehat{\alpha})) \qquad \text{and} \qquad S_q(\alpha) \;:=\; \tfrac{1}{2}\pi_*(\sigma(\widehat{\alpha})) \qquad \text{on } X.$$

Definition 7.4. The q**th secondary Ronga-Pontrjagin class** $\widetilde{p}_q \in \widetilde{H}^*(X)$ of a vector bundle E with metric connection D_E is defined to be the equivalence class of the qth Ronga-Pontrjagin spark, $S_q(\alpha)$, of any $\mathbb{LD}_2$ atomic collection α of $k = n+2-2q$ sections of $E \to X$.

Products of Ronga-Pontrjagin sparks and of secondary Ronga-Pontrjagin classes are defined as in the complex case above.

The total 2-dependency current and total Ronga-Pontrjagin spark are defined as follows. Let $n = \mathrm{Rk}\, E$ and define n_0 by $n = 2n_0$ if n is even and $n = 2n_0 + 1$ if n is odd. Recall that the total Pontrjagin form of a connection D_E on E is given by $p(\Omega_E) := 1 + p_1(\Omega_E) + \cdots + p_{n_0}(\Omega_E)$. Similarly the **total 2-dependency current** and **total Ronga-Pontrjagin spark** of a $\mathbb{TLD}_2$-atomic collection $\alpha = (\alpha^1, ..., \alpha^N)$ of $N \geq n$ sections of E are defined by

$$\mathbb{TLD}_2(\alpha) \;:=\; 1+\mathbb{LD}_2(\alpha^1, ..., \alpha^n)+\mathbb{LD}_2(\alpha^1, .., \alpha^{n-2})+\cdots+\begin{cases} \mathbb{LD}_2(\alpha^1, \alpha^2) & \text{if } n \text{ is even} \\ \mathbb{LD}_2(\alpha^1, \alpha^2, \alpha^3) & \text{if } n \text{ is odd,} \end{cases}$$

and

$$TS(\alpha) \;:=\; 1 + S_1(\alpha^1, ..., \alpha^n) + S_2(\alpha^1, .., \alpha^{n-2}) + \cdots + \begin{cases} S_{n_0}(\alpha^1, \alpha^2) & \text{if } n \text{ is even} \\ S_{n_0}(\alpha^1, \alpha^2, \alpha^3) & \text{if } n \text{is odd.} \end{cases}$$

Then equation (7.2) yields the fundamental current equation

$$dTS(\alpha) \;=\; p(\Omega_E) \;-\; \mathbb{TLD}_2(\alpha) \qquad \text{on } X.$$

Product Theorem 7.5. *Let E, F be real rank n, m vector bundles with metric connection over X and let α, β be $\mathbb{TLD}_2$-atomic collections of $n+m+2$ sections of E, F respectively. Suppose that the collection of sections (α, β) of $E \oplus F \to X$ is $\mathbb{TLD}_2$-atomic and that the total 2-dependency currents of α and β intersect properly. Then there is a locally integral current R and an L^1_{loc} current L on X so that*

$$\mathbb{TLD}_2(\alpha, \beta) \;=\; \mathbb{TLD}_2(\alpha) \wedge \mathbb{TLD}_2(\beta) \;-\; \tfrac{1}{2} dR, \tag{7.6}$$

and

$$TS(\alpha, \beta) \;=\; TS(\alpha) * TS(\beta) \;+ \tfrac{1}{2} R \;+\; dL. \tag{7.7}$$

In particular, the current $\frac{1}{2}dR$ is a d-closed locally rectifiable current whose cohomology class, $\mathrm{Tor}(E, F)$*, in $H^*(X, \mathbf{Z})$ is 2- torsion.*

Let $\mathbb{TLD}_2(E) := [\mathbb{TLD}_2(\alpha)] \in H^*(X, \mathbf{Z})$ denote the total Ronga-Pontrjagin class. Then, by (7.6),

$$\mathbb{TLD}_2(E \oplus F) \;=\; \mathbb{TLD}_2(E) \cdot \mathbb{TLD}_2(F) \;+\; \mathrm{Tor}(E, F) \qquad \text{in } H^*(X, \mathbf{Z}). \tag{7.8}$$

Consequently, the 2-torsion class $\mathrm{Tor}(E, F)$ *is independent of the choice of α, β. Furthermore, if $\widetilde{p}$ denotes the total secondary Ronga-Pontrjagin class, then by (7.7),*

$$2\widetilde{p}(D_{E \oplus F}) \;=\; 2\widetilde{p}(D_E) * \widetilde{p}(D_F) \qquad \text{in } \widetilde{H}^*(X). \tag{7.9}$$

Note. Recall that a current R is *locally integral* if R and dR are locally rectifiable. The locally integral current R in equation (7.6) is of the form $R = 2R_1 + R_2$ where R_1 and R_2 are locally integral. Therefore the 2-torsion class $\mathrm{Tor}(E, F)$ is given by $\mathrm{Tor}(E, F) = [\frac{1}{2} dR_2]$. Explicit formulae for the currents R_1 and R_2 are given in equation (7.13) below.

Remark. The 2-torsion class $\mathrm{Tor}(E, F)$. The class $\mathrm{Tor}(E, F)$ measures the extent to which the total 2-dependency class deviates from following the usual product rule for characteristic classes. In this remark we derive a formula for $\mathrm{Tor}(E, F)$ in terms of the integer Stiefel-Whitney and Pontrjagin classes. In particular this formula implies that the class $\mathrm{Tor}(E, F)$ is non zero.

Our formula relies on an analogous formula for the total integer Pontrjagin class which was found by Emory Thomas [T]. Thomas' formula states that the 2-torsion class $D(E, F) := p(E \oplus F) - p(E)p(F)$ is given by

$$D(E, F) \;=\; \sum_{q=1}^{n_0} \sum_{i+j=q-1} \delta_*\{w_{2i}(E)\, w_{2i+1}(E)\}\; \delta_*\{w_{2j}(F)\, w_{2j+1}(F)\},$$

where δ_* denotes the Bockstein coboundary, as above. Also, by equation (7.1),

$$\mathbb{TLD}_2(E) \;=\; p(E) \;+\; \mathrm{Tor}(E), \qquad \text{where } \mathrm{Tor}(E) \;:=\; \sum_{q=1}^{n_0} W_{2q-1}(E)\, W_{2q+1}(E).$$

Therefore, by equation (7.8), the required formula is
(7.10)
$$\mathrm{Tor}(E,F) \;=\; D(E,F) \;+\mathrm{Tor}(E\oplus F) \;-\; \mathrm{Tor}(E)p(F) \;-\; \mathrm{Tor}(F)p(E) \;-\; \mathrm{Tor}(E)\,\mathrm{Tor}(F).$$

A formula for the mod 2 reduction of $\mathrm{Tor}(E,F)$, which shows that $\mathrm{Tor}(E,F)$ is non zero, is easily obtained from (7.10).

Proof of Product Theorem 7.5. The proof of the Product Theorem is analogous to the one given in the complex case. The only major difference is that the statement and proof of Proposition 6.16 need to be modified in the following way. Let N be the smallest *even* integer greater than $n+m+1$. Consider the flag $\mathbf{R}\subset\mathbf{R}^2\subset\cdots\subset\mathbf{R}^N$, where $\mathbf{R}^k\subset\mathbf{R}^{k+1}$ by $x\mapsto(x,0)$, and its induced flag

$$G_2(\mathbf{R}^2)\;\subset\;G_2(\mathbf{R}^3)\;\subset\cdots\subset\;G_2(\mathbf{R}^N),$$

of Grassmannians of 2-planes. Recall that $G_2(\mathbf{R}^{2k})$ is orientable and that there are natural isomorphisms of orientation bundles $\mathcal{O}_{G_2(\mathbf{R}^{2k-1})}\cong\mathcal{O}_{G_2(\mathbf{R}^{2k+1})}$ for all k (cf. [HL3, A.8,10]).

Let V be one of the bundles E, F, $E\oplus F$. If $\mathrm{rank}(V)$ is even, we associate to V the **flag current**

$$\mathbf{F}_V\;=\;[G_2(\underline{\mathbf{R}}^N)]\;+\;[G_2(\underline{\mathbf{R}}^{N-2})]\;+\cdots+\;[G_2(\underline{\mathbf{R}}^\ell)]\;+\cdots+\;[G_2(\underline{\mathbf{R}}^2)].$$

on $G_2(\underline{\mathbf{R}}^N)$. If $\mathrm{rank}(V)$ is odd, we associate to V the **twisted flag current**

$$\mathbf{F}_V\;=\;[G_2(\underline{\mathbf{R}}^{N-1})]\;+\;[G_2(\underline{\mathbf{R}}^{N-3})]\;+\cdots+\;[G_2(\underline{\mathbf{R}}^\ell)]\;+\cdots+\;[G_2(\underline{\mathbf{R}}^3)].$$

on $G_2(\underline{\mathbf{R}}^N)$. Note that the top dimensional part of $\mathbf{F}_{E\oplus F}$ is $[G_2(\underline{\mathbf{R}}^{n+m+2})]$, which is of dimension $2(n+m)$. Consequently we introduce

$$\mathbf{G}\;:=\;\sum_{k=0}^{2(n+m)}\;\mathbf{G}_k\qquad\text{in } G_2(\underline{\mathbf{R}}^N)\times G_2(\underline{\mathbf{R}}^N),$$

where $\mathbf{G}_k$ is the dimension k part of $\mathbf{F}_E\times\mathbf{F}_F$.

Let $\Delta:G_2(\underline{\mathbf{R}}^N)\hookrightarrow G_2(\underline{\mathbf{R}}^N)\times G_2(\underline{\mathbf{R}}^N)$ denote the diagonal map. Note that, if the ranks of E and F have the same parity, then $\Delta_*(\mathbf{F}_{E\oplus F})$ and $\mathbf{G}$ are bona fide currents on $G_2(\underline{\mathbf{R}}^N)\times G_2(\underline{\mathbf{R}}^N)$. On the other hand, if the parities of the ranks of E and F are different, then $\Delta_*(\mathbf{F}_{E\oplus F})$ and $\mathbf{G}$ are $\mathcal{O}_{G_2(\underline{\mathbf{R}}^{N-1})}$-twisted currents on $G_2(\underline{\mathbf{R}}^N)\times G_2(\underline{\mathbf{R}}^N)$.

Proposition 7.11. *There are locally integral* $\mathbf{I}$ *and* $\mathbf{H}$ *of odd degree so that*

$$d\mathbf{I}\;=\;\Delta_*(\mathbf{F}_{E\oplus F})\;-\;\mathbf{G}\;-\;\tfrac{1}{2}d\mathbf{H}\tag{7.12}$$

as ($\mathcal{O}_{G_2(\underline{\mathbf{R}}^{N-1})}$-twisted) currents on $G_2(\mathbf{R}^N)\times G_2(\mathbf{R}^N)\times X$. In particular the current $\frac{1}{2}d\mathbf{H}$ is locally rectifiable and is 2-torsion.

The proof of Proposition 7.11 is given in the Appendix.

Irrespective of the parities of the ranks of E and F the products of $\mathbf{I}$, $\Delta_*(\mathbf{F}_{E\oplus F})$, $\mathbf{G}$ and $\mathbf{H}$ with $\mathrm{Div}(\widehat{\alpha},\widehat{\beta})$ and $\sigma(\widehat{\alpha},\widehat{\beta})$ are bona fide (untwisted) currents on the orientable manifold $G_2(\mathbf{R}^N)\times G_2(\mathbf{R}^N)\times X$. Therefore the proofs of equations (7.6) and (7.7) proceed exactly as in the complex case. This concludes the proof of the Product Theorem. $\square$

Remark. The locally integral current R in the Product Theorem is of the form $R = 2R_1 + R_2$ where

(7.13) $$R_1 \;=\; -(\pi_{EF})_*(\mathbf{I}\wedge \mathrm{Div}(\widehat{\alpha},\widehat{\beta})) \qquad \text{and} \qquad R_2 \;=\; (\pi_{EF})_*(\mathbf{H}\wedge \mathrm{Div}(\widehat{\alpha},\widehat{\beta}))$$

are locally integral currents. The current $\mathbf{I}$ is defined just as in the complex case, while an explicit formula for $\mathbf{H}$ can be derived from equation (10.9) below.

8. Sparks and Differential Characters.

This paper was motivated by a desire to understand the Cheeger-Simons theory of secondary characteristic classes from the point of view of characteristic currents. In their seminal papers [§], [CS], and [C], Cheeger and Simons introduced the ring of differential characters $\widehat{H}^*(X)$ on a smooth manifold X, and and then established a family of functorial invariants in $\widehat{H}^*(X)$ associated to vector bundles with connection on X. Their results strictly generalize the classical Chern-Weil homomorphism and have many applications in geometry and topology.

In this section we show that the sparks introduced in this paper are canonical representatives of the Cheeger-Simons classes. Furthermore, the product of two sparks is shown to represent the product of the classes in the ring $\widehat{H}^*(X)$. This together with the results of section 6give an alternate proof of Cheeger-Simons duality theorem: $\widehat{c}(E\oplus F) = \widehat{c}(E)\star\widehat{c}(F)$ for the total Chern differential character. Analogously, using results of section 7we obtain a duality theorem for the total Pontrjagin-Ronga differential character.

8A. Differential characters and the enhanced Chern-Weil Theorem..

Cheeger and Simons defined an $\mathbf{R}/\mathbf{Z}$-valued **differential character** of degree ℓ on X to be a homomorphism f from the space of smooth singular ℓ-dimensional cycles on X to $\mathbf{R}/\mathbf{Z}$, with the property that there is a smooth d-closed $(\ell+1)$-form ω on X so that, for any $(\ell+1)$-chain C,

(8.1) $$f(\partial C) \equiv \int_C \omega \quad \mathrm{mod}\,\mathbf{Z}.$$

The space of differential characters on X is denoted by $\widehat{H}^*(X,\mathbf{R}/\mathbf{Z})$.

Cheeger-Simons proved the fundamental uniqueness theorem for differential characters which states that the Weil homomorphism naturally factors through the space of differential characters. More precisely they proved the following. The ring $\widehat{H}^*(X;\mathbf{R}/\mathbf{Z})$ of differential characters is a functor on the category of topological spaces. There are two natural transformations:

$$\delta_1 : \widehat{H}^*(X;\mathbf{R}/\mathbf{Z}) \to H^*(X;\mathbf{R}) \qquad \text{and} \qquad \delta_2 : \widehat{H}^*(X;\mathbf{R}/\mathbf{Z}) \to H^*(X;\mathbf{Z})$$

with the property that the real reduction of $\delta_2 f$ is $\delta_1 f$. Let $\mathcal{B}$ denote the functor on the category of smooth manifolds which associates to X the set $\mathcal{B}(X)$ of equivalence classes of smooth complex vector bundles with unitary connection on X, and which associates to a smooth mapping $f : X \to Y$ the pull-back $\mathcal{B}(f) : \mathcal{B}(Y) \to \mathcal{B}(X)$. Let $\mathcal{B}_n(X) \subset \mathcal{B}(X)$ denote the subset of bundles of rank n.

Theorem 8.2. (Cheeger and Simons [CS]). *Consider the set $K^*(U_n, \mathbf{Z})$ of pairs (Φ, u) where $\Phi \in I^*_{U_n} \cong H^*(BU_n;\mathbf{R})$ is an Ad-invariant polynomial on the Lie algebra of U_n and where $u \in H^*(BU_n;\mathbf{Z})$ is a class with $u \otimes \mathbf{R} = [\Phi]$. For each such pair there exists a unique natural transformation of functors*

$$\mathcal{B}_n(X) \quad \xrightarrow{S_{\Phi,u}} \quad \widehat{H}^*(X;\mathbf{R}/\mathbf{Z})$$

such that

$$\delta_1 S(E, D_E) \;=\; \Phi(\Omega_E) \qquad \text{and} \qquad \delta_2 S(E, D_E) \;=\; u(E)$$

where $u(E)$ is the u characteristic class of E.

8B. Thom-Porteous characters. Consider now the functor $\mathcal{M}$ on the category of smooth manifolds which associates to X the set $\mathcal{M}(X)$ of isomorphism classes of smooth bundle maps $E \xrightarrow{\alpha} F$ between complex vector bundles with unitary connections on X, and which associates to a smooth mapping $f : X \to Y$ the pull-back $\mathcal{M}(f) : \mathcal{M}(Y) \to \mathcal{M}(X)$. Let $\mathcal{M}_k(X) \subset \mathcal{M}(X)$ denote the subset where α is k-atomic. To each element α of $\mathcal{M}_k(X)$ we have associated the kth degeneracy spark $S_k(\alpha)$.

Theorem 8.3. *There is a natural transformation of functors*

$$\mathcal{M}_k(X) \quad \xrightarrow{\widehat{TP}_k} \quad \widehat{H}^*(X;\mathbf{R}/\mathbf{Z})$$

which for $(E, F, \alpha) \in \mathcal{M}_k(X)$ is given on a cycle Z in X by

$$\widehat{TP}_k(\alpha)(Z) \;\equiv\; \int_Z S_k(\alpha') \quad \mod \mathbf{Z} \tag{8.10}$$

where $\alpha' : E \to F$ is any k-atomic map such that Z does not meet $\mathbb{D}_k(\alpha')$, and $S_k(\alpha')$ is the kth degeneracy spark of α'.

This transformation factors through the Cheeger-Simons transformation

$$\begin{array}{ccc} \mathcal{M}_k(X) & \xrightarrow{\phi} & \mathcal{B}(X) \\ \widehat{TP}_k \searrow & & \swarrow \\ & \widehat{H}^*(X;\mathbf{R}/\mathbf{Z}) & \end{array}$$

where $\phi(E, F, \alpha) = E^* \otimes F$ is the "forgetful" transformation which drops α.

Proof. Fix $(E, F, \alpha) \in \mathcal{M}_k(X)$ where $\operatorname{rank}(E) = m$ and $\operatorname{rank}(F) = n$. Let Z be a smooth singular cycle in X of dimension $\nu - 1$ where $\nu = 2(m-k)(n-k)$.

Lemma 8.5. *A generic bundle mapping $\alpha' : E \to F$ is k-atomic and satisfies the condition*

$$\operatorname{supp}(\mathbb{D}_k(\alpha')) \cap \operatorname{supp}(Z) \ = \ \emptyset \tag{8.6}$$

Proof. The first assertion is established in [HL3, §2]. The second is a straightforward application of the Thom Transversality Theorem. $\square$

Proposition 8.7. *Let $\alpha_1, \alpha_2 : E \to F$ be as in Lemma 8.5, and let Z be a smooth singular $(\nu - 1)$-cycle as above. Then*

$$\int_Z S_k(\alpha_1) \ \equiv \ \int_Z S_k(\alpha_2) \quad \mathrm{mod}\ \mathbf{Z} \tag{8.8}$$

where $S_k(\alpha')$ denotes the kth degeneracy spark of α'.

Furthermore, let C be a smooth singular ν-chain in X and set $Z = \partial C$. Then for any α' as in Lemma 8.5, we have

$$\int_{\partial C} S_k(\alpha') \ \equiv \ \int_C TP_k(\Omega_E, \Omega_F) \quad \mathrm{mod}\ \mathbf{Z} \tag{8.9}$$

where $TP_k(\Omega_E, \Omega_F)$ denotes the κth Thom-Porteous form of $\operatorname{Hom}(E, F)$ *given in (3.4).*

Proof. In [HZ2, 8.5] these equations were established in the special case of the top Chern spark of a single section. To prove the general case, we consider the pull-back cycle π^*Z in the total space of the bundle $\pi : G_r(E) \to X$ (where $r = m - k$, cf. §2) defined as follows. The pull-back cycle associated to a singular simplex $f : \Delta \to X$ is the current given by the canonical map into $G_r(E)$ of the fibre product of f with π. Observe now that if $\alpha : E \to F$ satisfies [8.6], then the section $\widehat{\alpha}$ of $\operatorname{Hom}(U, F) \to G_r(E)$ satisfies the corresponding condition with respect to π^*Z, and

$$\int_Z S_k(\alpha) \ = \ \int_{\pi^*Z} \sigma(\widehat{\alpha}) + s. \tag{8.10}$$

since by definition $S_k(\alpha) = \pi_*(\sigma(\widehat{\alpha}) + s)$. The general case now follows immediately by applying the special case to the top Chern spark $\sigma(\widehat{\alpha})$ of the single section $\widehat{\alpha}$ and then pushing down by π. $\square$

Corollary 8.11. *Equation (8.10) defines a differential character $\widehat{TP}_k$ on X which is independent of the choice of generic α'. Furthermore,*

$$\delta_1\widehat{TP}_k \ = \ TP_k(E,F) \qquad \text{and} \qquad \delta_2\widehat{TP}_k \ = \ [\mathbb{D}_k(\alpha)] \tag{8.12}$$

Proof. In the special case of the top Chern class of a cross-section of a bundle, the equations (8.12) are an immediate consequence of (3.2). Applying this to the section $\widehat{\alpha}$ and pushing down gives the general case. $\square$

Theorem 8.3 now follows from Theorem 8.2 and the functoriality given in Corollary 4.7. $\square$

8C. Spark product and the Cheeger product.

Theorem 8.13. *There is a natural transformation of functors*

$$\mathcal{M}_k(X) \times \mathcal{M}_\ell(X) \ \xrightarrow{\widehat{TP_k * TP_\ell}} \ \widehat{H}^*(X; \mathbf{R}/\mathbf{Z})$$

which for (α, β) is given on all cycles Z in X which do not meet $\mathbb{D}_k(\alpha) \wedge \mathbb{D}_\ell(\beta)$ by

$$(\widehat{TP_k * TP_\ell})(\alpha,\beta)(Z) \ \equiv \ \int_Z S_k(\alpha) * S_\ell(\beta) \quad \text{mod } \mathbf{Z} \tag{8.14}$$

This transformation factors through the Cheeger-Simons transformation

$$\begin{tikzcd}
\mathcal{M}_k(X) \times \mathcal{M}_\ell(X) \arrow[rr, "\phi"] \arrow[dr, "\widehat{TP_k * TP_\ell}"'] & & \mathcal{B}(X) \times \mathcal{B}(X) \arrow[dl, "\star"] \\
& \widehat{H}^*(X; \mathbf{R}/\mathbf{Z}) &
\end{tikzcd}$$

i.e., it is independent of the choice of sections and satisfies

$$(\widehat{TP_k * TP_\ell})(\alpha,\beta) \ = \ \widehat{TP}_k(\alpha) \star \widehat{TP}_\ell(\beta) \tag{8.15}$$

where $\star$ is the Cheeger product on the ring of differential characters (cf. [C]).

Proof. Fix $\alpha : E_1 \to F_1$ and $\beta : E_2 \to F_2$ over X where $\text{rank}(E_j) = m_j$ and $\text{rank}(F_j) = n_j$. Set $r_1 = m_1 - k$ and $r_2 = m_2 - \ell$. From (3.3) we have equations

$$\begin{aligned} d\sigma(\widehat{\alpha}) \ &= \ (\widehat{\Delta}_1 - ds) \ - \ \text{Div}(\widehat{\alpha}) \qquad \text{on } \ G_{r_1}(E) \quad \text{and} \\ d\sigma(\widehat{\beta}) \ &= \ (\widehat{\Delta}_2 - dt) \ - \ \text{Div}(\widehat{\beta}) \qquad \text{on } \ G_{r_2}(F) \end{aligned} \tag{8.16}$$

which we pull-back to the fibre product $G(E,F) := G_{r_1}(E) \times_X G_{r_2}(F)$. Now the spark product of top Chern forms satisfies

$$d(\sigma(\widehat{\alpha}) * \sigma(\widehat{\beta})) \;=\; (\widehat{\Delta}_1 - ds) \wedge (\widehat{\Delta}_2 - dt) - \mathrm{Div}(\widehat{\alpha}) \wedge \mathrm{Div}(\widehat{\beta}). \tag{8.17}$$

We define

$$S_k(\alpha) * S_\ell(\beta) \;:=\; \pi_* \left\{ \sigma(\widehat{\alpha}) * \sigma(\widehat{\beta}) + s \wedge \widehat{\Delta}_2 + \widehat{\Delta}_1 \wedge t - sdt. \right\} \tag{8.18}$$

Then from (8.17) and (8.18) we have

$$\begin{aligned} d\,\left(S_k(\alpha) * S_\ell(\beta)\right) \;&=\; \pi_* \left(\widehat{\Delta}_1 \wedge \widehat{\Delta}_2\right) - \pi_* \left(\mathrm{Div}(\widehat{\alpha}) \wedge \mathrm{Div}(\widehat{\beta})\right) \\ &=\; TP_k(E_1, F_1) \wedge TP_\ell(E_2, F_2) - \mathbb{D}_k(\alpha) \wedge \mathbb{D}_\ell(\beta) \end{aligned} \tag{8.19}$$

We now recall from section 3 that

(8.20) The form $s \wedge \widehat{\Delta}_2 + \widehat{\Delta}_1 \wedge t - sdt$ depends only on the choice of connections and not on the choice of the sections α and β.

Suppose now that X is a cycle of dimension $\mu - 1$ where $\mu = 2(m_1 - k)(n_1 - k) + 2(m_2 - \ell)(n_2 - \ell)$, and let $\pi^* Z$ be the pull-back of Z to $G(E,F)$ defined as in the proof of Proposition 8.7. Now the Thom Transversality Theorem implies that for generic bundle maps α' and β', $\mathrm{Div}(\widehat{\alpha}')$ and $\mathrm{Div}(\widehat{\beta}')$ intersect properly and $\mathrm{Div}(\widehat{\alpha}') \wedge \mathrm{Div}(\widehat{\beta}')$ will not meet $\pi^* Z$. Given another generic pair α'' and β'' we have by (8.20) that

$$\begin{aligned} &\int_{\pi^* Z} \left\{ \sigma(\widehat{\alpha}') * \sigma(\widehat{\beta}') + s \wedge \widehat{\Delta}_2 + \widehat{\Delta}_1 \wedge t - sdt \right\} - \int_{\pi^* Z} \left\{ \sigma(\widehat{\alpha}'') * \sigma(\widehat{\beta}'') + s \wedge \widehat{\Delta}_2 + \widehat{\Delta}_1 \wedge t - sdt \right\} \\ &\qquad = \int_{\pi^* Z} \sigma(\widehat{\alpha}') * \sigma(\widehat{\beta}') - \int_{\pi^* Z} \sigma(\widehat{\alpha}'') * \sigma(\widehat{\beta}'') \;\equiv\; 0 \qquad \mod BZ \end{aligned}$$

where the last assertion is proved in [HZ2, 8.5]. Since for any form ω one has

$$\int_{\pi^* Z} \omega \;=\; \int_Z \pi_* \omega,$$

we see that

$$\int_Z S_k(\alpha') \wedge S_\ell(\beta') \;\equiv\; \int_Z S_k(\alpha'') \wedge S_\ell(\beta'') \qquad \mod \mathbf{Z}.$$

Thus (8.14) gives a well-defined character $\widehat{TP_k * TP_\ell}$ independent of the choice of sections as asserted. From (8.19) we see that

$$\begin{aligned} \delta_1 \left(\widehat{TP_k * TP_\ell}\right) \;&=\; TP_k(E_1, F_1) \wedge TP_\ell(E_2, F_2) \\ \delta_2 \left(\widehat{TP_k * TP_\ell}\right) \;&=\; \mathbb{D}_k(\alpha) \wedge \mathbb{D}_\ell(\beta). \end{aligned}$$

Assertion (8.15) now follows from functoriality and Theorem 8.2. $\square$

8D. The Chern differential characters. The p**th Chern differential character** is the natural transformation $\widehat{c}_p : \mathcal{B}(X) \to H^{2p}(X, \mathbf{R}/\mathbf{Z})$ associated as above to the pth Chern form $c_p(\Omega_E)$ of the connection and the pth integer Chern class of the bundle E. This character can be canonically constructed as the Thom-Porteous spark of (any $(n-p)$-atomic) bundle map $\underline{\mathbf{C}}^{n-p+1} \longrightarrow E$. The spark construction in this case is particularly nice — the form s in (8.10) is zero. Furthermore, in this case there is an alternative construction of this spark associated to the singular push-forward connection given in [HL2]. This second construction yields precisely the same transgression form as we show in 10E.

The approach here is philosophically unifying. Recall that the pth integer Chern class arises as an obstruction to making a bundle map $\alpha : \underline{\mathbf{C}}^{n-p+1} \longrightarrow E$ everywhere injective. Here we see that the pth secondary Chern class and the pth Chern differential character arise in parallel fashion in the same problem.

To be specific, from §5 and Theorem 8.3 we immedately have the following. Suppose $E \to X$ is a smooth n-dimensional complex vector bundle with unitary connection D_E.

Theorem 8.21. *Let $\alpha : \underline{\mathbf{C}}^{n-p+1} \longrightarrow E$ be any $\mathbb{LD}$-atomic family of sections of E. Then the pth Chern differential character is given by*

$$\widehat{c}_p(D_E)(Z) \;=\; \int_Z S_p(\alpha) \mod \mathbf{Z}$$

on any $(2p-1)$-cycle Z which does not meet the linear dependency locus of α.

Remark 8.22. In particular, the value of the integral $\int_Z S_p(\alpha) \mod \mathbf{Z}$ is the same for all such α for which $\mathbb{LD}(\alpha)$ does not meet Z. By Lemma 8.5 this holds for generic α. Hence, the linear dependency sparks provide an alternate definition of the pth Chern differential character.

Remark 8.23. The fact that differential characters can be defined by forms with singularities was originally pointed out by Cheeger [C, pages 442-3]. The forms he uses are not canonically expressed in terms of the connection and do not necessarily give currents on X. They are smooth forms defined in the complement of a closed nowhere dense set, (c.f. [AE]), and can be constructed canonically from, say, a smooth triangulation.

The next result states that the spark product provides an alternative definition of the Cheeger product for Chern differential characters. It follows directly from Theorem 8.13.

Theorem 8.24. *Let $\alpha : \underline{\mathbf{C}}^{n-p+1} \longrightarrow E$ and $\beta : \underline{\mathbf{C}}^{m-q+1} \longrightarrow F$ be two $\mathbb{LD}$-atomic families of sections such that $\mathbb{LD}(\alpha)$ and $\mathbb{LD}(\beta)$ meet properly. Then the Cheeger product of differential characters is given by*

$$\left\{\widehat{c}_p(D_E) \star \widehat{c}_q(D_F)\right\}(Z) \;\equiv\; \int_Z S_p(\alpha) * S_q(\beta) \mod \mathbf{Z}$$

on any cycle Z which does not meet $\mathbb{LD}(\alpha) \wedge \mathbb{LD}(\beta)$.

Combining this result with the Product Theorem 6.8 for the total secondary Chern class we recover Cheeger-Simons' fundamental Product Theorem:

Corollary 8.25. (The Product Theorem of [CS]). *Let $\widehat{c}$ denote the total Chern differential character. Then*

$$\widehat{c}(D_E) \star \widehat{c}(D_F) \;=\; \widehat{c}(D_E \oplus D_F).$$

Remark 8.26. Cheegers' formula for the product of differential characters. In [C] Cheeger gives a precise formula for the product of two differential characters which are represented by forms with singularities. Suppose that $\widehat{c}_p(D_E)$ and $\widehat{c}_q(D_F)$ are represented by sparks $S_p(\alpha)$ and $S_q(\beta)$ as in Theorem 8.21. Cheeger's formula for the product differential character is

$$\left(\widehat{c}_p(D_E) \star \widehat{c}_q(D_F)\right)(Z) \;=\; \int_Z c_p(\Omega_E) \wedge S_q(\beta) \;+\; \int_{Z \cap \mathbb{LD}(\alpha)} S_p(\alpha),$$

where we suppose that the collections α and β have been chosen to be transverse to Z in such a way that both integrals are finite. This formula may be interpreted as the action of the current $c_p(\Omega_E) \wedge S_q(\beta) \;+\; S_p(\alpha) \wedge \mathbb{LD}(\beta)$ on certain transverse cycles such as Z. This current is not an L^1_{loc} form and so does not give a canonical representation of the product differential character as a "form with singularities". Furthermore its singular support is contained in the *union* of the linear dependency sets of α and β. In contrast to this the product $S_p(\alpha) * S_q(\beta)$ of the Chern sparks of α and β is an L^1_{loc} form which is smooth outside the *intersection* of the linear dependency sets of the two collections of sections.

Remark 8.27. (Cheeger-Simons construction of the pth Chern differential character). Cheeger and Simons [CS] give an explicit representative of the pth Chern differential character $\widehat{c}_p(D_E)$ which is related to the spark construction presented here. Their construction proceeds as follows. Let $k = n - p + 1$, where $n = \mathrm{Rk}(E)$, and let $\pi : \mathrm{St}_k(E) \to X$ be the Stiefel bundle of unitary k frames of E. Recall that there is a natural fibration $\rho : \mathrm{St}_k(E) \to \mathrm{St}_{k-1}(E)$ defined by sending a unitary frame $(e_1, ...e_k)$ to $(e_1, ..., e_{k-1})$. The fibres of this fibration are $(2p-1)$-dimensional spheres, S^{2p-1}. Now the pullback, $\pi^* c_p(\Omega_E)$, to $\mathrm{St}_k(E)$ of the pth Chern form of D_E is exact on $\mathrm{St}_k(E)$. In fact there is a natural transgression form $T_{2p-1}(D_E)$ of degree $2p-1$ on $\mathrm{St}_k(E)$ so that

$$(8.28) \qquad dT_{2p-1}(D_E) \;=\; \pi^* c_p(\Omega_E) \qquad \text{and} \qquad \rho_* T_{2p-1}(D_E) \;=\; \int_{S^{2p-1}} T_{2p-1}(D_E) \;=\; 1.$$

Using the homology sequence associated to the Stiefel bundle $\pi : \mathrm{St}_k(E) \to X$ one can show that there is a $(2p-1)$-cycle Y on $\mathrm{St}_k(E)$ and a $2p$- chain W on X so that $Z = \pi_*(Y) + \partial W$. Then Cheeger and Simons show that the pth Chern differential character is given by

$$(8.29) \qquad \widehat{c}_p(D_E)(Z) \;\equiv\; \int_Y T_{2p-1}(D_E) \;+\; \int_W c_p(\Omega_E) \quad \operatorname{mod} \mathbf{Z}.$$

A natural transgression form $T_{2p-1}(D_E)$ on $\mathrm{St}_k(E)$ satisfying equations (8.28) can be constructed as follows. First note that the tautological rank k bundle U in π^*E over $\mathrm{St}_k(E)$ is trivial, $U = \underline{\mathbf{C}}^k$. Let E_0 be the orthogonal complement of $U = \underline{\mathbf{C}}^k$ in π^*E, so that $\pi^*E = E_0 \oplus \underline{\mathbf{C}}^k$ over $\mathrm{St}_k(E)$. There are two natural connections on $\pi^*E \to \mathrm{St}_k(E)$, namely the pullback connection, D_E, of the given connection on E, and the direct sum connection $D_{E_0} \oplus D_{\underline{\mathbf{C}}^k}$, where $D_{\underline{\mathbf{C}}^k} = d$ is the trivial connection on $\underline{\mathbf{C}}^k$ and D_{E_0} is the connection on E_0 induced by D_E. Let D_s be any smooth family of smooth connections interpolating between the two connections D_E and $D_{E_0} \oplus D_{\underline{\mathbf{C}}^k}$. The required form $T_{2p-1}(D_E)$ is defined to be the classical Chern-Weil transgression form associated with the family of connections D_s, [BCh1]. To see that this form $T_{2p-1}(D_E)$ satisfies equations (8.28) first observe that $dT_{2p-1}(D_E) = \pi^*c_p(\Omega_E) - c_p(\Omega_{E_0\oplus\underline{\mathbf{C}}^k}) = \pi^*c_p(\Omega_E)$, since $c_p(\Omega_{E_0\oplus\underline{\mathbf{C}}^k}) = 0$. Secondly, a straightforward argument from Chern-Weil theory shows that $\rho_*T_{2p-1}(D_E) = 1$.

Remark 8.30. (The relationship between the two constructions of the pth Chern differential character). Cheeger-Simons construction (8.29) of the pth Chern differential character is related to the spark definition 8.21 as follows.

To begin we recall that there is a natural embedding $\mathrm{St}_k(E) \hookrightarrow \mathrm{Hom}(\underline{\mathbf{C}}^k, E)$ and that over $\mathrm{Hom}(\underline{\mathbf{C}}^k, E)$ there is a tautological or universal bundle map $\alpha : \underline{\mathbf{C}}^k \to E$. Note that α has full rank k over $\mathrm{St}_k(E)$ and that the subbundle E_0 of E is the orthogonal complement of the image of α. The universal bundle map α can be used to construct a special family of connections, denoted by $\overrightarrow{D}_{\alpha,s}$, which interpolates between the connections D_E and $D_{E_0} \oplus D_{\underline{\mathbf{C}}^k}$ on E over $\mathrm{St}_k(E)$. The family $\overrightarrow{D}_{\alpha,s}$, which was constructed in [HL2, I.8.2], is called the **universal approximating family of pushforward connections**.

Secondly, observe that the tautological collection α of sections of E over $\mathrm{Hom}(\underline{\mathbf{C}}^k, E)$ is $\mathbb{LD}$-atomic and that the restriction of the pth Chern spark $S_p(\alpha)$ to $\mathrm{St}_k(E)$ is smooth.

The following result, which is of independent interest, is proved and discussed further in the Appendix.

Theorem 8.31. *Suppose that the transgression form $T_{2p-1}(D_E)$ on $\mathrm{St}_k(E)$ is chosen to be the one which is associated with the universal approximating family of pushforward connections $\overrightarrow{D}_{\alpha,s}$. Then*

$$T_{2p-1}(D_E) \;=\; S_p(\alpha) \qquad \text{as smooth forms on } \mathrm{St}_k(E). \tag{8.32}$$

Finally, given a $(2p-1)$-cycle Z on X, let α be any $\mathbb{LD}$-atomic collection of k sections of $E \to X$ which is transverse to Z. Furthermore assume that the restriction of α to Z is unitary, and so defines a section α of $\mathrm{St}_k(E) \to Z$. Define $Y = \alpha_*(Z)$ to be the pushforward of the cycle Z to $\mathrm{St}_k(E)$. Then, by construction, $Z = \pi_*(Y)$ and so Theorem 8.31 implies that

$$\int_Z S_p(\alpha) \;=\; \int_Y T_{2p-1}(D_E).$$

Consequently, in this case, Cheeger-Simons construction (8.29) of the pth Chern differential character and the construction given in Theorem 8.21 are in exact agreement.

8E. Ronga-Pontrjagin differential characters and the Chern-Simons 3-form. The qth Ronga-Pontrjagin differential character of a real vector bundle E with metric connection D_E is defined to be the unique functorial differential character, $\widehat{p}_q(D_E)$, associated with the qth Pontrjagin form $p_q(\Omega_E)$ of the connection and the qth Ronga-Pontrjagin cohomology class $p_q + W_{2q-1}W_{2q+1}$ which was introduced in Section 7. Just as in Theorem 8.21 above, the qth Ronga-Pontrjagin differential character of D_E is given by the formula

$$\widehat{p}_q(D_E)\,(Z) \;=\; \int_Z S_q(\alpha) \quad \mathrm{mod}\,\mathbf{Z}, \tag{8.33}$$

where Z is any smooth singular $(4q-1)$-cycle in X and where $S_q(\alpha)$ is the qth Ronga-Pontrjagin spark of any $\mathbb{LD}_2$-atomic collection α of sections of E which is transverse to Z. In other words, the qth Ronga- Pontrjagin sparks determine the qth Ronga-Pontrjagin differential character via equation (8.33).

Remark 8.34. (The Pontrjagin differential character of [CS]). The qth Pontrjagin differential character of a real vector bundle E with connection D_E is the unique functorial differential character associated with the qth Pontrjagin form of D_E and the qth *integer* Pontrjagin class. Cheeger-Simons' explicitly construct the qth Pontrjagin differential character as the $2q$th Chern differential character of the complexification of E. Obviously the Pontrjagin differential character does not equal the Ronga-Pontrjagin differential character.

Let $F \to X$ be an oriented real rank 3 vector bundle with metric connection D_F and let $\pi : \mathcal{F} \to X$ denote the oriented orthonormal frame bundle of F. The Chern-Simons 3-form of D_F is a canonically defined smooth transgression form $T(D_F)$ on the frame bundle whose exterior derivative equals the pullback to $\mathcal{F}$ of the 1st Pontrjagin form of the connection: $dT(D_F) = \pi^* p_1(\Omega_F)$, see [ChS].

Let α be the tautological collection of 3 sections of the pullback of F over $\mathcal{F}$. Since the sections α are linearly independent on $\mathcal{F}$ the 1st Ronga-Pontrjagin spark $S_1(\alpha)$ of α and D_F (see Definition 7.3) is a smooth 3-form on $\mathcal{F}$ which, by equation (7.2), also satisfies $dS_1(\alpha) = \pi^* p_1(\Omega_F)$. The following result is proved in the Appendix.

Proposition 8.35.

$$S_1(\alpha) \;=\; T(D_F) \qquad \textit{as differential forms on } \mathcal{F}.$$

Question. More generally, let $F \to X$ be a real rank n vector bundle and let α be the tautological collection of $n+2-2q$ sections of F over the orthonormal frame bundle $\mathcal{F}$. Is the qth Ronga-Pontrjagin spark $S_q(\alpha)$ equal to Chern-Simons' qth Pontrjagin transgression form on $\mathcal{F}$?

Note. The orthonormal frame bundle $\mathcal{F} \to X$ is a subbundle of the bundle $\mathrm{Hom}(\mathbf{R}^3, F) \to X$ of collections of 3 sections of F, and the collection α over $\mathcal{F}$ extends to a tautological $\mathbb{LD}_2$-atomic collection α of 3 sections of the pullback of F over $\mathrm{Hom}(\mathbf{R}^3, F)$. Consequently the L^1_{loc} form $S_1(\alpha)$ on $\mathrm{Hom}(\mathbf{R}^3, F)$ can be regarded as an extension to all of $\mathrm{Hom}(\mathbf{R}^3, F)$ of the Chern-Simons 3-form on $\mathcal{F}$.

Suppose now that $F = TX$ is the tangent bundle to a compact oriented Riemannian 3- manifold. Chern-Simons' constructed a numerical invariant $\Phi(X)$ of the Riemannian 3-manifold X from the Chern-Simons 3-form as follows. Since an oriented 3-manifold is parallelizable there is a global section α of the frame bundle $\mathcal{F}(TX) \to X$. The invariant $\Phi(X)$ is defined to be the element of $\mathbf{R}/\mathbf{Z}$ given by $\Phi(X) \equiv \int_X \alpha^* T(D_{TX}) \bmod \mathbf{Z}$, (which is independent of the choice of α). It was this construction which led to the concept of a differential character. Proposition 8.35 above easily implies that $\Phi = \widehat{p}_1(D_{TX})$ is the 1st Ronga-Pontrjagin differential character of D_{TX}.

9. Differential characters via Singularities of Maps.

Differential characters, which are invariants of a vector bundle with connection, are most interesting when the associated primary characteristic form of the connection vanishes. In this case a result of Cheeger-Simons' [CS, Cor 2.4] states that the differential character defines a primary cohomology class in $H^{\mathrm{odd}}(X, \mathbf{R}/\mathbf{Z})$. In this section we present some geometrically interesting examples in which differential characters are naturally defined in terms of the degeneracies of a vector bundle map. Furthermore, in certain special cases, we will show that these differential characters define *non-zero* $\mathbf{R}/\mathbf{Z}$ cohomology classes of odd degree.

Some of our examples reproduce well known differential characters, such as the Chern and Ronga-Pontryjagin characters discussed in Section 8. However other examples yield the so called Ronga-Thom-Porteous differential characters, which have not been discussed in the literature. In any case our main point is that the degeneracy sparks of a bundle map provide functorial, local representatives for many differential characters. In particular we will construct differential characters from the degeneracy sparks associated to: collections of sections, singularities of projections, CR-singularities, and the high-order intersection of a pair of foliations.

The differential characters in this section all arise as follows, (c.f. [HL3, Section 5]). Let E and F be *real* vector bundles of ranks m and n respectively and let k be an integer with $0 \le k \le \min\{m, n\}$. (Similar results hold for complex bundles, see Remark 5.14.) We assume that one of the following two cases holds:

(Even case) The integers $m = 2m_0$, $n = 2n_0$ and $k = 2k_0$ are all even,

(Odd case)

The integers $m = 2m_0 + 1$, $n = 2n_0 + 1$ and $k = 2k_0 + 1$ are all odd.

Then, as in Remark 5.14, there is a canonically defined L^1_{loc} form $S_k(\alpha)$ called the **kth degeneracy spark of** α, so that

$$(9.1) \qquad \widetilde{\Delta}^{(m_0-k_0)}_{n_0-k_0}\{p(\Omega_F)p(\Omega_E)^{-1}\} \;-\; \mathbb{D}_k(\alpha) \;=\; dS_k(\alpha) \qquad \text{on } X.$$

Here $\mathbb{D}_k(\alpha)$ denotes the kth degeneracy current of α and the Shur polynomial $\widetilde{\Delta}^{(b)}_a(\eta)$ is defined by

$$\widetilde{\Delta}^{(b)}_a(\eta) \;=\; \det(\eta_{a-i+j})_{1\le i,j\le b},$$

where $\eta = \eta_1 + \eta_2 + \dots$ is a differential form with each η_k of pure degree $4k$. Equation (9.1) holds whenever the bundle map is k-atomic (see [HL3]).

Ronga first observed that the $\mathbf{Z}$-cohomology class of the current $\mathbb{D}_k(\alpha)$ is not equal to $\widetilde{\Delta}^{(m_0-k_0)}_{n_0-k_0}\{p(F)p(E)^{-1}\}$ (where p denotes the total integer Pontryjagin class); rather it differs from $\widetilde{\Delta}^{(m_0-k_0)}_{n_0-k_0}\{p(F)p(E)^{-1}\}$ by a 2-torsion term. (An explicit formula for this 2-torsion term was calculated in [HZ1] in the case that E is trivial.) Consequently we will call the differential character determined by the kth degeneracy spark $S_k(\alpha)$ the kth Ronga-Thom-Porteous differential character of the connections D_E and D_F. In summary we have

Definition 9.2. Let E^m and F^n be real vector bundles with metric connections D_E and D_F, let k be an integer with $0 \le k \le \min\{m, n\}$ and assume that either the even or the odd case above holds. The **kth Ronga-Thom-Porteous differential character of the connections D_E and D_F** is defined to be the degree $(m-k)(n-k)-1$ differential character

$$(9.3) \qquad \widehat{S}_k(D_E, D_F)(Z) \equiv \int_Z S_k(\alpha) \qquad \mathrm{mod}\, \mathbf{Z},$$

where Z is a smooth singular cycle of dimension $(m-k)(n-k)-1$ in X and where $S_k(\alpha)$ is the kth degeneracy spark of any k-atomic bundle map $\alpha : E \to F$ which is transverse to Z.

The following examples were first discussed at the level of primary characteristic forms and degeneracy currents in [HL3].

9A. Higher Linear Dependencies.

Let $\alpha : \mathbf{R}^{k+\ell} \to F$ be a k-atomic family of sections of a rank n real vector bundle F. Suppose $\ell = 2\ell_0$ is even and $0 \le k < n$. The **ℓ-dependency current** $\mathbb{LD}_\ell(\alpha) := \mathbb{D}_k(\alpha)$ of α is supported on the subset of X where at least ℓ of these sections are linearly dependent on the rest. The ℓ-dependency spark, $LD_\ell(\alpha) := S_k(\alpha)$, satisfies

$$\det_{\ell_0\times\ell_0}\left(p_{n_0-k_0-i+j}(\Omega_F)\right) \;-\; \mathbb{LD}_\ell(\alpha) \;=\; dS_\ell(\alpha),$$

and the corresponding Ronga-Thom-Porteous differential character

$$\widehat{LD}_\ell(k+\ell, D_F) := \widehat{S}_k(d_{\underline{\mathbf{R}}^{k+\ell}}, D_F)$$

is of degree $\ell(n-k)-1$.

Two simple ways to ensure that $\widehat{LD}_\ell(k+\ell, D_F)$ is an $\mathbf{R}/\mathbf{Z}$-cohomology class are to insist that either the differential character is of top degree on X, or that $F = TX$ where X is a constant curvature manifold (since then $p_j(\Omega_{TX}) = 0$ for all j).

Example 9.4. (Top degree characters). Let X be a Riemannian manifold.
(1) If dim $X = 3$, the 2-dependency spark $S_2(\alpha)$ of a collection of 3 vector fields on X represents the differential character $\widehat{LD}_2(3, D_{TX})$, which is an element of $H^3(X, \mathbf{R}/\mathbf{Z})$, since it is of top degree. The differential character $\widehat{LD}_2(3, D_{TX})$ is simply the 1st Ronga-Pontryjagin differential character, and so is non-zero by Proposition 8.35.
(2) If dim $X = 15$, the 4-dependency spark $S_4(\alpha)$ of a collection of 15 vector fields on X represents the differential character $\widehat{LD}_4(15, D_{TX})$, which is an element of $H^{15}(X, \mathbf{R}/\mathbf{Z})$, since it is also of top degree. The corresponding primary characteristic form is $(p_2^2 - p_1p_3)(\Omega_{TX})$.
(3) Similarly, the 4-dependency spark of a collection of 9 vector fields on a 7-manifold X represents the top degree differential character $\widehat{LD}_4(9, D_{TX}) \in H^7(X, \mathbf{R}/\mathbf{Z})$ and the corresponding characteristic form is $(p_1^2 - p_2)(\Omega_{TX})$.

Example 9.5. (Characters of constant curvature manifolds).
(1) Suppose that X is a Riemannian 4-manifold with $p_1(\Omega) \equiv 0$ (e.g., suppose X has constant curvature or X is a riemannian product $S^1 \times M$). Then the 2-dependency spark of any collection of 4 vector fields on X represents the 1st Ronga-Pontryjagin character $\widehat{p}_1 \in H^3(X, \mathbf{R}/\mathbf{Z})$. To see that $\widehat{p}_1$ can be non-zero consider $X = S^1 \times M$, where M is a 3-manifold whose Chern-Simons invariant is non-zero (cf. [Mill]). The value of $\widehat{p}_1$ on the 3-cycle $\{pt\} \times M \subset X$ is exactly the Chern- Simons invariant of M.
(2) Let X be a riemannian 8-manifold of constant curvature, or more generally a product of constant curvature spaces. Then the 4- dependency spark of a collection of 10 vector fields on X represents the character $\widehat{LD}_4(10, D_{TX}) \in H^7(X, \mathbf{R}/\mathbf{Z})$.

9B. Singularities of Projections. Let X be an m-manifold with connection D_{TX} and suppose $j : X \to \mathbf{R}^N$ is any immersion of X into Euclidean space. Each linear homomorphism $P : \mathbf{R}^N \to \mathbf{R}^n$ restricts to a smooth mapping $\widehat{P} = P \circ j : X \to \mathbf{R}^n$. The kth degeneracy current of the projection P on X is the current $\mathbb{D}_k(P) := \mathbb{D}_k(d\widehat{P})$, which measures where the differential $d\widehat{P} : TX \to \underline{\mathbf{R}}^n$ has rank $\leq k$. The corresponding kth degeneracy spark, $S_k(P)$, represents the differential character $\widehat{S}_k(D_{TX}, d_{\underline{\mathbf{R}}^n})$, which is independent of the choice of projection P, and the dimension of the Euclidean space $\mathbf{R}^N$ into which X is immersed.

Remark 9.6. (The adjoint problem). Let $\alpha : E \to F$ and consider the adjoint mapping $\alpha^* : F^* \to E^*$ of the dual bundles. In [HL3] it was shown that, if the dual bundles are equipped with the dual metric connections, then

$$\widetilde{\Delta}_{m_0-k_0}^{(n_0-k_0)}\{p(\Omega_{E^*})p(\Omega_{F^*})^{-1}\} \;=\; (-1)^d\widetilde{\Delta}_{n_0-k_0}^{(m_0-k_0)}\{p(\Omega_F)p(\Omega_E)^{-1}\},$$

where $d = (m_0 - k_0)(n_0 - k_0)$, and (at least for generic maps α) that $\mathbb{D}_k(\alpha^*) = (-1)^d \mathbb{D}_k(\alpha)$. Consequently a version of Cheeger-Simons' uniqueness theorem implies that the differential characters

$$\widehat{S}_k(D_{F^*}, D_{E^*}) \;=\; (-1)^d \widehat{S}_k(D_E, D_F)$$

are equal.

In particular the differential character, $\widehat{S}_k(D_{TX}, d_{\underline{\mathbf{R}}^n})$, represented by the kth degeneracy spark of a projection of X on $\mathbf{R}^n$ is equal (up to sign) to the differential character $\widehat{S}_k(d_{\underline{\mathbf{R}}^n}, D_{T^*X})$ which is represented by the $(n-k)$th-dependency spark of a collection of n sections of the cotangent bundle T^*X. Consequently Examples 9.4 and 9.5 have natural analogues in which differential characters in $H^{\text{odd}}(X, \mathbf{R}/\mathbf{Z})$ are represented by the degeneracy sparks of singularities of projections.

9C. CR-Singularities. Let X be a real Riemannian m-manifold endowed with it's Riemannian connection D_{TX}, and let $f : X \to \mathbf{C}^n$ be an immersion, where $m \le n$. (Recall that such immersions exist by Whitney's Theorem.) The differential $df : TX \to \underline{\mathbf{C}}^n$ extends to a *complex* vector bundle map $df_{\mathbf{C}} : TX \otimes_{\mathbf{R}} \underline{\mathbf{C}} \to \underline{\mathbf{C}}^n$ over X. Fix an integer k with $0 \le k < m$ and let $r = m - k$. The **rth complex tangency current** $\mathbb{C}r_r(f) := \mathbb{D}_k(df_{\mathbf{C}})$ is supported on the set of points $x \in X$ where $f_* T_x X$ contains a complex subspace of dimension at least r. By [HL3, 10.12, 4.3] there is a canonically defined L^1_{loc} form $Cr_r(f)$ on X called the **rth complex tangency spark** of the immersion f such that

$$(9.7) \qquad (-1)^d \Delta_r^{(n-m+r)}\{\widetilde{p}(\Omega_{TX})\} \;-\; \mathbb{C}r_r(f) \;=\; dCr_r(f),$$

where $d = r(n - m + r)$, where $\widetilde{p} = 1 - p_1 + p_2 - p_3 + \ldots$ and where $\Delta_a^{(b)}(\eta)$ is the Shur polynomial of Remark 5.14.

Since $Cr_r(f)$ is the spark of a *complex* bundle map it represents the differential character uniquely determined by the differential form $(-1)^d \Delta_r^{(n-m+r)}\{\widetilde{p}(\Omega_{TX})\}$ and the **Z**-cohomology class $(-1)^d \Delta_r^{(n-m+r)}\{\widetilde{p}(TX)\}$, (where the p_j in $\widetilde{p}$ now denote the integer Pontryjagin classes). The associated differential character is $\widehat{Cr}_r(f) = (-1)^d \Delta_r^{(n-m+r)}\{\widehat{\widetilde{p}}(D_{TX})\}$.

As above, if X has constant curvature, then

$$\widehat{Cr}_r(f) \;\in\; H^{\text{odd}}(X, \mathbf{R}/\mathbf{Z}).$$

The following example shows that this cohomology class is non- trivial.

Example 9.8. Let X be a real n manifold, let $f : X \to \mathbf{C}^n$ be an immersion and let $r = 2$. In this case equation (9.7) states that

$$p_1^2(\Omega_{TX}) \;-\; \mathbb{C}r_2(f) \;=\; dCr_2(f) \qquad \text{on } X,$$

and so the 2nd complex tangency spark $Cr_2(f)$ is an L^1_{loc} form of degree 7 which represents the p_1^2-integer Pontrjagin differential character, $\widehat{p}_1^2(D_{TX})$.

Specializing further, let $X = S^1 \times M$ be a Riemannian 8- manifold, where M is a 7-dimensional Lens space of constant curvature with $\widehat{p}_1^2(D_{TM}) \neq 0 \bmod \mathbf{Z}$ (cf. [Mill]). By construction, $\widehat{p}_1^2(D_{TX})$ is non-zero since it's action on the 7-cycle M is equal to $\widehat{p}_1^2(D_{TM})$. Also note that the differential character $\widehat{p}_1^2(D_{TX})$ is an $\mathbf{R}/\mathbf{Z}$- cohomology class, since X has constant curvature.

9D. Relative invariants for connections. Suppose D_1 and D_2 are two connections on a complex vector bundle $E \to X$ with the same characteristic forms:

$$c(\Omega^{D_1}) \;\equiv\; c(\Omega^{D_2}).$$

Then for any k, $0 \leq k < n = \mathrm{rank}(E)$ and any k-atomic bundle map $\alpha : E \to E$ we have the equation

$$dS_k(\alpha) \;=\; \Delta_{n-k}^{(n-k)}\left(c(\Omega^{D_1})c(\Omega^{D_2})^{-1}\right) - \mathbb{D}_k(\alpha) \;=\; -\mathbb{D}_k(\alpha).$$

Thus the spark $S_k(\alpha)$ represents a primitive for the degeneracy current $\mathbb{D}_k(\alpha)$ and also represents the secondary cohomology class

$$\delta_2\left[\widehat{S}_k(D_1, D_2)\right] \;\in\; H^{2k}(X; \mathbf{R}/\mathbf{Z})$$

determined by the differential character.

A classical case of this occurs when E is a hermitian line bundle on a complex manifold X and D_1, D_2 are the cononical connections associated to distinct holomorphic structures on E. If E_1, E_2 denote the corresponding holomorphic line bundles, then $\mathrm{Hom}(E_1, E_2) = E_1^* \otimes E_2 \in \mathrm{Pic}_0(X)$ is a bundle which is topologically (but not holomorphically) trivial and carries a natural flat connection. The differential character $\widehat{S}_1(D_1, D_2) : \{1\text{−chains}\} \to \mathbf{R}/\mathbf{Z}$ is the holonomy of this connection.

9E. Orthogonality currents for foliations. Let $\mathcal{E}$, $\mathcal{F}$ be smooth foliations of codimension e and f on a riemannian n-manifold X, and let $N(\mathcal{E})$, $N(\mathcal{F})$ denote their normal bundles. The inclusion $N(\mathcal{E}) \hookrightarrow TX$ followed by orthogonal projection $TX \to N(\mathcal{F})$ gives a bundle map

$$\alpha : N(\mathcal{E}) \longrightarrow N(\mathcal{F}) \qquad \text{with} \quad \ker\alpha = N(\mathcal{E}) \cap T(\mathcal{F}).$$

The degeneracies of this map correspond to "excess" or "unexpected" orthogonalities between $\mathcal{E}$ and $\mathcal{F}$, and so for each k, $0 \leq k < \min\{e, f\}$ the current

$$\mathbb{O}r_k(\mathcal{E}, \mathcal{F}) \;:=\; \mathbb{D}_k(\alpha)$$

will be called the k^{th}**-orthogonality current**.

We now introduce Bott connections $D_{\mathcal{E}}$ and $D_{\mathcal{F}}$ on $N(\mathcal{E})$ and $N(\mathcal{F})$ (cf. [B2]), and consider the Pontrjagin forms

$$p\left(\Omega_{N(\mathcal{F})}\right) \;=\; 1 + p_1 + p_2 + \dots \qquad \text{and} \qquad p\left(\Omega_{N(\mathcal{E})}\right)^{-1} \;=\; 1 + \bar{p}_1 + \bar{p}_2 + \dots.$$

By the Bott Vanishing Theorem [B2] we know that the monomials

$$p^{\alpha} = p_1^{\alpha_1} p_2^{\alpha_2} \cdots \equiv 0 \quad \text{on} \quad X \qquad \text{whenever} \quad \deg(p^{\alpha}) = \sum_j 4j\alpha_j > 2e$$

and similarly

$$\bar{p}^{\beta} \equiv 0 \quad \text{on} \quad X \qquad \text{whenever} \quad \deg(p^{\alpha}) = \sum_j 4j\beta_j > 2f$$

Suppose now that $e \equiv f \equiv k(\text{mod}2)$ and write $e = 2e_0, f = 2f_0, k = 2k_0$ in the even case and $e = 2e_0+1, f = 2f_0+1, k = 2k_0+1$ in the odd case. Consider the Thom-Porteous form

$$\Delta_{f_0-k_0}^{(e_0-k_0)}(p \cdot \bar{p})$$

which is of total degree $N = (e-k)(f-k) = 4(e_0-k_0)(f_0-k_0)$. Notice that this form is a linear combination of products $p^{\alpha}\bar{p}^{\beta}$ where $\deg(p^{\alpha}) + \deg(\bar{p}^{\beta}) = N$. Since $\max\{\deg(p^{\alpha}), \deg(\bar{p}^{\beta})\} \geq N/2$, the Bott Vanishing Theorem above says that $p^{\alpha}\bar{p}^{\beta} \equiv 0$ if $N/2 > \max\{2e, 2f\}$. Therefore we have the following.

Proposition 9.15. *If $(e-k)(f-k) > 4max\{e, f\}$, then $\Delta_{f_0-k_0}^{(e_0-k_0)}(p \cdot \bar{p}) \equiv 0$ on X. The degeneracy spark $S_k(\alpha)$ therefore satisfies the equation*

$$dS_k(\alpha) \;=\; \mathbb{O}r_k(\mathcal{E}, \mathcal{F}).$$

The differential character $\widehat{S}_k(D_{\mathcal{E}}, D_{\mathcal{F}})$ represented by this spark corresponds to a cohomology class in $H^{(e-k)(f-k)}(X; \mathbf{R}/\mathbf{Z})$.

When $e = f$ the condition in Proposition 9.15is satisfied when $e > \sqrt{k+2}\{\sqrt{k+2}+\sqrt{2}\}$.

10. Appendix.

10A. Proof of Proposition 6.16. Since the base manifold X does not play a role in the proof of the proposition we omit all reference to it with the understanding that all currents $[S]$ should be replaced by the cartesian product $[S \times X]$. We will construct the current $\mathbf{I}$ on $\mathbf{P}^{n+m} \times \mathbf{P}^{n+m}$ as a sum $\mathbf{I} = \sum_{k=1}^{n+m} \mathbf{I}_k$ where $\mathbf{I}_k$ is a current on $\mathbf{P}^k \times \mathbf{P}^k$ which satisfies the equation

$$\partial \mathbf{I}_k \;=\; [\Delta_k] \;-\; \sum_{j=0}^{k} [\mathbf{P}^{k-j} \times \mathbf{P}^j] \tag{10.1}$$

where $[\Delta_k] \subset \mathbf{P}^k \times \mathbf{P}^k$ is the diagonal.

Equation (10.1) is proved as follows. Consider the embeddings $\mathbf{P}^j_{(0,1,\dots,j)} \hookrightarrow \mathbf{P}^k$ defined by $[z] \mapsto [z,0]$ and $\mathbf{P}^{k-j}_{(j,j+1,\dots,k)} \hookrightarrow \mathbf{P}^k$ defined by $[z] \mapsto [0,z]$. Using Morse theory we will show that there is an integral current T_k so that

$$(10.2) \qquad \partial T_k \;=\; [\Delta_k] \;-\; \sum_{j=0}^{k} [\mathbf{P}^{k-j}_{(j,\dots,k)} \times \mathbf{P}^j_{(0,\dots,j)}] \qquad \text{on } \mathbf{P}^k \times \mathbf{P}^k.$$

Equation (10.1) then follows from (10.2) and the standard fact that there are integral currents R^j_k so that

$$\partial R^j_k \;=\; [\mathbf{P}^{k-j}_{(j,\dots,k)}] \;-\; [\mathbf{P}^{k-j}_{(0,\dots,k-j)}] \qquad \text{on } \mathbf{P}^k$$

for $0 \le j \le k$.

The proof of equation (10.2) relies on the following result of [HL4]. Let $\varphi_s : M \to M$ be a Morse-Stokes flow on a manifold M with $\varphi_1 = \mathrm{Id}$. Then there exists an integral current T on $M \times M$ so that

$$(10.3) \qquad \partial T \;=\; [\Delta] \;-\; \sum_{p \in \mathrm{Stat}(\varphi_s)} [U_p] \times [S_p] \qquad \text{on } M \times M,$$

where Δ is the diagonal and $U_p := \{x \in M \;:\; \varphi_s(x) \to p \text{ as } s \to 0\}$ and $S_p := \{x \in M \;:\; \varphi_s(x) \to p \text{ as } s \to \infty\}$ are the unstable and stable manifolds of a stationary point p of the flow.

To prove equation (10.2) we consider the flow $\varphi_s : \mathbf{P}^k \to \mathbf{P}^k$ defined by

$$(10.4) \qquad \varphi_s[z_0, \dots, z_k] \;=\; [z_0, sz_1, .., s^k z_k].$$

This flow is Morse-Stokes by [M, p26] and [HL4, 1.11]. Let $e_0, e_1, \dots, e_k$ denote the standard basis of $\mathbf{C}^{k+1}$. The stationary points of the flow φ_s are the $k+1$ points $[e_0], [e_1], \dots, [e_k]$ and the closure of the unstable and stable manifolds of the stationary point $[e_j]$ are

$$\overline{U}_{[e_j]} \;=\; \mathbf{P}^{k-j}_{(j,j+1,\dots,k)} \qquad \text{and} \qquad \overline{S}_{[e_j]} \;=\; \mathbf{P}^j_{(0,1,..,j)}.$$

The integral current T_k in (10.2) is defined as follows. Let

$$(10.5) \qquad \mathcal{T} \;=\; \{(\varphi_s([z]), [z]) \;:\; 1 \le s < \infty,\; [z] \in \mathbf{P}^k\} \;\subset\; \mathbf{P}^k \times \mathbf{P}^k.$$

Since the flow φ_s is algebraic, one sees that $\mathcal{T}$ is a semi-analytic subvariety of dimension $2(k)+1$ and locally finite volume in $\mathbf{P}^k \times \mathbf{P}^k$. T_k is just the current given by integration over $\mathcal{T}$. $\square$

10B. Proof of Proposition 7.11. Let N be even and let $\Delta \subset G_2(\mathbf{R}^N)\times G_2(\mathbf{R}^N)$ denote the diagonal copy of $G_2(\mathbf{R}^N)$. Recall that $G_2(\mathbf{R}^N)$ is orientable. We will show that there are integral (untwisted) currents T and R so that

$$(10.6)\quad \partial T \;=\; [\Delta] \;-\; \sum_{j=1}^{N/2} G_2(\mathbf{R}^{N-2j+2})\times G_2(\mathbf{R}^{2j}) \;-\; \tfrac{1}{2}\partial R \qquad \text{on } G_2(\mathbf{R}^N)\times G_2(\mathbf{R}^N).$$

An explicit formula for R is given in equation (10.9) below.

Given equation (10.6) the proof of the proposition is completed as follows. If the ranks of E and F are both even then equation (7.12) is the sum from $k=0$ to $k=\frac{n+m}{2}$ of current equations of dimension $4k$. The dimension $4k$ part of equation (7.12) is obtained by intersecting equation (10.6) (when $N=n+m+2$) with the current $[G_2(\mathbf{R}^{2k+2})\times G_2(\mathbf{R}^{2k+2})]$.

If the rank of E is odd and the rank of F is even the equation (7.12) is the sum from $k=0$ to $k=\frac{n+m-1}{2}$ of $\mathcal{O}_{G_2(\underline{\mathbf{R}}^{N-1})}$-twisted current equations of dimension $4k+2$. The dimension $4k+2$ part of equation (7.12) is obtained by intersecting equation (10.6) (when $N=n+m+3$) with $[G_2(\mathbf{R}^{2k+3})\times G_2(\mathbf{R}^{2k+3})]$.

Finally, if the ranks of E and F are both odd then equation (7.12) is the sum from $k=0$ to $k=\frac{n+m}{2}$ of current equations of dimension $4k$. The dimension $4k$ part of equation (7.12) is obtained by intersecting equation (10.6) (when $N=n+m+4$) with the current $[G_2(\mathbf{R}^{2k+2}_{(2,\dots,2k+3)})\times G_2(\mathbf{R}^{2k+2}_{(2,\dots,2k+3)})]$ where $\mathbf{R}^{2k+2}_{(2,\dots,2k+3)}\subset \mathbf{R}^N$ by $(x_2,\dots,x_{2k+3})\hookrightarrow (0,x_2,\dots,x_{2k+3},0\dots,0)$.

The proof of equation (10.6) proceeds as follows. Let $e_1,\dots,e_N$ be the standard basis for $\mathbf{R}^N$. Define flags

$$\{0\}\ \subset V_1\ \subset\cdots\subset\ V_N=\mathbf{R}^N \qquad\text{and}\qquad \mathbf{R}^N=W_N\ \supset W_{N-1}\ \supset\cdots\supset\ \{0\}$$

by $V_i=\mathrm{Span}\{e_1,..,e_i\}$ and $W_{N-i}=V_i^{\perp}$. As in the complex case it suffices to show that there are integral currents T_1 and R so that

$$(10.7)\quad \partial T_1 \;=\; [\Delta] \;-\; \sum_{j=1}^{N/2} G_2(W_{N-2j+2})\times G_2(V_{2j}) \;-\; \tfrac{1}{2}\partial R \qquad \text{on } G_2(\mathbf{R}^N)\times G_2(\mathbf{R}^N).$$

This equation can be proved as follows. The linear flow on $\mathbf{R}^N$ defined by $\varphi_s(x_1,\dots,x_n)=(x_1,sx_2,\dots,s^{N-1}x_N)$ induces a Morse-Stokes flow, also denoted by φ_s, on $G_2(\mathbf{R}^N)$. This flow produces the standard Schubert cell decomposition of $G_2(\mathbf{R}^N)$. More specifically, the stationary points of φ_s are the 2-planes $P_{ij}=\mathrm{Span}\{e_i,e_j\}$, where $1\le i<j\le N-1$, and the closures of the stable and unstable manifolds of the stationary point P_{ij} are given by the Schubert cycles

$$\overline{S}_{ij} \;=\; \{P\in G_2(\mathbf{R}^N)\ /\ P\cap V_{i+1}\neq\{0\}\ ,\ P\subset V_{j+1}\}$$

of dimension $i+j-1$, and

$$\overline{U}_{ij} \;=\; \{P\in G_2(\mathbf{R}^N)\ /\ P\cap W_{N-j}\neq\{0\}\ ,\ P\subset W_{N-i}\}$$

of complementary dimension $2(N-2)-(i+j-1)$. Note that the unstable manifold of φ_s is equal to the stable manifold of the time reversed flow $\varphi_{\frac{1}{s}}$. By [HL4] there is an integral current T_1 on $G_2(\mathbf{R}^N)\times G_2(\mathbf{R}^N)$, so that

$$\partial T_1 \;=\; [\Delta]\;-\;P \qquad \text{where} \qquad P\;=\;\sum_{0\leq i<j\leq N-1}[U_{ij}]\times[S_{ij}].$$

Therefore equation (10.7) follows from the fact that

$$P \;=\; \sum_{k=1}^{N/2} G_2(W_{N-2k+2})\times G_2(V_{2k}) \;+\; \tfrac{1}{2}\partial R, \tag{10.8}$$

where

$$R \;:=\; \sum_{i,j\ \text{even}}[U_{ij}]\times[S_{i+1\,,\,j}] \;-\; \sum_{i,j\ \text{odd}}[U_{i-1\,,\,j}]\times[S_{i,j}]. \tag{10.9}$$

To prove equation (10.8) note that, since $\overline{S}_{2k-2,2k-1} \;=\; G_2(V_{2k})$ and $\overline{U}_{2k-2,2k-1} \;=\; G_2(W_{N-2k+2})$, it suffices to show that

$$\tfrac{1}{2}\partial R \;=\; \left(\sum_{i,j\ \text{even}} + \sum_{\substack{i\ \text{odd}\\ j\,\text{even}}} + \sum_{\substack{i\ \text{even}\\ j\neq i+1,\ \text{odd}}} + \sum_{i,j\ \text{odd}}\right)[U_{ij}]\times[S_{ij}]. \tag{10.10}$$

Thus we must calculate the boundaries of the stable and unstable manifolds. Ehresmann ([E],[Ch2]) inductively oriented the cells S_{ij} and calculated that $\partial S_{ij} \;=\; nS_{i-1,j}\;+mS_{i,j-1}$ where $n,m\in\{0,\pm 2\}$. In fact he showed that

$$\partial S_{ij} \;=\; \begin{cases} 0 & i,j\ \text{even}\\ 0 & i\ \text{even},\ j=i+1\\ 2S_{i,j-1} & i\ \text{even},\ j\neq i+1\ \text{odd}\\ -2S_{i-1,j} & i\ \text{odd},\ j\ \text{even}\\ -2S_{i-1,j}-2S_{i,j-1} & i,j\ \text{odd}\end{cases} \tag{10.11}$$

The calculation of ∂U_{ij} is somewhat more subtle. First recall that S_{ij} and U_{ij} intersect transversely in the point P_{ij} and that in [HL4] the orientations on U_{ij} were chosen so that

$$S_{ij}\bullet U_{kl} \;=\; \delta_{ik}\ \delta_{jl},$$

where the Kronecker index $A \bullet B$ counts the signed algebraic number of intersection points of A and B. It then follows [HL4, 5.1] that

$$\partial S_{ij} \bullet U_{kl} \;=\; (-1)^{\dim U_{kl}}\; S_{ij} \bullet \partial U_{kl}.$$

This fact, together with Ehresmann's calculation that $\partial U_{ij} \;=\; nU_{i+1,j} \;+ mU_{i,j+1}$, implies that

$$\partial U_{ij} \;=\; \begin{cases} 2U_{i+1,j} - 2U_{i,j+1} & i, j \text{ even} \\ 0 & i \text{ even, } j = i+1 \\ -2U_{i+1,j} & i \text{ even, } j \neq i+1 \text{ odd} \\ -2U_{i,j+1} & i \text{ odd, } j \text{ even} \\ 0 & i, j \text{ odd} \end{cases} \tag{10.12}$$

Equation (10.10) now follows from equations (10.9), (10.11) and (10.12) by a simple but tedious calculation. This concludes the proof of the proposition. $\square$

10C. Technical results required for the proof of the Product Theorem 6.4. In this subsection we present two technical results, Lemmas 10.19 and 10.26 below, which are required to complete the proof of the Product Theorem. We begin with a remark.

Remark 10.13. Products and slices of currents. In the proof of the Product Theorem it was necessary to multiply currents by divisors and by top Chern sparks of atomic sections. These products are defined by a smoothing process which we now describe.

In [HL2,III.4] the authors constructed a canonical family of **Thom forms**, τ_s, for $0 < s \leq \infty$, on a vector bundle $\pi : V \to X$ using a given connection D_V on V. Associated to this family of Thom forms is a family of smooth forms σ_s on V satisfying the equation $d\sigma_s = \pi^* c_n(\Omega_V) - \tau_s$ on V. (This equation is an application of the classical transgression formula of Chern-Weil theory [BCh1].) Because $\sigma_s \to \sigma$ in $L^1_{\text{loc}}(V)$ as $s \to 0$, the forms σ_s are called smooth **approximate spherical Chern potentials**.

Let γ be an atomic section of $V \to X$. Then, by [HL2],

$$\mathrm{Div}(\gamma) \;=\; \lim_{s\to 0} \gamma^*(\tau_s),$$

is the current limit of smooth forms on X, and

$$\sigma(\gamma) \;=\; \lim_{s\to 0} \gamma^*(\sigma_s) \qquad \text{in } L^1_{\text{loc}}(X).$$

The smooth approximations $\gamma^*(\tau_s)$ and $\gamma^*(\sigma_s)$ to the currents $\mathrm{Div}(\gamma)$ and $\sigma(\gamma)$ are related by the equation

$$d\gamma^*(\sigma_s) \;=\; c_n(\Omega_V) \;-\; \gamma^*(\tau_s) \tag{10.14}$$

involving smooth forms on X. Note that the local Gauss-Bonnet-Chern formula (2.2) is derived by taking the current limit as $s \to 0$ of equation (10.14).

Consequently it is natural to define the product, $S \wedge \mathrm{Div}(\gamma)$, of a current S with the divisor of an atomic section γ to be the current limit

$$(10.15) \qquad S \wedge \mathrm{Div}(\gamma) \ := \ \lim_{s \to 0} S \wedge \gamma^*(\tau_s),$$

if it exists; that is $S \wedge \mathrm{Div}(\gamma)$ is defined to be the **slice** of S by γ at 0, (see [F]). Similarly the product, $S \wedge \sigma(\gamma)$, of a current S with the top Chern spark, $\sigma(\gamma)$, of γ is defined to be the current limit

$$S \wedge \sigma(\gamma) \ := \ \lim_{s \to 0} S \wedge \gamma^*(\sigma_s),$$

if it exists.

The products of a current S with a divisor and with a top Chern spark of an atomic section γ of $V \to X$ are of particular interest in the case that S is the current of integration over a submanifold. Suppose that $S = [Z]$ is the current of integration over a smooth, oriented submanifold Z which is closed (as a subset of X), and suppose that the restriction $\gamma\,\big|_Z$ of γ to Z is an atomic section of $V \to Z$. Then the product of $[Z]$ with the divisor of γ exists and equals

$$(10.16) \qquad [Z] \wedge \mathrm{Div}(\gamma) \ = \ i_*(\mathrm{Div}(\gamma\,\big|_Z)) \qquad \text{on } X,$$

where $i : Z \hookrightarrow X$ is inclusion. Consequently, $[Z] \wedge \mathrm{Div}(\gamma)$ is a *locally integrally flat current* on X, by [HS,4.27]. Similarly, the product of $[Z]$ with the top Chern spark of γ exists and equals

$$(10.17) \qquad [Z] \wedge \sigma(\gamma) \ = \ i_*(\sigma(\gamma\,\big|_Z)),$$

and so is a *flat current* on X. These results follow immediately from the functorality of the Thom forms τ_s and of the approximate spherical Chern potentials σ_s.

More generally, suppose that $S = \mathrm{Div}(\alpha)$ is itself the divisor of an atomic section α. In this case we have two possible definitions of the product $\mathrm{Div}(\alpha) \wedge \mathrm{Div}(\gamma)$, the one given by equation (10.15) and, in the case that the section (α, γ) is atomic, the definition $\mathrm{Div}(\alpha) \wedge \mathrm{Div}(\gamma) := \mathrm{Div}(\alpha, \gamma)$ of equation (2.5). One would hope that, whenever (α, γ) is atomic, these two definitions would agree, i.e. that the slice of the current $\mathrm{Div}(\alpha)$ by γ at 0 exists and equals $\mathrm{Div}(\alpha, \gamma)$. Although we do not know if this is true in general some partial results, which are sufficient for our purposes, were obtained in [HZ2].

One situation in which the two definitions agree is the case in which the section α has zero as a regular value. In this case $S = \mathrm{Div}(\alpha) = [Z]$ is the current of integration over the zero submanifold Z of α (by [HS]), and so the results of the penultimate paragraph hold. In addition, by Theorem 9.12 of [HZ2], the section (α, γ) is atomic if and only if $\gamma\,\big|_Z$ is atomic, in which case the product of $[Z]$ with the divisor of γ exists and equals

$$[Z] \wedge \mathrm{Div}(\gamma) \ = \ i_*(\mathrm{Div}(\gamma\,\big|_Z)) \ = \ \mathrm{Div}(\alpha, \gamma).$$

Definition 10.18. Two collections α and β of sections of bundles E and F over X are said to be **cleanly related** if
(1) they satisfy the hypotheses of the Product Theorem 6.4,
(2) the slice, $\mathbf{I} \wedge \mathrm{Div}(\widehat{\alpha}, \widehat{\beta})$, of the current $\mathbf{I}$ of Proposition 6.16 by the section $(\widehat{\alpha}, \widehat{\beta})$ at 0 exists and is locally rectifiable, and
(3) the flat currents $\mathbf{I} \wedge (\widehat{\alpha}, \widehat{\beta})^*(\tau_s)$ converge to the slice $\mathbf{I} \wedge \mathrm{Div}(\widehat{\alpha}, \widehat{\beta})$ in the flat topology as $s \to 0$.

The following result is used in Step Two of the proof of the Product Theorem.

Lemma 10.19. *Let α, β be cleanly related collections of sections of E and F.*
(1) The product of the diagonal flag current $\Delta_*(\mathbf{F})$ with the divisor $\mathrm{Div}\,(\widehat{\alpha}, \widehat{\beta})$ and with the top Chern spark $\sigma(\widehat{\alpha}, \widehat{\beta})$ exist and equal

$$\Delta_*(\mathbf{F}) \wedge \mathrm{Div}\,(\widehat{\alpha}, \widehat{\beta}) \;=\; \Delta_*(\mathbf{F} \wedge \mathrm{Div}\,\widehat{(\alpha, \beta)}) \tag{10.20}$$

$$\Delta_*(\mathbf{F}) \wedge \sigma(\widehat{\alpha}, \widehat{\beta}) \;=\; \Delta_*(\mathbf{F} \wedge \sigma\widehat{(\alpha, \beta)}) \tag{10.21}$$

on $\mathbf{P}_E^{n+m} \times \mathbf{P}_F^{n+m} \times X$.
(2) The current equation

$$d(\mathbf{I} \wedge \mathrm{Div}(\widehat{\alpha}, \widehat{\beta})) \;=\; \Delta_*(\mathbf{F}) \wedge \mathrm{Div}(\widehat{\alpha}, \widehat{\beta}) \;-\; \mathbf{G} \wedge \mathrm{Div}(\widehat{\alpha}, \widehat{\beta}) \tag{10.22}$$

holds on $\mathbf{P}_E^{n+m} \times \mathbf{P}_F^{n+m} \times X$.
(3) Let $\sigma = \sigma(\widehat{\alpha}, \widehat{\beta})$, $D = \mathrm{Div}(\widehat{\alpha}, \widehat{\beta})$, $c_n = p_E^* c_n(\Omega_{\mathrm{Hom}(U_E, E)})$ and $c_m = p_F^* c_m(\Omega_{\mathrm{Hom}(U_F, F)})$. Then there is an L^1_{loc} current K so that

$$dK \;=\; \Delta_*(\mathbf{F}) \wedge \sigma \;-\; \mathbf{G} \wedge \sigma \;-\; \mathbf{I} \wedge c_n \wedge c_m \;+\; \mathbf{I} \wedge D \tag{10.23}$$

on $\mathbf{P}_E^{n+m} \times \mathbf{P}_F^{n+m} \times X$.

Proof. Part (1) follows immediately from the fact that $\widehat{(\alpha, \beta)} = \Delta^*(\widehat{\alpha}, \widehat{\beta})$ as sections of $\mathrm{Hom}(U, E \oplus F) \to \mathbf{P}^{n+m} \times X$, and from the functorality of the Thom forms τ_s and of the approximate spherical Chern potentials σ_s (see [HL2]). Part (2) follows by multiplying equation (6.17) by the approximation $(\widehat{\alpha}, \widehat{\beta})^*(\tau_s)$ to $\mathrm{Div}\,(\widehat{\alpha}, \widehat{\beta})$ and taking the current limit as $s \to 0$.

The proof of Part (3) proceeds as follows. Let T denote the right hand side of equation (10.23). First we will show that there is a flat current J so that $dJ = T$. Then, by equations (6.17) and (10.14),

$$d(\mathbf{I} \wedge (\widehat{\alpha}, \widehat{\beta})^*(\sigma_s)) \;=\; \Delta_*(\mathbf{F}) \wedge (\widehat{\alpha}, \widehat{\beta})^*(\sigma_s) \;-\; \mathbf{G} \wedge (\widehat{\alpha}, \widehat{\beta})^*(\sigma_s) \;-\; \mathbf{I} \wedge c_n \wedge c_m \;+\; \mathbf{I} \wedge (\widehat{\alpha}, \widehat{\beta})^*(\tau_s). \tag{10.24}$$

Now, by assumption, the right hand side, T_s, of equation (10.24) is a locally flat current which converges to T in the flat topology as $s \to 0$. Furthermore equation (10.24) states that the current T_s belongs to the range of the exterior derivative operator d mapping the space of locally flat currents to itself. Now it is a standard fact that the operator d acting on the space of locally flat currents on a paracompact smooth manifold has closed range. Consequently there is a flat current J so that $T = dJ$. (This standard fact is part of theorem that the cohomology of the complex of locally flat currents is dual to the de Rham cohomology of smooth forms on a paracompact manifold. The proof, which is via Cech theory, follows from the fact that all subspaces of the infinite product $\prod_{j=1}^{\infty} \mathbf{R}$ are closed.)

To complete the proof of Part (3) simply let K be an $L^1_{\rm loc}$ current which has the same exterior derivative as the locally flat current J. (Such an $L^1_{\rm loc}$ current K exists since flat currents are those of the form $J = K + d\widetilde{K}$, where $K, \widetilde{K}$ are $L^1_{\rm loc}$.) $\square$

The second technical result which is used in the proof of the Product Theorem that "almost all" collections α, β of sections of E and F over X are cleanly related. More precisely, we proceed as follows.

Since a complex vector bundle is locally trivial a partition of unity on the base manifold can be used to construct a bundle surjection $A : \underline{\mathbf{C}}^K \to E$ from a trivial bundle $\underline{\mathbf{C}}^K$ to E. Each fixed point $u \in \mathbf{C}^K$ determines a constant section, also denoted by u, of $\underline{\mathbf{C}}^K$ over X. A global section α of E can now be embedded into a K-dimensional family of global sections by

$$\alpha_u \ := \ \alpha \ + \ \ Au \qquad \text{on } E.$$

Consequently a collection $\alpha = (\alpha^1, \ldots, \alpha^{n+m+1})$ of $n+m+1$ sections of E can be embedded into a family of collections of global sections

$$\alpha_u \ := (\alpha^1 + Au^1, \ldots, \alpha^{n+m+1} + Au^{n+m+1}) \ = \ \alpha + Au, \tag{10.25}$$

where each $u^j \in \mathbf{C}^K$ and $u = (u^1, \ldots, u^{n+m+1}) \in \mathbf{C}^N$, with $N = K(n+m+1)$.

Similarly, a collection β of sections of F can be embedded into a family $\beta_v = \beta + Bv$, where $v \in \mathbf{C}^M$ and $B : \mathbf{C}^L \to F$ is a bundle surjection.

Then we have the following lemma.

Lemma 10.26. *Let α, β be collections of $n+m+1$ sections of E, F. Then, for almost all $u \in \mathbf{R}^N$ and $v \in \mathbf{R}^M$, the collections α_u and β_v of sections of E and F are cleanly related.*

Proof of Lemma 10.26. The proof relies on the fact that there are local coordinates z on $\mathbf{P}^k$ and $u = (u', u'') \in \mathbf{C}^n \times \mathbf{C}^{N-n}$ on $\mathbf{C}^N$ so that the section $\widehat{\alpha}_u$ of $\operatorname{Hom}(U, E) \to \mathbf{P}^k \times X$ is locally of the form

$$\widehat{\alpha}_u(z, x) \ = \ F_{u''}(z, x) \ - \ u',$$

where $F_{u''}(z, x)$ is an $\mathbf{C}^n$-valued function.

Part (1) of Definition 10.18 now follows immediately from Sard's Theorem and the fact that a function which has zero as a regular value is atomic. Parts (2) and (3) of the Definition follows from a theorem of Federer [F, 4.3.6] which states that, for almost all $u' \in \mathbf{C}^n$, the slice of a rectifiable current $\mathbf{I}$ by an $\mathbf{C}^n$-valued function F at u' exists and is rectifiable, and furthermore that the approximations to the slice converge in the flat topology. $\square$

10D. Proof of Proposition 8.35.

Let $\omega = (\omega_{ij})$ be the 3×3 skew symmetric matrix of connection 1-forms of D_F on the orthonormal frame bundle $\mathcal{F}$, and let $\Omega = d\omega - \omega \wedge \omega$ denote the curvature matrix. The Chern-Simons 3-form is defined by

$$TP_1(D_F) \;=\; \tfrac{1}{4\pi^2}\left(\omega_{12}\,\Omega_{12} \;+\; \omega_{13}\,\Omega_{13} \;+\; \omega_{23}\,\Omega_{23} \;-\; \omega_{12}\,\omega_{13}\,\omega_{23}\right).$$

Recall that the first Ronga-Pontrjagin spark, $S_1(\alpha)$, is defined as follows. Let $G_2(\mathbf{R}^3)$ denote the Grassmannian of *oriented* 2-planes in $\mathbf{R}^3$ and let $U \to G_2(\mathbf{R}^3)$ be the tautological rank 2 bundle. As in Section 7 let $\sigma(\widehat{\alpha})$ denote the Euler spark of the induced section $\widehat{\alpha}$ of $\mathrm{Hom}(U, F) \to G_2(\mathbf{R}^3) \times X$. Then

$$S_1(\alpha) \;=\; \tfrac{1}{2}\pi_*\sigma(\widehat{\alpha}),$$

where $\pi : G_2(\mathbf{R}^3) \times X \to X$ is projection.

The proof of the proposition is simply an explicit calculation of $S_1(\alpha)$ which will be performed by deriving an explicit local expression for $\sigma(\widehat{\alpha})$, factoring out the volume form of the fibre $G_2(\mathbf{R}^3)$ from $\sigma(\widehat{\alpha})$ and then calculating the fibre integral.

Let $\epsilon = (\epsilon_1, ..., \epsilon_6)^t$ be a positively oriented orthonormal local frame for the rank 6 bundle $H := \mathrm{Hom}(U, F)$ and define $u = (u_1, ..., u_6)$ by $\widehat{\alpha} = u\epsilon$. Define the connection 1- form ω_H by $D_H\epsilon = \omega_H\epsilon$ and set $Du = du + u\omega_H$ and $\lambda(\epsilon) = \epsilon_1 \wedge \cdots \wedge \epsilon_6$. Then by [HL2, p186]

$$\sigma(\widehat{\alpha})\,\lambda(\epsilon) \;=\; \tfrac{1}{(2\pi)^3}\;\left(J_0 \;+\; J_1 \;+\; J_2\right),$$

where

$$\begin{aligned} J_0 \;&=\; -\tfrac{1}{8}\tfrac{1}{|u|^2}(u\epsilon)\,(Du\epsilon)\,(\epsilon^t\Omega_H\epsilon)^2 \\ J_1 \;&=\; \tfrac{1}{6}\tfrac{1}{|u|^4}(u\epsilon)\,(Du\epsilon)^3\,(\epsilon^t\Omega_H\epsilon) \\ J_2 \;&=\; -\tfrac{1}{15}\tfrac{1}{|u|^6}(u\epsilon)\,(Du\epsilon)^5. \end{aligned} \tag{10.27}$$

Let $I_k\,\lambda(\epsilon) = \frac{1}{2}\frac{1}{(2\pi)^3}\int_{G_2(\mathbf{R}^3)} J_k$, so that

$$S_1(\alpha) \;=\; I_0 \;+\; I_1 \;+\; I_2. \tag{10.28}$$

Next we choose a local frame ϵ for H. First recall the isomorphism of $G_2(\mathbf{R}^3)$ with the unit sphere S^2 in $\mathbf{R}^3$ given by sending an oriented 2-plane to its unit normal vector. We will use the local coordinate systems

$$\begin{aligned} \mathbf{R}^2 &\to G_2(\mathbf{R}^3) \\ t &\mapsto (t\,,\,\pm 1)^{\perp} = U_t. \end{aligned}$$

Let $t = (t_1, t_2) = r\,e^{i\theta}$, and note that $U_t = \operatorname{span}\{e_1, e_2\}$ where $e_1 = (1, 0, \mp t_1)$ and $e_2 = (0, 1, \pm t_2)$. Consequently the sections

$$E_1 = \frac{t_2 e_1 - t_1 e_2}{|t_2 e_1 - t_1 e_2|} \qquad \text{and} \qquad E_2 = \frac{t_1 e_1 + t_2 e_2}{|t_1 e_1 + t_2 e_2|}$$

form a positively oriented orthonormal frame for U. Let

$$\begin{aligned} c &= \cos\theta & s &= \sin\theta \\ C = \cos\psi &= \frac{1}{\sqrt{1+r^2}} & S = \sin\psi &= \frac{r}{\sqrt{1+r^2}}, \end{aligned}$$

and note that almost all of $G_2(\mathbf{R}^3)$ is parametrized by (θ, ψ) in the range $0 \le \theta < 2\pi$ and $0 < \psi < \pi$. Then

$$E_1 = (s\,,\,-c\,,\,0) \qquad E_2 = (Cc\,,\,Cs\,,\,-S), \tag{10.29}$$

and the position vector $E_3 = U^{\perp}$ of S^2 is given by

$$E_3 = (Sc\,,\,Ss\,,\,C). \tag{10.30}$$

Finally, since α is a global frame for F over $\mathcal{F}$ it follows that $\epsilon = \begin{pmatrix} \epsilon' \\ \epsilon'' \end{pmatrix}$ is a local frame for H where $\epsilon' = \alpha \otimes E_1^*$ and $\epsilon'' = \alpha \otimes E_2^*$. Consequently the local expression u for $\widehat{\alpha}$ defined by $\widehat{\alpha} = u\epsilon$ is

$$u = (E_1, E_2),$$

and so $|u|^2 = 2$.

A straightforward calculation shows that

$$\omega_H = \begin{pmatrix} \omega & C d\theta\, I_3 \\ -C d\theta\, I_3 & \omega \end{pmatrix},$$

and so

$$\epsilon^t \Omega_H \epsilon = 2({\epsilon''}^t \epsilon')\, S d\psi d\theta + {\epsilon'}^t \Omega \epsilon' + {\epsilon''}^t \Omega \epsilon'', \tag{10.31}$$

$$D u \epsilon = (E_3 \epsilon') S d\theta - (E_3 \epsilon'') d\psi + E_1 \omega \epsilon' + E_2 \omega \epsilon''. \tag{10.32}$$

Note that $\Theta := S d\psi d\theta$ is the standard volume form on $G_2(\mathbf{R}^3) \cong S^2$. Next we calculate J_0, J_1 and J_2 discarding those terms which do not have the volume form Θ as a factor.

First we show that $J_2 = 0$. Since J_2 has $(Du\epsilon)^5$ as a factor it suffices to show that $(Du\epsilon)^5 = 0$. Now by equation (10.32)

$$(Du\epsilon)^5 \;=\; 60(E_3\epsilon')(E_3\epsilon'')\{\,(E_1\omega\epsilon')^2\,(E_2\omega\epsilon'') \;+\; (E_1\omega\epsilon')\,(E_2\omega\epsilon'')^2\}\;\Theta,$$

which is zero because if A and B are rows of three functions then

$$(A\epsilon')\,(B\omega\epsilon')^2 \;=\; 0 \qquad \text{whenever } AB^t = 0.$$

To calculate J_0 and J_1 we use the following lemma, whose proof is left to the reader.

Lemma 10.33. *Let A and B be rows of three functions, let G be a row of three 1-forms and let $Z = (\omega_{23}, -\omega_{13}, \omega_{12})$.*
If $|A| = 1$ then

(1) $(A\epsilon')\,(A\omega\epsilon')\,(B\epsilon') \;=\; A\omega\,(B\times A)^t\;\lambda(\epsilon')$,
(2) $(A\epsilon')\,(A\omega\epsilon')\,(\epsilon''^t\epsilon') \;=\; (Z-(AZ^t)A)\,\epsilon''\;\lambda(\epsilon')$, *and*
(3) $(G\epsilon')\,(A\omega\epsilon')^2 \;=\; 2\,GA^t\,(A_1\,\omega_{12}\,\omega_{13} \;+\; A_2\,\omega_{12}\,\omega_{23} \;+\; A_3\,\omega_{13}\,\omega_{23})\;\;\lambda(\epsilon')$.

By equations (10.27, 10.31, 10.32) and part (2) of the Lemma,

$$\begin{aligned}
J_0 \;&=\; \tfrac14\{(E_1\epsilon'')\,(E_1\omega\epsilon'') \;+\; (E_2\epsilon'')\,(E_2\omega\epsilon'')\}(\epsilon'^t\epsilon'')\,(\epsilon'^t\Omega\epsilon')\;\Theta\\
&=\; \tfrac14\{2Z \;-\; (E_1Z^t)E_1 \;-\; (E_2Z^t)E_2\}\epsilon'\,(\epsilon'^t\Omega\epsilon')\;\Theta\;\lambda(\epsilon'')\\
&=\; \tfrac14\{\big((1+S^2c^2)\omega_{23},\; -(1+S^2s^2)\,\omega_{13},\; (1+C^2)\omega_{12}\big)\,\epsilon'\}\,(\epsilon'^t\Omega\epsilon')\;\Theta\;\lambda(\epsilon'')\\
&=\; \tfrac12\{\,(1+C^2)\omega_{12}\,\Omega_{12} \;+\; (1+S^2s^2)\omega_{13}\,\Omega_{13} \;+\; (1+S^2c^2)\omega_{23}\,\Omega_{23}\}\Theta\;\lambda(\epsilon).
\end{aligned}$$

Consequently

$$I_0 \;=\; \tfrac12\,\tfrac{1}{(2\pi)^3}\int_{S^2} J_0 \;=\; \tfrac{1}{6\pi^2}\,(\omega_{12}\,\Omega_{12} \;+\; \omega_{13}\,\Omega_{13} \;+\; \omega_{23}\,\Omega_{23})\,. \tag{10.34}$$

Finally we calculate J_1. By equations (10.27, 10.31, 10.32),

$$J_1 \;=\; (J_1^A \;+\; J_1^B)\;\Theta,$$

where

$$\begin{aligned}
J_1^A \;&=\; \tfrac14(u\epsilon)\,(E_3\epsilon')\,(E_3\epsilon'')\,\{E_1\omega\epsilon' \;+\; E_2\omega\epsilon''\}\,(\epsilon'^t\Omega\epsilon' \;+\; \epsilon''^t\Omega\epsilon'')\\
&= \tfrac14\{\,(E_2\epsilon'')\,(E_2\omega\epsilon'')\,(\epsilon'^t\Omega\epsilon') \;+\; (E_1\epsilon')\,(E_1\omega\epsilon')\,(\epsilon''^t\Omega\epsilon'')\,\}\;(E_3\epsilon')\,(E_3\epsilon'')\\
&=\; \tfrac14\{\,(E_1\epsilon'')\,(E_1\omega\epsilon'') \;+\; (E_2\epsilon'')\,(E_2\omega\epsilon'')\,\}\;(E_3\epsilon')\,(E_3\epsilon'')\,(\epsilon'^t\Omega\epsilon'),
\end{aligned}$$

and

$$J_1^B \;=\; \tfrac{1}{12}(u\epsilon)\,(E_1\omega\epsilon' \;+\; E_2\omega\epsilon'')^3\;\;({\epsilon''}^t\epsilon')$$

$$= \tfrac{1}{4}\{\,(E_2\epsilon'')\,(E_2\omega\epsilon'')\,(E_1\omega\epsilon')^2 \;+\; (E_1\epsilon')\,(E_1\omega\epsilon')\,(E_2\omega\epsilon'')^2\}({\epsilon''}^t\epsilon'). \tag{10.35}$$

To calculate J_1^A we use part (1) of the Lemma to show that

$$\begin{aligned}
&\{\,(E_1\epsilon'')\,(E_1\omega\epsilon'') \;+\; (E_2\epsilon'')\,(E_2\omega\epsilon'')\,\}\;E_3\epsilon'' \\
&\quad= \;\{E_1\omega(E_3\times E_1)^t \;+\; E_2\omega(E_3\times E_2)^t\}\;\lambda(\epsilon'') \\
&\quad= \;2\,E_1\omega E_2^t\;\lambda(\epsilon'') \\
&\quad= \;2\,\{(E_1\times E_2)_1\,\omega_{23} \;+\; (E_1\times E_2)_2\,\omega_{31} \;+\; (E_1\times E_2)_3\,\omega_{12}\} \\
&\quad= \;2\,(E_{31}\,\omega_{23} \;+\; E_{32}\,\omega_{31} \;+\; E_{33}\,\omega_{12}).
\end{aligned}$$

Since $E_3 = (t_1, t_2, t_3)$ is the position vector of S^2 it follows that

$$J_1^A \;=\; t_3^2\,\omega_{12}\,\Omega_{12} \;+\; t_2^2\,\omega_{13}\,\Omega_{13} \;+\; t_1^2\,\omega_{23}\,\Omega_{23},$$

and so

$$I_1^A \;=\; \tfrac{1}{2}\tfrac{1}{(2\pi)^3}\int_{S^2} J_1^A\,\Theta \;=\; \tfrac{1}{12\pi^2}\,(\omega_{12}\,\Omega_{12} \;+\; \omega_{13}\,\Omega_{13} \;+\; \omega_{23}\,\Omega_{23})\,. \tag{10.36}$$

Finally the first term of J_1^B in equation (10.35) is

$$\begin{aligned}
&\tfrac{1}{4}\,(E_2\epsilon'')\,(E_2\omega\epsilon'')\,(E_1\omega\epsilon')^2\,({\epsilon''}^t\epsilon') \\
&= \tfrac{1}{4}\{\big((E_2Z^t)E_2 \;-\; Z\big)\,\epsilon'\}\;(E_1\omega\epsilon')^2\;\lambda(\epsilon'') \qquad \text{by (2) of Lemma,} \\
&= \tfrac{1}{2}\{\,(E_2Z^t)E_2 \;-\; Z\}E_1^t\;(s\omega_{12}\,\omega_{13} \;-\; c\omega_{12}\,\omega_{23})\;\lambda(\epsilon) \qquad \text{by (3) of Lemma and (10.29),} \\
&= \;-\;\tfrac{1}{2}(s\omega_{23} \;+c\omega_{13})\;(s\omega_{12}\,\omega_{13} \;-\; c\omega_{12}\,\omega_{23})\;\lambda(\epsilon) \\
&= \;-\;\tfrac{1}{2}\,\omega_{12}\,\omega_{13}\,\omega_{23}.
\end{aligned}$$

The calculation of the second term of J_1^B is similar and also equals $-\,\tfrac{1}{2}\omega_{12}\,\omega_{13}\,\omega_{23}$. Consequently

$$I_1^B \;=\; \tfrac{1}{2}\tfrac{1}{(2\pi)^3}\int_{S^2} J_1^B\Theta \;=\; -\,\tfrac{1}{4\pi^2}\omega_{12}\,\omega_{13}\,\omega_{23}. \tag{10.37}$$

The proposition now follows from equations (10.28) and (10.34, 10.36, 10.37). $\square$

10E. Further Discussion and Proof of Theorem 8.31.

Let $\alpha : \underline{\mathbf{C}}^m \to F$ be a collection of m sections of a rank n complex bundle F with connection D_F, and let $p = n - m + 1$. The singular pushforward connection $\overrightarrow{D}$ associated with α (which was defined in [HL2, I.2.10]) is a smooth connection on the complement of the linear dependency set Σ of α in X. Rather than attempting to make sense of $\overrightarrow{D}$ across Σ Harvey and Lawson defined a smooth approximating family of pushforward connections $\overrightarrow{D}_s$ (for $0 < s \le \infty$) on F over all of X such that $\overrightarrow{D}_s \to \overrightarrow{D}$ over $X \sim \Sigma$ as $s \to 0$, and $\overrightarrow{D}_\infty = D_F$ over all of X. The construction of the family $\overrightarrow{D}_s$ relies on the choice of an approximation mode. Here we choose the algebraic approximation mode.

If the bundle map α is geometrically atomic (see [HL5]), then the **pth Chern current of the pushforward connection of** α,

$$c_p(\!(\overrightarrow{D})\!) \;:=\; \lim_{s\to 0} c_p(\overrightarrow{\Omega}_s),$$

exists and is a flat current on X. Furthermore there is a canonically defined functorial $L^1_{\rm loc}$ transgression form T such that

$$c_p(\Omega_F) \;-\; c_p(\!(\overrightarrow{D})\!) \;=\; dT \qquad \text{on } X.$$

One of the basic unresolved problems of [HL5] is to calculate the pth Chern current $c_p(\!(\overrightarrow{D})\!)$ for an arbitrary geometrically atomic α. We propose that this could be done as follows.

Recall that, if the collection α is $\mathbb{LD}$-atomic, then the pth Chern spark, $S_p(\alpha)$, is an $L^1_{\rm loc}$ form such that

$$c_p(\Omega_F) \;-\; \mathbb{LD}(\alpha) \;=\; dS_p(\alpha) \qquad \text{on } X.$$

Therefore it is very natural to formulate the following conjecture.

Conjecture 10.38. *Suppose that α is both $\mathbb{LD}$-atomic and geometrically atomic. Then*

$$T \;=\; S_p(\alpha) \qquad \textit{in } L^1_{\rm loc}(X).$$

Consequently the pth Chern current of the pushforward connection of α is equal to the linear dependency current of α:

$$c_p(\!(\overrightarrow{D})\!) \;=\; \mathbb{LD}(\alpha) \qquad \textit{on } X.$$

Note. Calculations by the third author have established this conjecture in some special cases. For instance the conjecture is true if either $m = n$ or $m = 2$. Rather than presenting these calculations we prove the conjecture in the case that α is a collection of *orthonormal* sections of F, i.e. we prove Theorem 8.31.

To prove Theorem 8.31 it is sufficient to establish the conjecture in the case that α is a collection of *orthonormal* section of F, in which case T and $S_p(\alpha)$ are smooth forms. The transgression T and the spark $S_p(\alpha)$ are both defined as integrals. Our approach to proving Theorem 8.31 is to do both integrals and show that the resulting explicit formulae for T and $S_p(\alpha)$ are equal. We begin by calculating the transgression form T.

Calculation of the transgression form T.

Let $e = (e_1, \dots e_m)^t$ and $f = (f_1, \dots, f_n)^t$ be orthonormal (local) frames for $\underline{\mathbf{C}}^m$ and F and define a local $m \times n$ matrix a for α by $\alpha e = af$. Since α is an orthonormal collection of sections the frames e and f can be chosen so that $a = (\mathrm{Id}_m \mid 0)$. Let $a^* := a^t$. Then the curvature matrix of $\overrightarrow{D}_s$ is given by

$$(10.39) \qquad \overrightarrow{\Omega}_s \;=\; s^2(s^2 + a^*a)^{-1} \left(\Omega_F \;-\; Da^*(s^2 + aa^*)^{-1}Da\right),$$

where

$$(10.40) \qquad Da \;=\; da + a\omega_F \qquad \text{and} \qquad Da^* \;=\; da^* - \omega_F a^*.$$

Also let $\lambda(f) = f_1 \wedge f_1^* \wedge \cdots \wedge f_n \wedge f_n^*$, where f^* is the dual frame for F^*. Then

$$(10.41) \quad T\,\lambda(f) \;=\; (\tfrac{i}{2\pi})^p \tfrac{p}{n!} \binom{n}{p} \int_0^\infty (f^*f)^{m-1} \left[f^*a^*(s^2 + aa^*)^{-2}Daf\right] (f^*\overrightarrow{\Omega}_s f)^{p-1}\, d(s^2).$$

To utilize the simple structure of the matrix a we block f and ω as

$$(10.42) \qquad f \;=\; \begin{pmatrix} f_V \\ f_W \end{pmatrix} \qquad\qquad f^* \;=\; \begin{pmatrix} f_H & f_K \end{pmatrix}$$

$$(10.43) \qquad \omega \;=\; \begin{pmatrix} \omega_V \\ \omega_W \end{pmatrix} \;=\; \begin{pmatrix} \omega_H & \omega_K \end{pmatrix},$$

where f_V is $m \times 1$, ω_V is $m \times n$, etc. After the change of variables $x = \frac{1}{1+s^2}$ equation (10.41) becomes

(10.44)

$$T\,\lambda(f) \;=\; (\tfrac{i}{2\pi})^p \tfrac{p}{n!} \binom{n}{p} \int_0^\infty (f^*f)^{m-1}\, [f_H^*\omega_V f]\, \{((1-x)f_H^*, f_K^*)(\Omega_F \;-\; x\omega_H\omega_V)f\}^{p-1} dx.$$

Because the expression we will obtain for $S_p(\alpha)$ does not naturally include the terms involving f_K^* we need the following identity whose proof is left to the reader.

Lemma 10.45. *Let Y be $m\times 1$ and $X = \begin{pmatrix} X_V \\ X_W \end{pmatrix}$, where X_V is $m\times 1$ and X_W is $(n-m)\times 1$. Suppose that X_i and Y_j commute. Then*

(10.46)

$$(f^*f)^{m-1} f_H^* Y\{(1-x)f_H^* X_V + f_K^* X_W\}^{p-1} \;=\; \{(1-x)f^*f + xf_H^* f_V\}^{m-1} f_H^* Y\,(f^*X)^{p-1}.$$

It is convenient to define

$$\Omega \;:=\; f^*\Omega_F f \qquad R \;:=\; f_H^*\omega_V f \qquad Q \;:=\; f^*\omega_H\omega_V f. \tag{10.47}$$

Then, applying Lemma 10.45 to equation (10.44), expanding out and using the fact that $\int_0^1 (1-x)^k\, x^{r-k} dx = \frac{1}{r+1-k}\frac{1}{\binom{r+1}{k}}$ we obtain the required formula

$$T\;\lambda(f) \;=\; \sum_{k=0}^{m-1}\sum_{\ell=0}^{p-1} C_{k\ell}(f^*f)^k\,(f_H^* f_V)^{m-1-k}\,\Omega^\ell\, R\, Q^{p-1-\ell}, \tag{10.48}$$

where

$$C_{k\ell} \;=\; (-1)^{p-1-\ell}(\tfrac{i}{2\pi})^p\,\tfrac{p}{n!}\binom{n}{p}\;\frac{\binom{m-1}{k}\binom{p-1}{\ell}}{(n-\ell-k)\binom{n-\ell}{k}}. \tag{10.49}$$

Calculation of the pth Chern spark $S_p(\alpha)$.

Recall from equation (5.3) that the pth Chern spark is defined by

$$S_p(\alpha) \;=\; \int_{\mathbf{P}(\mathbf{C}^m)} \sigma(\widehat{\alpha}), \tag{10.50}$$

where $\widehat{\alpha}$ is the induced section of $\mathrm{Hom}(U,F) \to \mathbf{P}(\mathbf{C}^m)\times X$. The calculation of the fibre integral in equation (10.50) has three main steps, which can be outlined as follows.

Step 1. Let z be a local homogeneous coordinate on $\mathbf{P}(\mathbf{C}^m)$ and set

$$z^* \;=\; \frac{\overline{z}^{\,t}}{|z|^2}, \tag{10.51}$$

so that $zz^* = 1$. Then the volume form on $\mathbf{P}(\mathbf{C}^m)$ is given by $dV(z) = (\frac{i}{2\pi}\,dzdz^*)^{m-1}$. Let $\alpha e = af$ define the local matrix a for α as above. The first step is to express $\sigma(\widehat{\alpha})$ in terms of Ω_F and z, z^*, a, a^* and their differentials.

Step 2. Isolate the terms in $\sigma(\widehat{\alpha})$ which involve the volume form $dV(z)$ of $\mathbf{P}(\mathbf{C}^m)$, and thereby reduce the problem to that of calculating the integral

$$\int_{\mathbf{P}(\mathbf{C}^m)} (zMz^*)^q dV(z), \tag{10.52}$$

where M is an $m\times m$ matrix.

Step3. Use a theorem of Bott [B1] to evaluate the integral (10.52) and so obtain the required explicit expression for $S_p(\alpha)$.

In more detail we proceed as follows.

Step 1. If e is a local frame for $\underline{\mathbf{C}}^m$ then $E = ze$ is a local frame for the tautological bundle $U \to \mathbf{P}(\underline{\mathbf{C}}^m)$. Let $u = (u_1, \dots u_n)$ be defined by $\widehat{\alpha} = u(E^* \otimes f)$, and let Ω_H be the curvature of the induced connection on $\mathrm{Hom}(U, F)$. Then, by [HL2, III.4.28],

$$\sigma(\widehat{\alpha})\lambda(f) \;=\; (\tfrac{i}{2\pi})^n \tfrac{1}{(n-1)!} \int_0^1 I(x)dx, \tag{10.53}$$

where

$$I(x) \;=\; \frac{f^*u^*Duf}{|u|^2} \left\{ f^*\Omega_H f \;-\; x\frac{f^*Du^*Duf}{|u|^2} \right\}^{n-1}. \tag{10.54}$$

Using the notation of equations (10.40) and (10.51) we see that

$$\begin{array}{ccc} u = za \qquad u^* = a^*z^* & & \Omega_H \;=\; \Omega_F \;-\; dzdz^* \\ Du = zDa + dz(1 - z^*z)a & & Du^* = Da^*z^* + a^*(1 - z^*z)dz^*. \end{array}$$

For simplicity set

$$\begin{array}{ccc} b \;=\; \frac{f^*a^*z^*}{|za|} & \Omega \;=\; f^*\Omega_F f & \beta \;=\; dzdz^*\, f^*f \\ A \;=\; \frac{zDaf}{|za|} & & A^* \;=\; \frac{f^*Da^*z^*}{|za|} \\ Z \;=\; \frac{dz(1-z^*z)af}{|za|} & & Z^* \;=\; \frac{f^*a^*(1-z^*z)dz^*}{|za|}. \end{array}$$

Then

$$I(x) \;=\; b(A + Z)\left\{\Omega + \beta - x(A^* + Z^*)(A + Z)\right\}^{n-1}. \tag{10.55}$$

Step 2. Our aim here is to isolate the terms of top degree $m-1$ (in both dz and dz^*) in the integrand $I(x)$. First, by carefully chosen repeated applications of the Binomial Theorem, we find that the top degree term of $I(x)$ is

$$I(x) \;=\; \sum_{k=0}^{m-1}\sum_{\ell=0}^{p-1} \gamma_{k\ell}\, x^{n-1-k-\ell}\, b\, \Omega^\ell\, (A^*)^{p-1-\ell} A^{p-\ell} \beta^k (Z^*Z)^{m-1-k}, \tag{10.56}$$

where

$$\gamma_{kl} \;=\; (-1)^{n-1-k-\ell} \binom{n-1}{k+\ell} \binom{k+\ell}{\ell} \binom{n-\ell-k}{m-1-k} \binom{n-1-\ell-k}{m-1-k}. \tag{10.57}$$

To factor the volume form out of equation (10.56) we perform a long calculation which shows that

$$\beta^k (Z^*Z)^{m-1-k} \;=\; \frac{(-1)^{m-1-k}}{\binom{m-1}{k}} \,(f^*f)^k \,\left\{\frac{f^*a^*(1-z^*z)af}{|za|^2}\right\}^{m-1-k} dV(z). \tag{10.58}$$

(This calculation uses versions of the formula $(dzCdz^*)^m = \det(C)\,(dzdz^*)^m$, where C is $m\times m$, to swap the roles of dz and f.) Plugging equation (10.58) into equation (10.56) and doing the integral (10.53) we conclude that the top degree term of $\sigma(\widehat{\alpha})$ is given by

$$\sigma(\widehat{\alpha}) \;=\; \sum_{k=0}^{m-1}\sum_{\ell=0}^{p-1} \binom{n-\ell}{m-1} C_{kl}\, b\, \Omega^\ell (f^*f)^k \left\{\frac{f^*a^*(1-z^*z)af}{|za|^2}\right\}^{m-1-k} A(A^*A)^{p-1-\ell}\, dV(z), \tag{10.59}$$

where C_{kl} is given by equation (10.49).

At this stage we recall that $a = (\mathrm{Id}_m \mid 0)$, i.e. that α is an orthonormal collection of sections of F. Using the notation introduced in equations (10.42) and (10.43) set

$$D \;=\; \delta\delta^* \;:=\; (\omega_V f) f^* \omega_H \qquad E \;=\; \delta\epsilon^* \;:=\; (\omega_V f) f_H^*, \tag{10.60}$$

and

$$J_r \;:=\; \int_{\mathbf{P}(\mathbf{C}^m)} (zEz)^*(zDz^*)^r\, dV(z). \tag{10.61}$$

Then, since $(f_H^* z^*)^2 = 0$, equations (10.50) and (10.59) imply that

$$S_p(\alpha)\lambda(f) \;=\; \sum_{k=0}^{m-1}\sum_{\ell=0}^{p-1} \binom{n-\ell}{m-1} C_{k\ell}\; \Omega^\ell (f^*f)^k \,(f_H^* f_V)^{m-1-k}\, J_{p-1-\ell}.$$

Step 3. The final step in the calculation of the spark $S_p(\alpha)$ uses the following lemma to evaluate the integral (10.61).

Lemma 10.62. *Using the notation of equations (10.60) and (10.47),*

$$J_r \;=\; \frac{1}{\binom{m+r}{m-1}}(\epsilon^*\delta)\,(\delta^*\delta)^r \;=\; \frac{1}{\binom{m+r}{m-1}}\; RQ^r.$$

Given the lemma we conclude that

$$S_p(\alpha)\lambda(f) \;=\; \sum_{k=0}^{m-1}\sum_{\ell=0}^{p-1} C_{k\ell}(f^*f)^k\,(f_H^* f_V)^{m-1-k}\;\Omega^\ell R Q^{p-1-\ell}. \tag{10.63}$$

Comparing equations (10.48) and (10.63) we see that $S_p(\alpha) = T$ as required! $\square$

Proof of Lemma 10.62. The proof follows from the following two facts. The first fact is the following theorem of Bott [B1]. Let $E \to X$ be a complex vector bundle with connection, let $\pi : \mathbf{P}(E) \to X$ denote the projectivization of E and let $U \to \mathbf{P}(E)$ denote the tautological bundle. Then,

$$\pi_*(c(\Omega_U)^{-1}) \;=\; c(\Omega_E)^{-1}.$$

This result can be rephrased as saying that, for any $m \times m$ matrix M,

$$\int\limits_{\mathbf{P}(\mathbf{C}^m)} \frac{1}{1 + zMz^* - \frac{i}{2\pi} dzdz^*} \;=\; \frac{1}{\det(1+M)},$$

and hence that

$$\sum_{q=0}^{\infty} \binom{q+m-1}{m-1} \;(-1)^q \int\limits_{\mathbf{P}(\mathbf{C}^m)} (zMz^*)^q \; dV(z) \;=\; \frac{1}{\det(1+M)}.$$

The second fact is that, if v and w are column vectors whose entries commute with each other, then

$$\det(1 + vw^t) \;=\; 1 + w^t v.$$

Of course we cannot use this fact directly since ϵ^* does not commute with δ or δ^*. To circumvent this problem we introduce an additional variable t and redefine ϵ^* to be $\epsilon^* = dt f_H^*$. Now ϵ^*, δ and δ^* all commute and we can calculate J_r using the formula

$$J_r \;=\; \frac{\partial}{\partial t} \llcorner \int\limits_{\mathbf{P}(\mathbf{C}^m)} (zEz^*)\,(zDz^*)^r \; dV(z). \qquad \square$$

References

[AE] C.B. Allendoerfer and J. Eells, *On the cohomology of smooth manifolds*, Comment. Math. Helv. **51** (1950), 551–570.

[Bo] A. Borel, *Topics in the homology theory of fibre bundles*, Lect. Notes in Math. **36** (1967), 1–95.

[B1] R. Bott, *A residue formula for holomorphic vector fields*, J. Diff. Geom. **1** (1967), 311–330.

[B2] ———, *On a topological obstruction to integrability*, Proc. of Symp. in Pure Math. **16** (1970), 127-131.

[BCh1] R. Bott and S.S. Chern, *Hermitian vector bundles and the equidistribution of the zeros of their holomorphic sections*, Acta. Math. **114** (1968), 71–112.

[BCh2] ———, *Some formulas related to complex transgression*, Essays on Topology and Related Topics (Mémoires dédiés à Georges de Rham, Springer-Verlag, New York, 1970, pp. 48–57.

[C] J. Cheeger, *Multiplication of Differential Characters*, Instituto Nazionale di Alta Mathematica, Symposia Mathematica **XI** (1973), 441–445.

[CS] J. Cheeger and J. Simons, *Differential Characters and Geometric Invariants*, Lect. Notes in Math. **1167** (1985), Springer–Verlag, New York, 50–80.

[Ch1] S.S. Chern, *A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds*, Ann. Math. **45**, 747-752.

[Ch2] ———, *On the multiplication in the characteristic ring of a sphere bundle*, Ann. Math. **49**, 362-372.

[ChS] S.S. Chern and J. Simons, *Characteristic forms and geometric invariants*, Ann. Math. **45**, 747-752.

[E] C. Ehresmann, *Sur la topologie des certaines variétés algébriques réelles*, Journal de Math. Pures **104** (1939), 69–100.

[F] H. Federer, *Geometric Measure Theory*, Springer–Verlag, New York, 1969.

[Fu] W. Fulton, *Intersection Theory*, Ergobnisse der Mathematik und ihrer Grenzgebeite 3, Folge, Band 2, Springer Verlag, Berlin-Heidelberg, 1984.

[HS] F.R. Harvey and S. Semmes, *Zero divisors of atomic functions*, Ann. Math. **135** (1992), 567–600.

[HL1] F.R. Harvey and H.B. Lawson, *A theory of characteristic currents associated with a singular connection — Announcement*, B.A.M.S. **31** (1994), 54–63.

[HL2] ———, *A theory of characteristic currents associated with a singular connection*, Astérisque **213** (1993), 1–268.

[HL3] ———, *Geometric residue theorems*, Amer. J. Math. **117** (1995), 829– 873.

[HL4] ———, *Morse theory and Stokes' theorem*, (to appear), 1–35.

[HL5] ———, *The local MacPherson Formula*, (to appear), 1–32.

[HZ1] F.R. Harvey and J. Zweck, *Stiefel–Whitney Currents*, J. Geometric Analysis (to appear), 1–39.

[HZ2] ———, *Divisors and Euler sparks of atomic sections*, Indiana Math Jour. (to appear), 1–46.

[Mill] John J. Millson, *Examples of nonvanishing Chern-Simons invariants*, J. Diff. Geom. **10** (1975), 589–600.

[M] J.W. Milnor, *Morse Theory*, Annals of Math. Studies, vol. 51, Princeton Univ. Press., 1963.

[NR] H. S. Narasimhan and S. Ramanan, *Existence of universal connections*, Amer. J. Math. **85** (1963), 223–231.

[P] I.R. Porteous, *Simple singularities of maps – Proceedings of Liverpool Singularities Symposium, I*, Lecture Notes in Mathematics **192** (1971), 286–307.

[R] F. Ronga, *Le calcul de la classe de cohomologie duale a* $\overline{\Sigma}^k$ *– Proceedings of Liverpool Singularities Symposium, I*, Lecture Notes in Mathematics **192** (1971), 313–315.

[S] J. Simons, *Characteristic forms and transgression: characters associated to a connection*, Stony Brook preprint, (1974).

[T] R. Thom, *Les singularitiés des applications différentiables*, Ann. Inst. Fourier, Grenoble **6** (1955–56), 43–87.

[Ton] P. Tondeur, *Foliations on Riemannian manifolds*, Springer-Verlag, New York, 1988.

[W] H. Whitney, *Sphere-spaces*, Proc. Nat. Acad Sci. **21** (1935), 64-68.