An Analysis of the Methods Used to Predict the Flight Path of Malaysia Airlines Flight 370

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A Tragic Disappearance

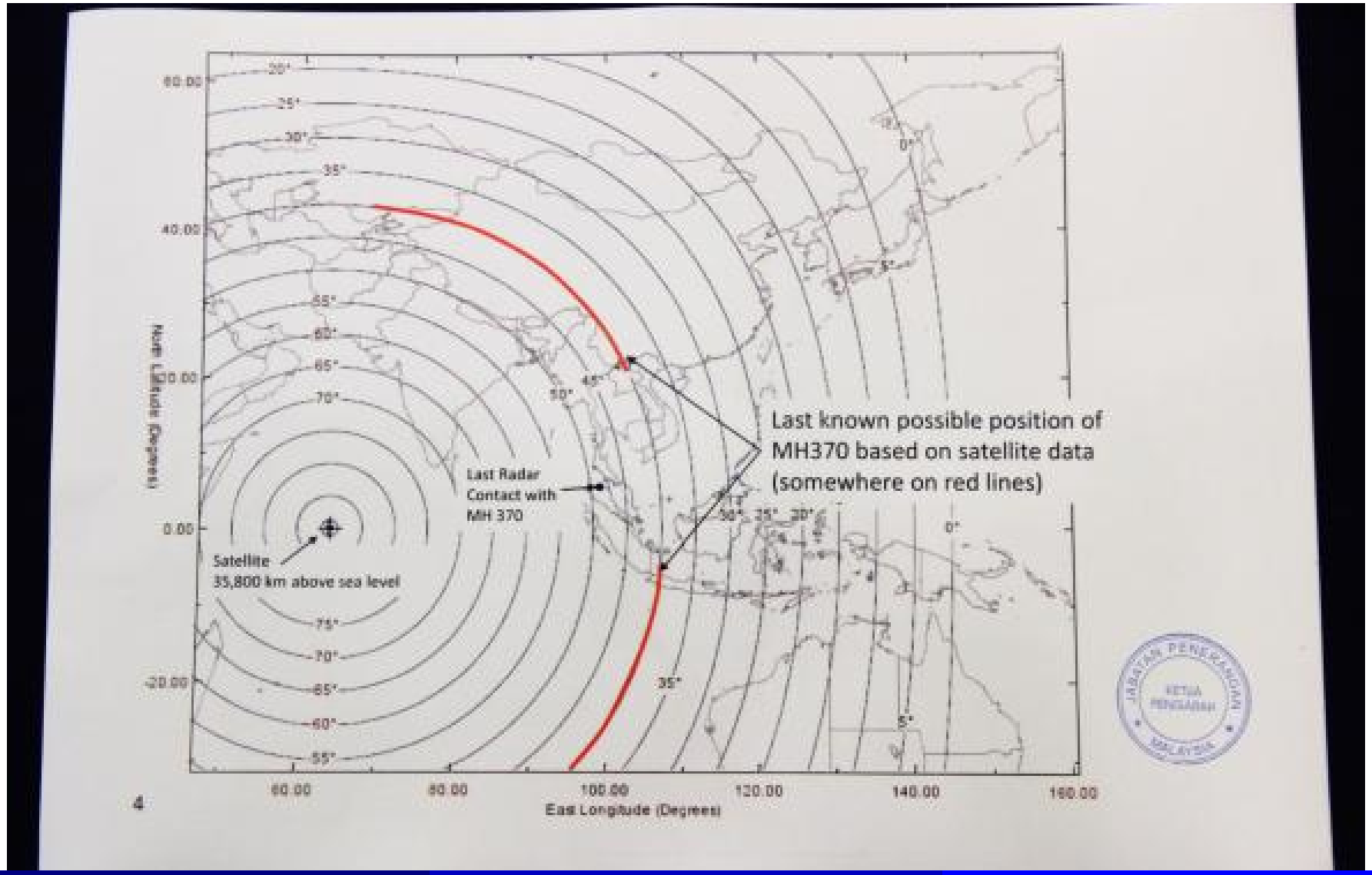
Malaysia Airlines flight 370 disappeared on March 8 en route to Beijing from Kuala Lumpur.

To date there is no confirmed debris from the aircraft and no survivors have been found.

An international search effort is trying to provide some resolution for the family members of the 239 passengers, crew, and pilots of MH370.

Flight Path [Washington Post, Mar 17]

8:11 am Arc [Malaysian Govt., Mar 15]

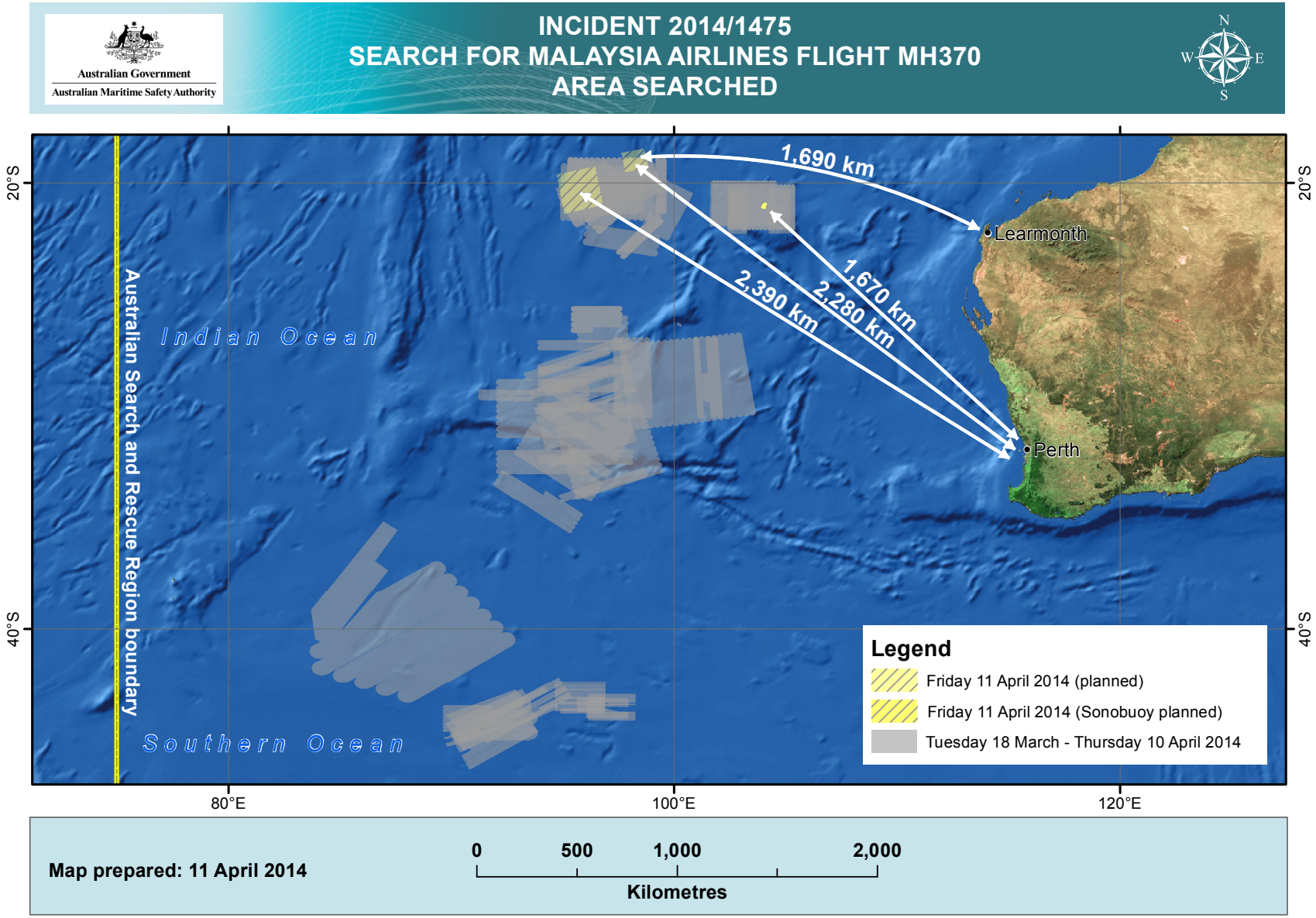


Where did MH370 go?

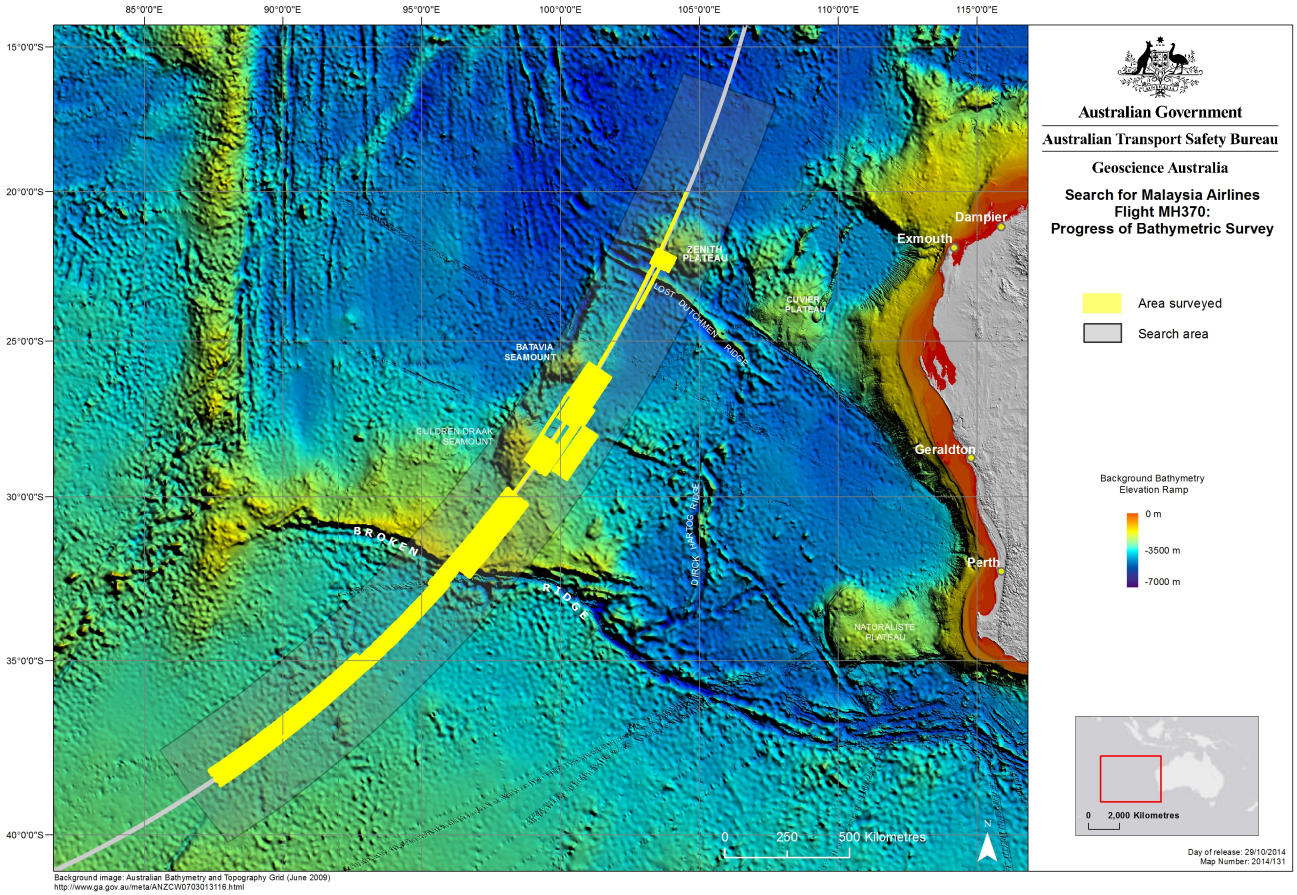
- The Malaysian Govt set up a **Working Group** of experts to determine the flight path of MH370.
- Mar 17: Search refocused to a region in the Indian Ocean, 3000 km S.W. of Perth.
- Mar 28: Search area shifted to a region 1500 km west of Perth.
- Apr 7: Possible black box signals detected 2300 km northwest of Perth. Area searched without success.
- June 26: Search area 1500 km west of Perth refined.
- Oct 8: Search area moved 800 km further SW.

Why was the search area changed five times?

MH 370 Surface Search Areas [Apr 11]

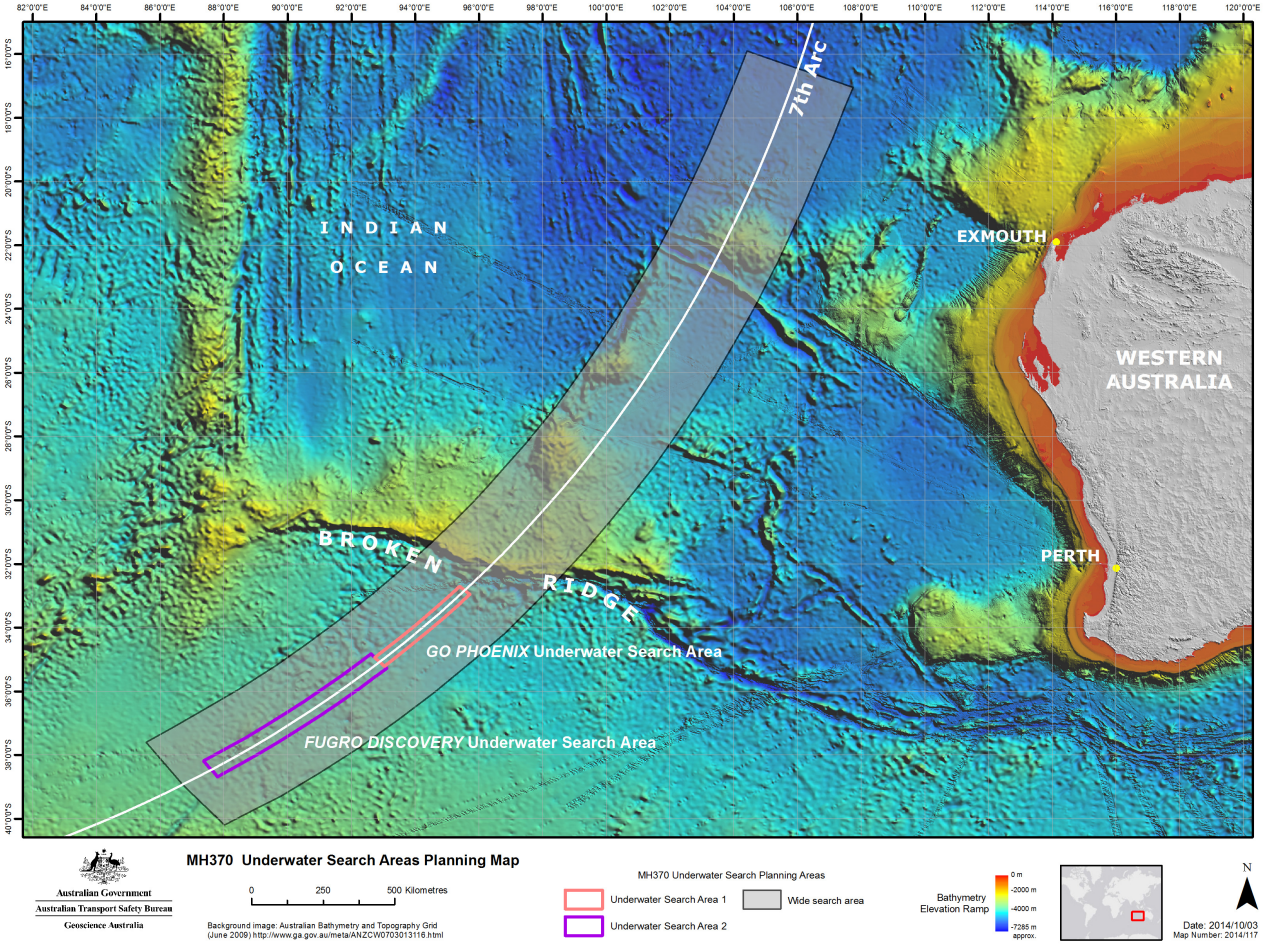


BathymetricSurvey [ATSB, Oct 27]



A wide range of proposed search areas!

Underwater Search Areas [ATSB, Oct 8]

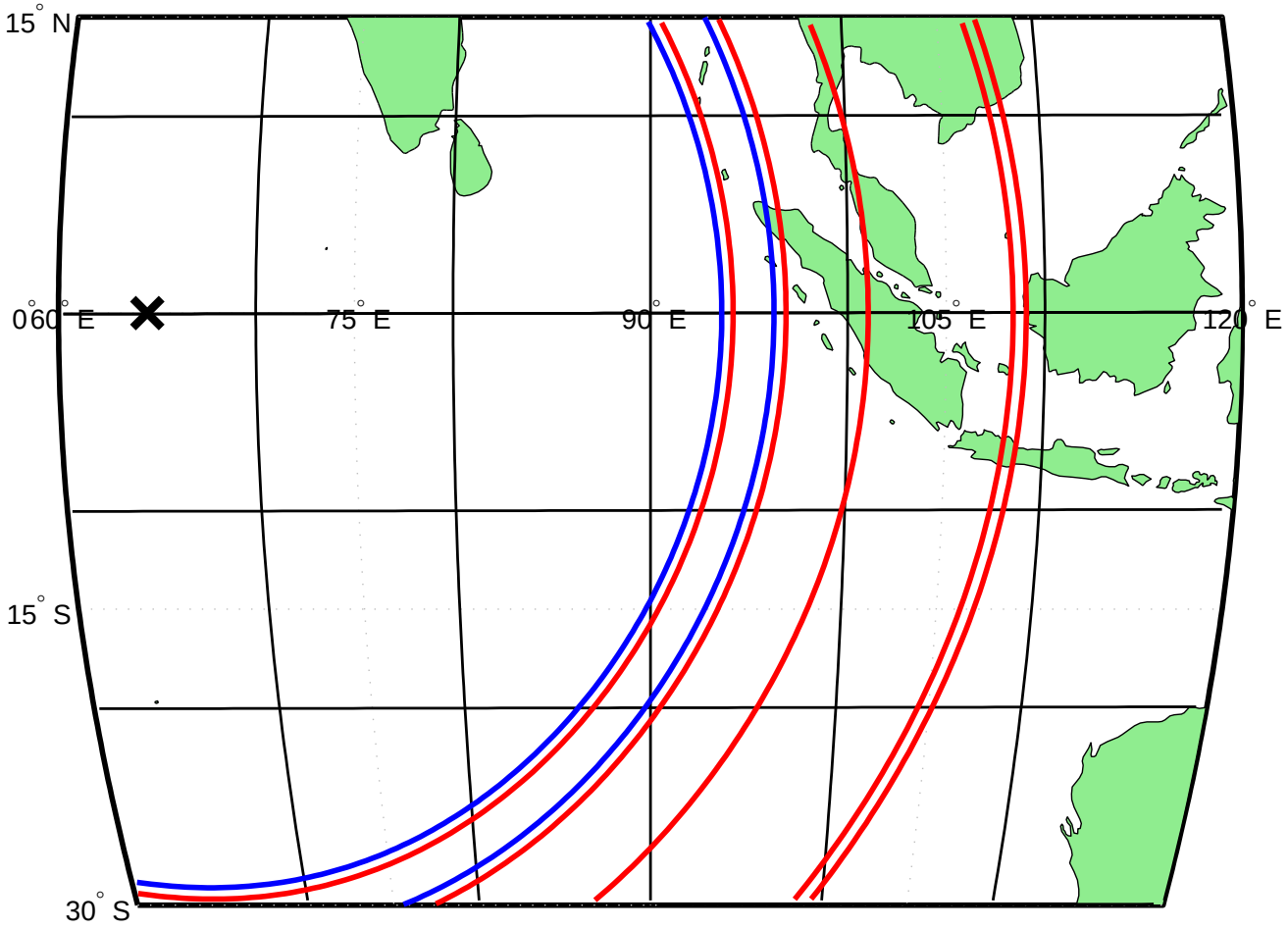


Goal: Explain the math!

Overview of Radar and Satellite Data

- Radar data suggest that the flight path of MH370 was very erratic during the first 2-3 hours with
  - sharp turns,
  - large changes in speed and elevation.
- After last radar contact, the Inmarsat 3-F1 satellite "pinged" the aircraft 7 times over a 6 hour period.
- At each ping time, the engineers could determine
  - an arc of possible locations of the aircraft,
  - the **burst frequency offset [BFO]** which is a quantity related to the aircraft-satellite Doppler shift.

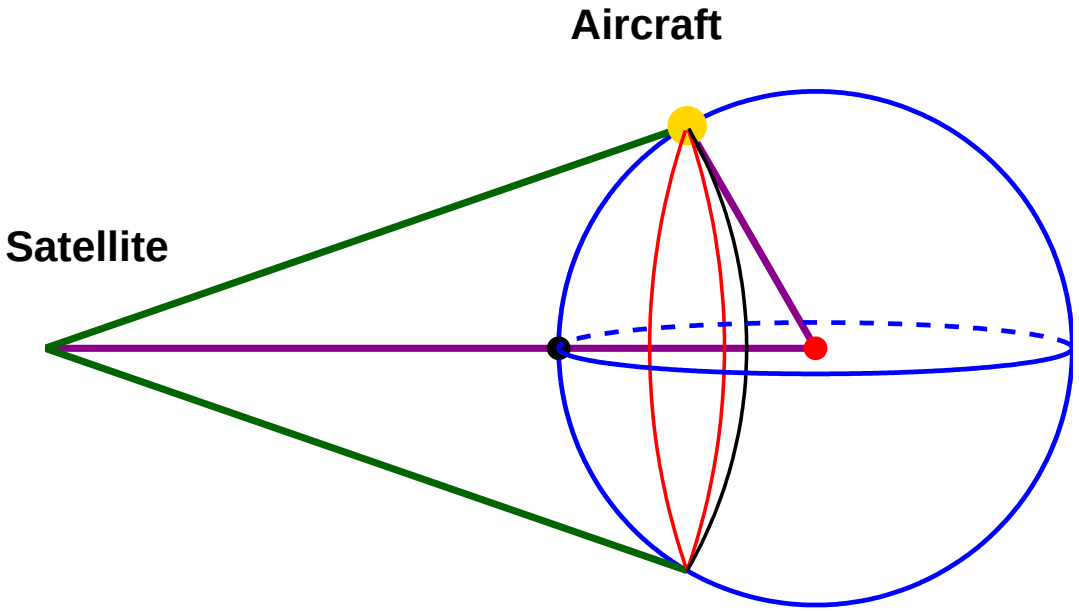
Ping Arcs[1]



[1] Derived from May 27 data release and June 26 ATSB report.

Whence the Arcs?

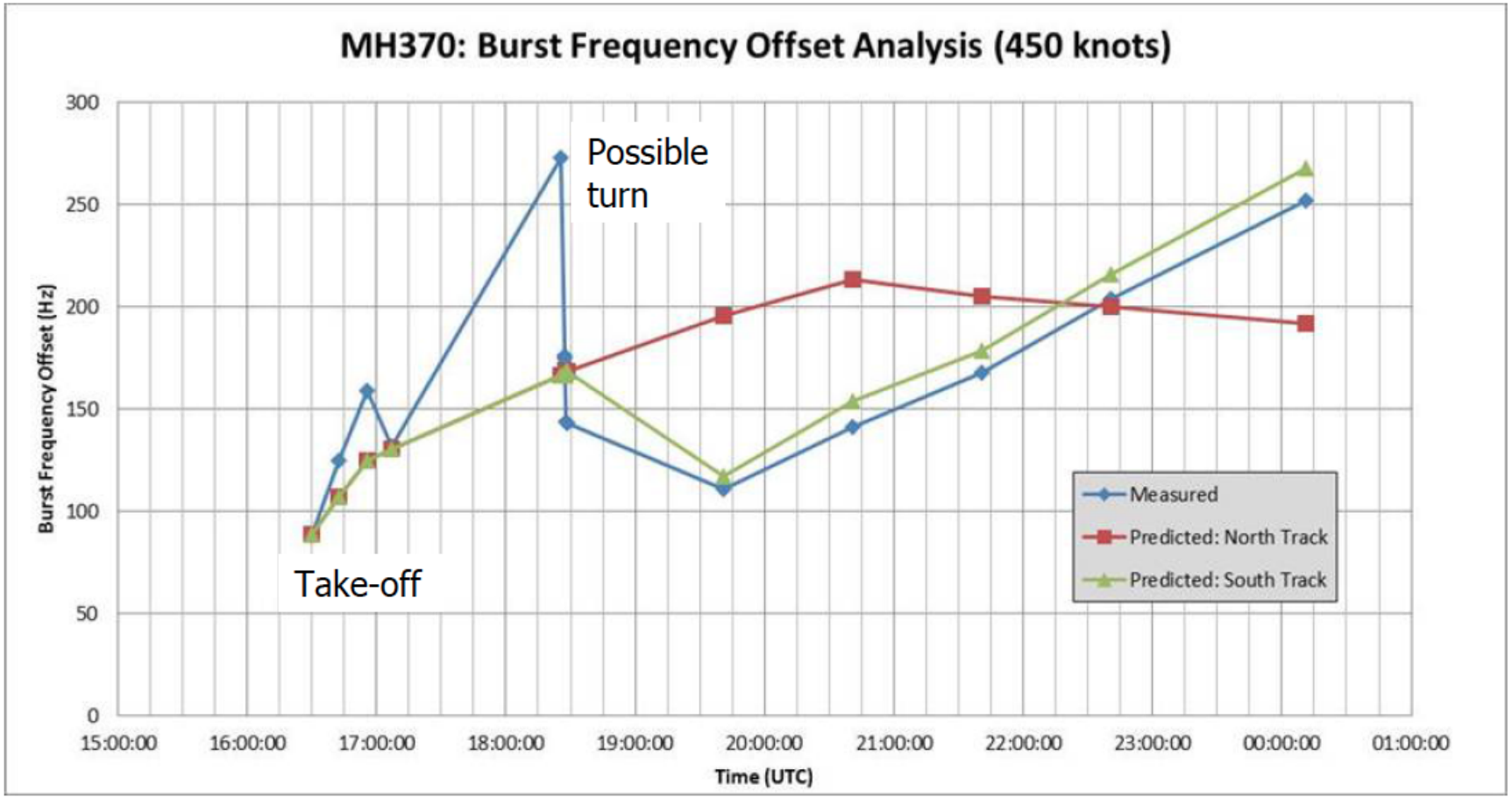
The time that it takes the signal to be sent and received, via the satellite, to the ground station can be used to establish the distance of the aircraft from the satellite.



All points on the red circle are at the same distance from the satellite as is the aircraft.

BFO Data [AAIB, Mar 25]

MH370 measured data against predicted tracks



The Math Problem

- Determine flight path from
  - Ping times and ping arc angles
  - BFO/Doppler data
  - Position on first arc
- Requires a **flight path model:**
  1. Known-speed, concatenation-of-geodesics model
  2. Unknown-speed, concatenation-of-geodesics model
  3. Concatenation of small circles, BFO model

Why was the search area changed?

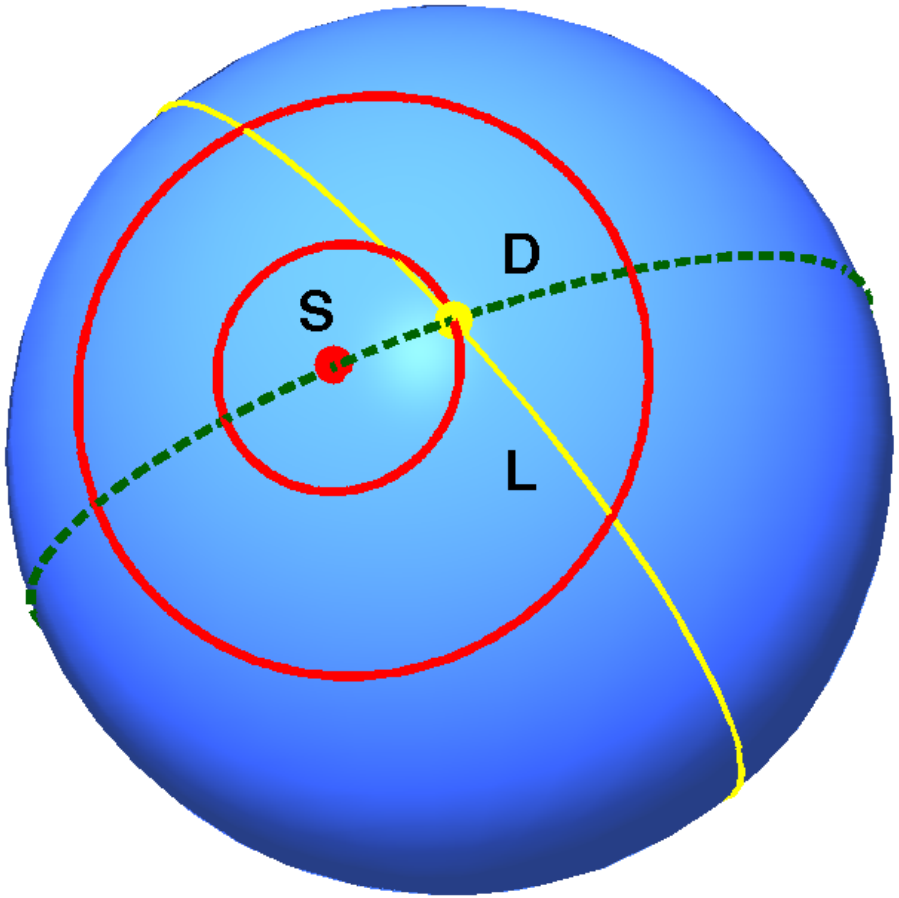
1. Use of BFO in addition to ping arc data
2. More sophisticated engineering analysis of data
3. Improvements to math models and algorithms

(I) Known-speed, concat^n-of-geodesics

Modeling Assumptions

1. The earth is a perfect sphere.
2. Aircraft has constant elevation and ground speed.
3. Input parameters:
   - Aircraft initial position,
   - Aircraft speed.
4. Flight path is a concat^n of **great circle segments**.
5. The satellite is in a **geostationary** orbit.
6. Doppler shift data was not used.

Spherical Triangle Geometry



Rotate great circle path about initial position to reach next ping arc at next ping time.

Solution in Spherical Satellite Coords.

1. **Spherical satellite coordinates**, (Θ, Φ), of $\mathbf{y}$:

$$\mathbf{y} \;=\; \mathbf{y}(\Theta, \Phi) \;=\; \cos\Phi\,\mathbf{i}_S + \cos\Theta\sin\Phi\,\mathbf{j}_S + \sin\Theta\sin\Phi\,\mathbf{k}_S.$$

2. Ping arcs are circles of latitude, $\Phi = \Phi_n$ (known).
3. Initial Position on 1st ping arc, $\Theta = \Theta_1$ (known).
4. Solve for $\Theta_2 \cdots \Theta_N$ in succession.

Great Circle Parametrization

1. Great circle flight path with position, $\mathbf{r}_n$, and velocity, $\mathbf{v}_n$, at the time, t_n, is

$$\mathbf{r}_A(t) \;=\; \cos\left[\frac{v_n(t-t_n)}{R_A}\right]\mathbf{r}_n \;+\; \frac{R_A}{v_n}\sin\left[\frac{v_n(t-t_n)}{R_A}\right]\mathbf{v}_n.$$

2. Unknown **heading direction** , β_n, of the aircraft:

$$\mathbf{v}_n \;=\; v_n\cos\beta_n\,\mathbf{y}_\Theta \;+\; v_n\sin\beta_n\,\mathbf{y}_\Phi.$$

The Ping Arc Equation

1. Condition that aircraft crosses the $(n+1)$-st ping arc at time t_{n+1}:

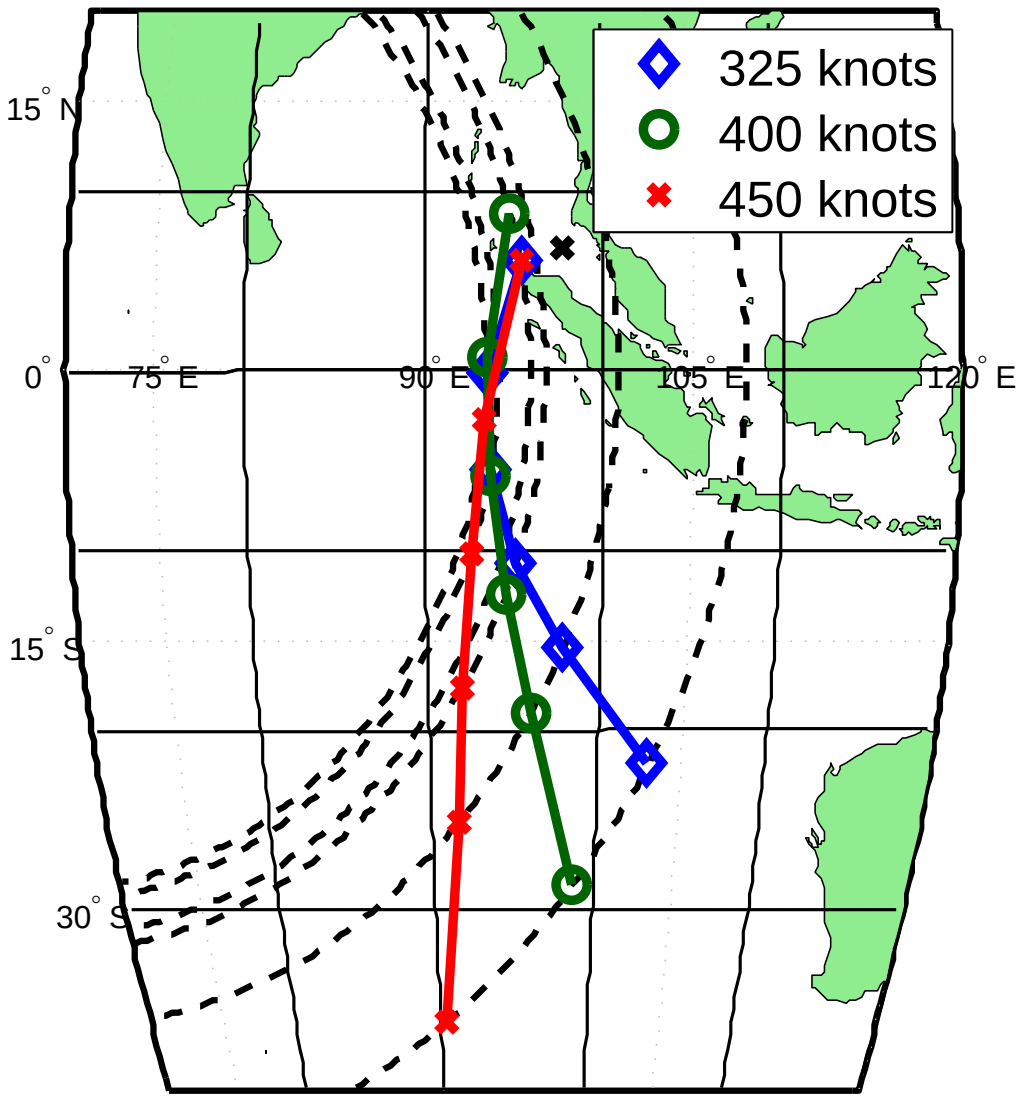
$$\mathbf{r}_A(t_{n+1}) \;=\; \mathbf{y}(\Theta_{n+1}, \Phi_{n+1}).$$

2. Two independent equations for β_n and Θ_{n+1}.

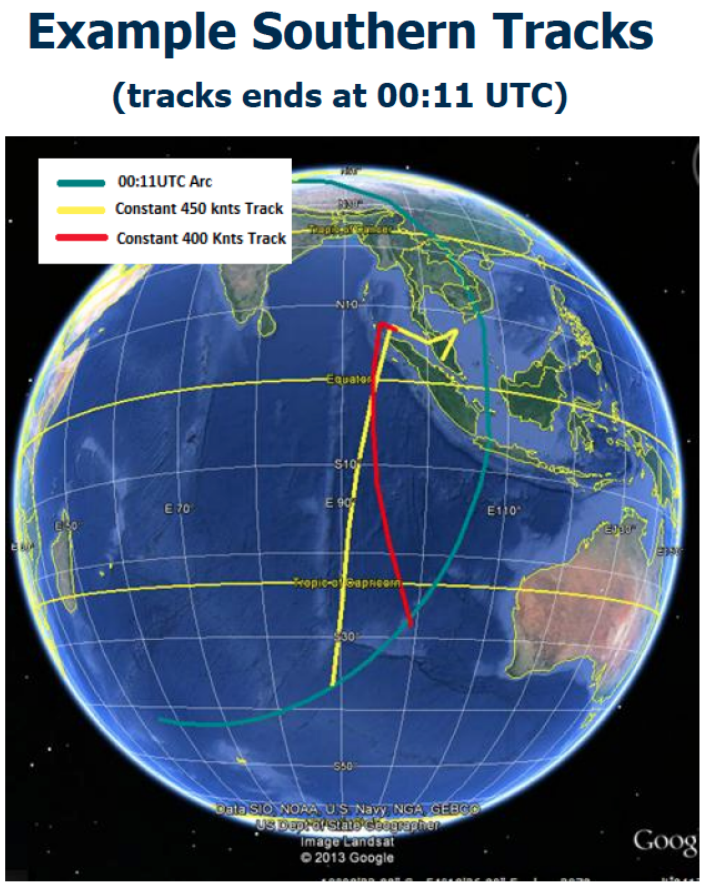
3. Eliminate Θ_{n+1} to get

$$\sin\beta_n \;=\; \frac{\cos\varphi_n \cos\left(\frac{\Delta t_n v_n}{R_A}\right) \;-\; \cos\varphi_{n+1}}{\sin\varphi_n \sin\left(\frac{\Delta t_n v_n}{R_A}\right)}.$$

Constant Speed Flight Paths [MH370]



AAIB Possible Southern Tracks [Mar 25]

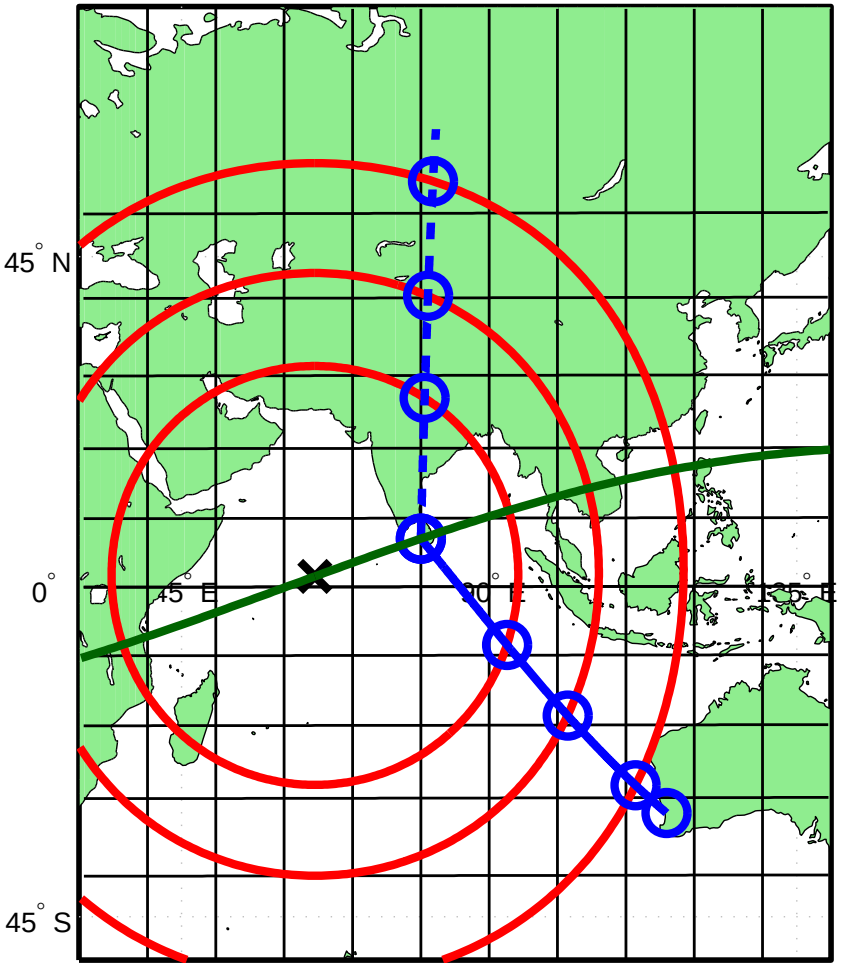


The search area for MH370 was moved 1000 km N.E. because radar data showed the plane had initially travelled faster and further west.

Positive and Negative Flight Paths

- If $\beta_n \neq \pm\pi/2$ there are two solutions, $\beta_n^{\pm}$:
  1. $\cos\beta_n^{+} > 0$,
  2. $\cos\beta_n^{-} < 0$.
- For $n > 1$ you can often choose between $\beta_n^{\pm}$.
- Upshot: There are two plausible flight paths:
  1. **Positive:** $\beta_1 = \beta_1^{+}$,
  2. **Negative:** $\beta_1 = \beta_1^{-}$.

Two Tracks (XX123; Geostationary)

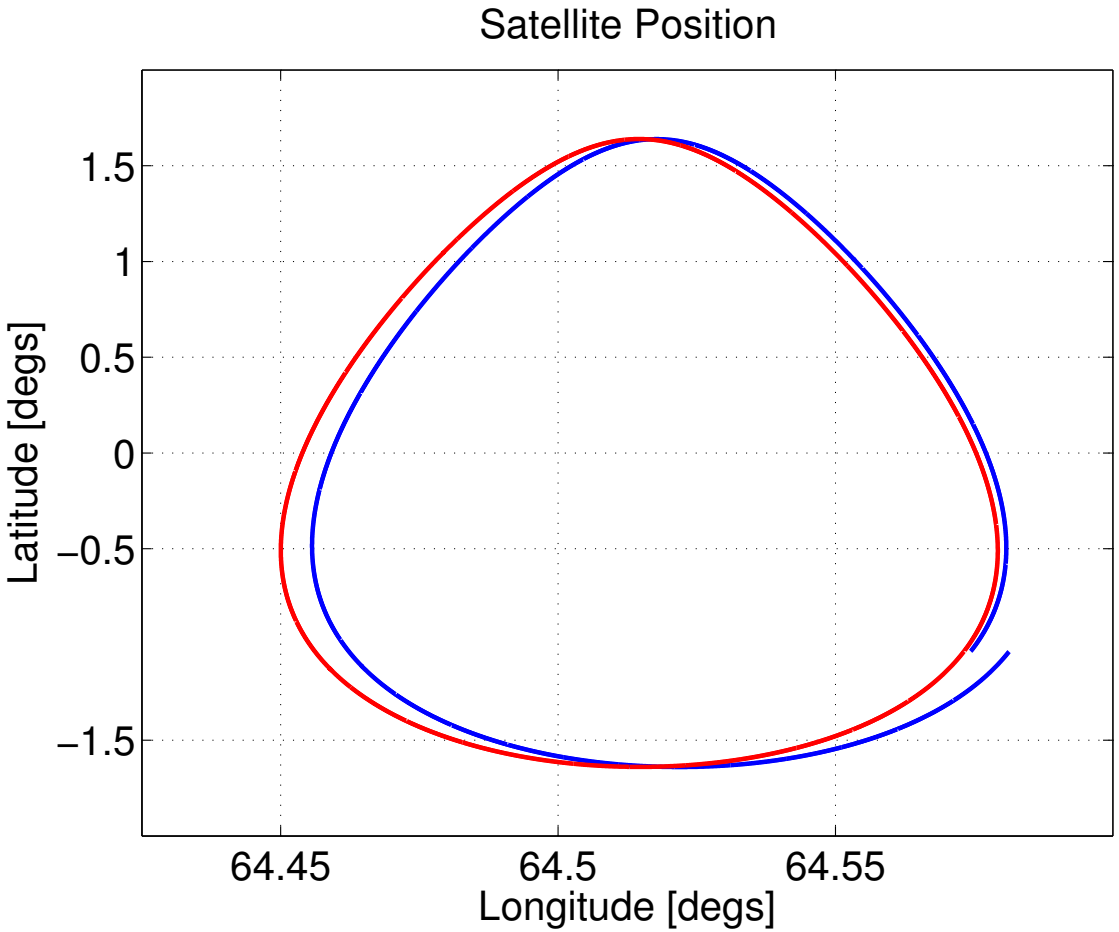


The problem is symmetric about the great circle containing the satellite and starting location.

Types of Satellite Orbits

1. **Geostationary orbit:** Satellite is located at a fixed point in the sky.
   - Doppler shifts for $+$ and $-$ paths are identical.
2. **Geosynchronous orbit:** Satellite returns to the same point in the sky at the same time each day.
   - The slight motion of INMARSAT 3-F1 satellite in the sky **breaks the symmetry** between $+$ and $-$ paths.

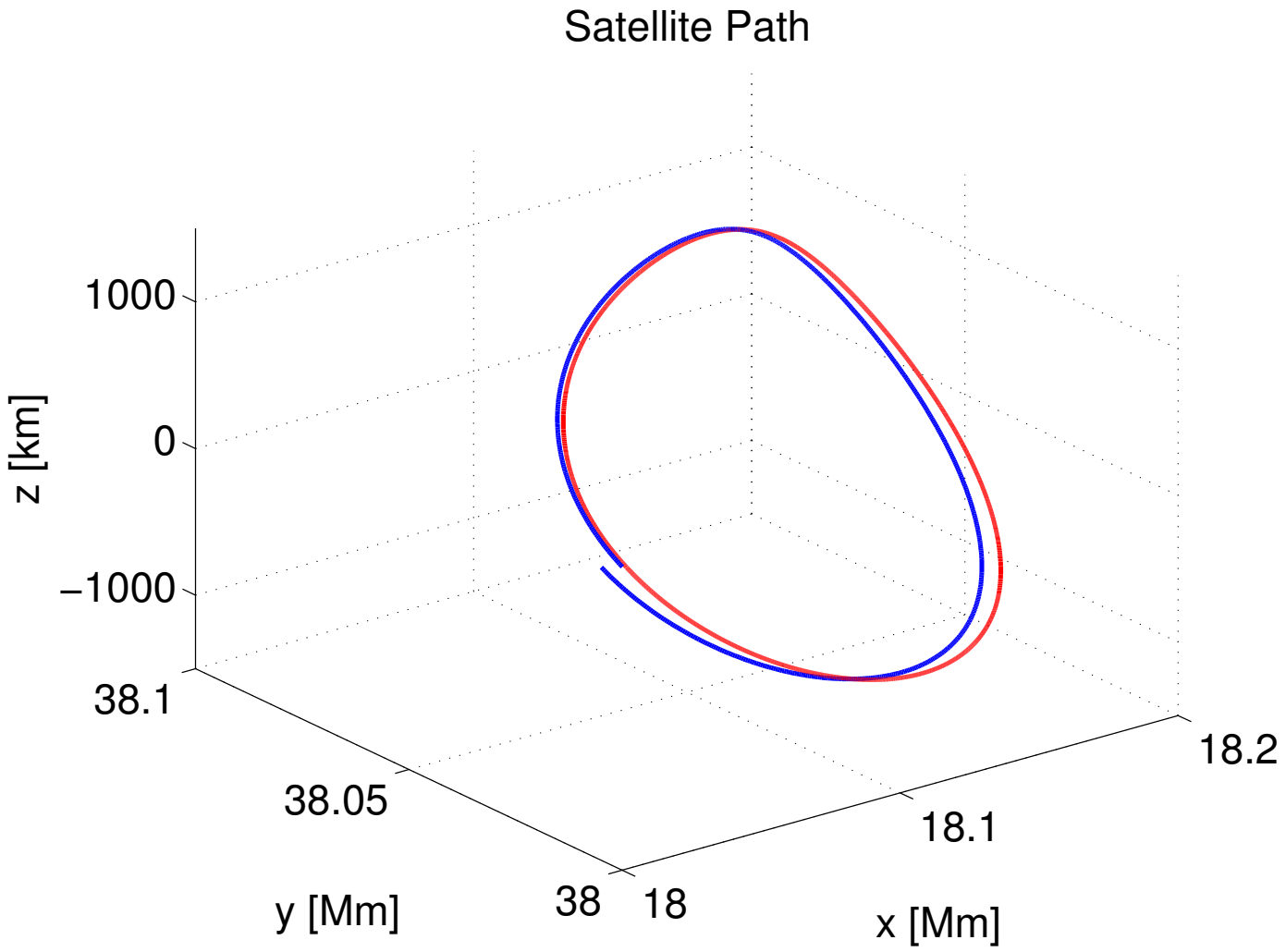
Geosynchronous Satellite Orbit



Data (blue) from aqqa.org

Model (red) used below [Formulae not shown].

Geosynchronous Satellite Orbit



Position error < 6 km. Velocity error < 0.5 km/hr.

(II) Concat^n-of-geodesics [unknown v]

Modeling Assumptions

1. Input parameter: Aircraft initial position.
2. Flight path is a concat^n of **great circle segments**.
3. The satellite is in a **geosynchronous** orbit.
4. Doppler shift data is used.

Different great circle segments can have different *a priori* unknown speeds.

The Main Ideas in the Model

- What model to get from one ping arc to the next?
- Simplest choice is a constant speed great circle:

$$\mathbf{r}(t) = \cos\left(\frac{v_n(t-t_n)}{R_A}\right)\mathbf{r}_n + \frac{R_A}{v_n}\sin\left(\frac{v_n(t-t_n)}{R_A}\right)\mathbf{v}_n$$

- Given current location, $\mathbf{r}_n$ on n-th ping arc, determine the velocity vector, $\mathbf{v}_n$, that gets us to the next ping arc at the right time.

The Main Ideas in the Model

We need three constraints to determine $\mathbf{v}_n \in \mathbf{R}^3$

1. Constant elevation constraint: $\mathbf{v}_n \in T_{\mathbf{r}_n} S^2$.
2. Ping arc traversal constraint: $F(\mathbf{v}_n) = 0$.
3. The Doppler shift data gives a third constraint.

The Doppler Effect

- Suppose an electromagnetic signal that is sent by the aircraft to the satellite.
- If the aircraft is moving **towards** the satellite the light received will be **more blue** than the light sent.
- If the aircraft is **moving away**, the received light will be **more red**.

$$\frac{\Delta f}{f} = -\frac{1}{c}\,\widehat{\mathbf{u}}\cdot\mathbf{v}$$

Doppler Shift $\propto$ Speed along line of sight

Doppler Shift (Geostationary Case)

Doppler shift provides partial velocity information

$$\frac{\Delta f}{f} = -\frac{R_S}{c\,D_{SA}} v_\Phi \sin\Phi$$

- R_S is distance from satellite to center of earth.
- Φ is ping arc angle.
- $D_{SA} = D_{SA}(\Phi)$ is distance from satellite to aircraft.
- v_Φ is component of aircraft velocity $\perp$ to ping arc.

Δf cannot distinguish between $\pm$ tracks:

v_Φ is same for both.

Doppler Shift (Geosynchronous Case)

$$\frac{\Delta f}{f} = -\frac{R_S}{c\,D_{SA}}\left[v_\Phi \sin\Phi - \widehat{\mathbf{u}}(\Phi,\Theta)\cdot\mathbf{v}_S\right]$$

- $\widehat{\mathbf{u}}$ is unit vector in direction of aircraft from satellite
- $\mathbf{v}_S$ is satellite velocity w.r.t. frame rotating with earth

Because of Θ-dependence

Δf can distinguish between $\pm$ tracks!

The Moving Satellite Frame

1. Choose orthonormal basis (frame)

$$\left\{\widehat{\mathbf{i}}_n, \widehat{\mathbf{j}}_n, \widehat{\mathbf{k}}_n\right\} = \{\mathbf{i}_S(t_n), \mathbf{j}_S(t_n), \mathbf{k}_S(t_n)\}$$

   so that satellite position is

$$\mathbf{r}_S(t_n) = R_S\,\mathbf{i}_S(t_n).$$

2. **Spherical satellite coordinates** of aircraft:

$$\mathbf{r}_A(t_n) = R_A\mathbf{y}_n$$

$$\mathbf{y}_n = \cos\Phi_n\,\widehat{\mathbf{i}}_n + \cos\Theta_n\sin\Phi_n\,\widehat{\mathbf{j}}_n + \sin\Theta_n\sin\Phi_n\,\widehat{\mathbf{k}}_n.$$

3. Aircraft velocity in 2-D tangent space to sphere:

$$\mathbf{v}_n = \mathbf{v}_A(t_n) = v_{\Theta,n}\mathbf{y}_\Theta + v_{\Phi,n}\mathbf{y}_\Phi.$$

Ping Arc Equation

1. Satellite frame: $F_n = \begin{bmatrix} \widehat{\mathbf{i}}_n & \widehat{\mathbf{j}}_n & \widehat{\mathbf{k}}_n \end{bmatrix} \in \mathrm{SO}(3)$
2. $Y_n = F_n^T \begin{bmatrix} \mathbf{y}_n & \mathbf{y}_{\Phi,n} & \mathbf{y}_{\Theta,n} \end{bmatrix} \in \mathrm{SO}(3)$
3. Position of aircraft at t_{n+1} on great circle:

$$\mathbf{r}_{n+1} = R_A F_n Y_n \mathbf{x}_n, \tag{1}$$

where $\mathbf{x}_n$ is the unit column vector

$$\mathbf{x}_n = \frac{1}{v_n} \begin{bmatrix} v_n \cos\alpha_n & v_{\Phi,n} \sin\alpha_n & v_{\Theta,n} \sin\alpha_n \end{bmatrix}^T, \tag{2}$$

with $\alpha_n = v_n \Delta t_n / R_A$.

Ping Arc Equation

1. For aircraft to cross the next ping arc at t_{n+1} need

$$\mathbf{r}_{n+1} = R_A F_{n+1}(Y_{n+1})_{*1}.$$

2. So, the **ping arc equation** is

$$\mathbf{x}_n \;=\; A_n(Y_{n+1})_{*1}, \qquad \text{where } A_n = Y_n^T F_n^T F_{n+1} \in \mathrm{SO}(3).$$

3. Two independent equations in $v_{\Phi,n}$, $v_{\Theta,n}$, and Θ_{n+1}.
4. Eliminate Θ_{n+1} to obtain

$$f_{\mathrm{Ping}}(v_{\Phi,n}, v_{\Theta,n}) \;=\; 0.$$

Summary for Model II

For exact data

1. Given $\mathbf{r}_n$ on current ping arc find $(v_{\Phi,n}, v_{\Theta,n})$ so that great circle path ends at $\mathbf{r}_{n+1}$ on next ping arc.
2. Solve Doppler Shift Equation for $v_{\Phi,n}$.
3. Solve Ping Arc Equation for $v_{\Theta,n}$.

For uncertain data

Numerically minimize

$$F(v_{\Phi,n}, v_{\Theta,n}) = f_{\mathrm{DS},1}^2(v_{\Phi,n}) + f_{\mathrm{Ping}}^2(v_{\Phi,n}, v_{\Theta,n}) + f_{\mathrm{DS},2}^2(v_{\Phi,n}, v_{\Theta,n})$$

using unconstrained, gradient-based optimization.

Model III: Small Circles and BFO

Great circle scenario:

Constant aircraft heading relative to the air.

BUT

Due to wind, path of aircraft over earth may not have exactly been a great circle.

Modeling Assumptions

1. Flight path is a concat^n of **small circle segments.**[2]
2. BFO data is used.

[2] A small circle is any circle on a sphere.

Burst Frequency Offset

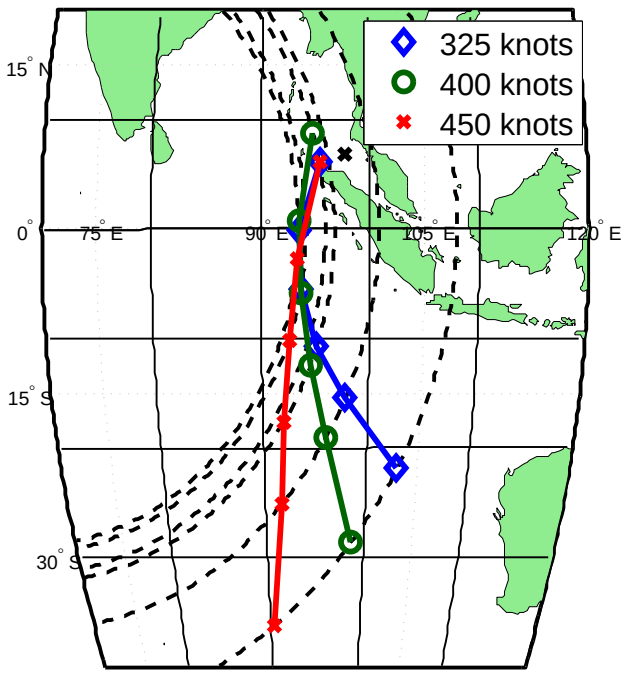
$$\mathbf{BFO} = f_{\mathbf{Received}} - f_{\mathbf{Design}}$$

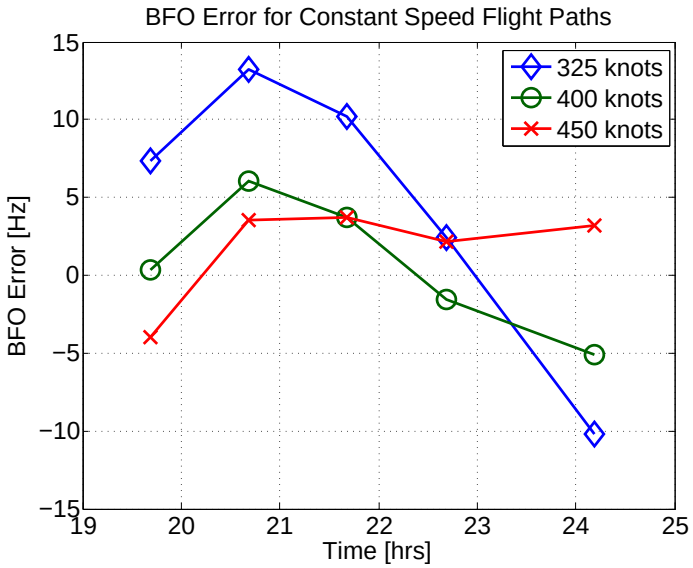
$$\text{BFO} = \Delta f_{\text{AS}} - \Delta f_{\text{AS-Comp}} + \Delta f_{\text{SG}} + \delta f_{\text{AI}} + \delta f_{\text{Bias}}.$$

- $\Delta f_{\text{AS}} =$ Aircraft-Satellite Doppler shift (DS)
- $\Delta f_{\text{AS-Comp}}$ Aircraft-Geostationary Satellite DS[3]
- $\Delta f_{\text{SG}} =$ Satellite-Ground Station DS
- $\delta f_{\mathbf{AI}} = \delta f_{\text{AI}}(t) =$ Aircraft-Independent frequency shift
- $\delta f_{\mathbf{Bias}} = 150 \pm 5$ Hz $=$ Aircraft Dependent bias

[3] Made on aircraft to partially compensate for Δf_{AS}

The Reasons Search is so Challenging





1. **Uncertainty in $\delta f_{\mathbf{AI}}$ and $\delta f_{\mathbf{Bias}}$**
2. **Small BFO error of these three flights**
3. **Uncertainty in initial position on 2nd ping arc**

Small Circles

- The **geodesic curvature** of a small circle is

$$\kappa_g = \pm v \sqrt{\frac{1}{R_C^2} - \frac{1}{R_A^2}}.$$

- κ_g quantifies how much C is turning in S.
- **Extra parameter:** κ_g.
- **Extra constraint:** Δf at the **next** ping arc.

If you know κ_g of C on S^2, then you know C.[4]

[4] By Darboux Frame Formulae

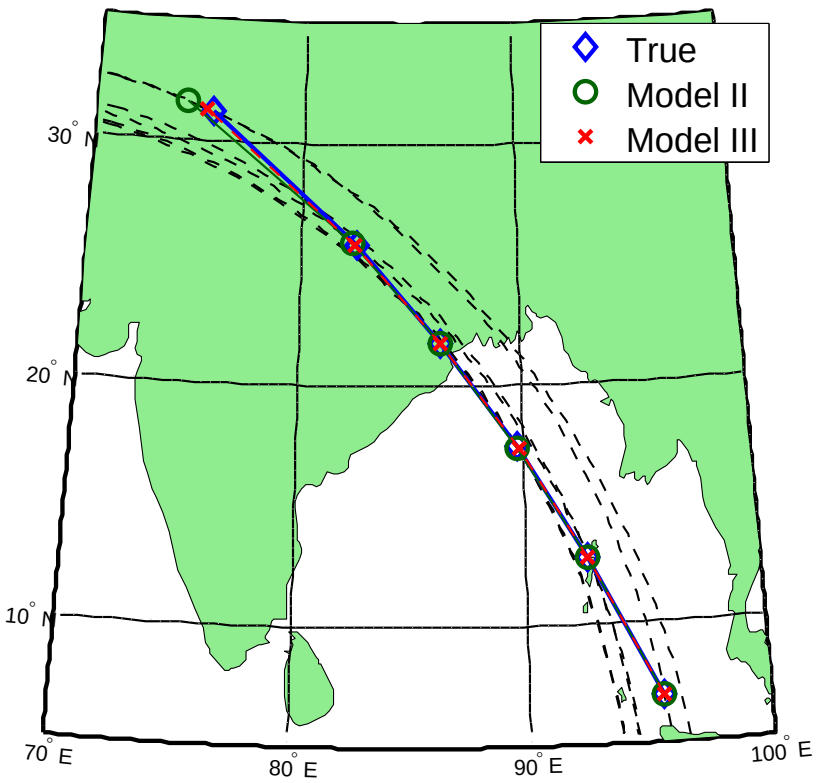
Objective Function to Minimize

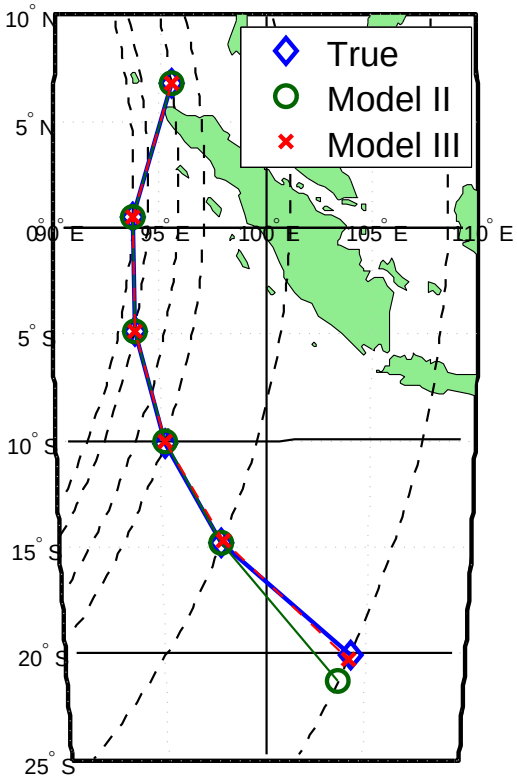
$$
\begin{aligned}
f_{\mathrm{BFO},1}(v_{\Phi,n}, v_{\Theta,n}) &= 0, \\
f_{\mathrm{Ping}}(v_{\Phi,n}, v_{\Theta.n}, K_{g,n}) &= 0, \\
f_{\mathrm{BFO},2}(v_{\Phi,n}, v_{\Theta,n}, K_{g,n}) &= 0.
\end{aligned}
$$

$$
F = f_{\mathrm{Ping}}^2 + f_{\mathrm{BFO},1}^2 + f_{\mathrm{BFO},2}^2 + f_{\mathrm{Penalty}}^2.
$$

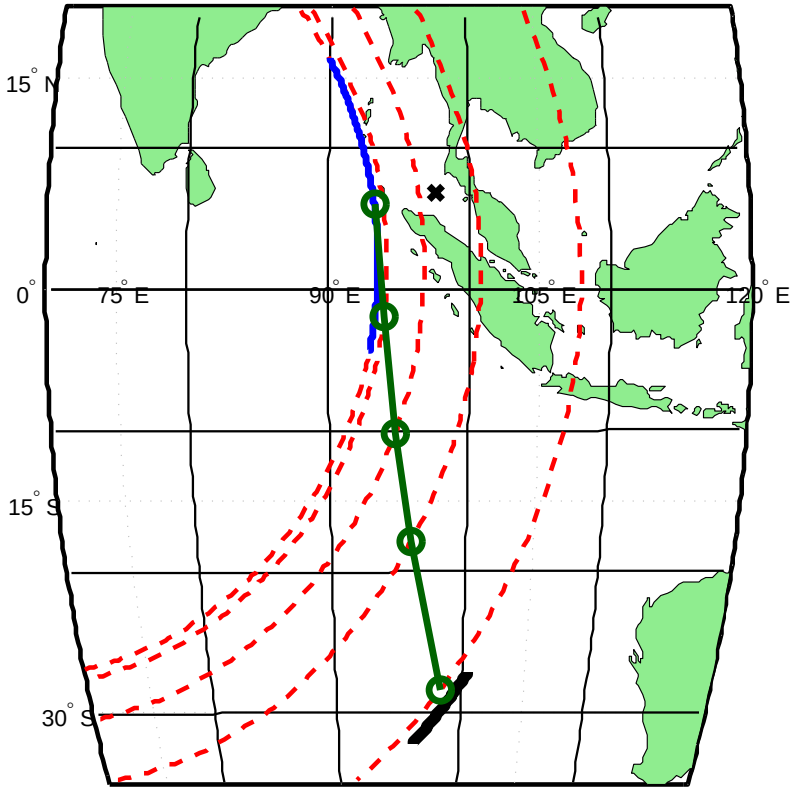
Solve using a gradient-descent method.

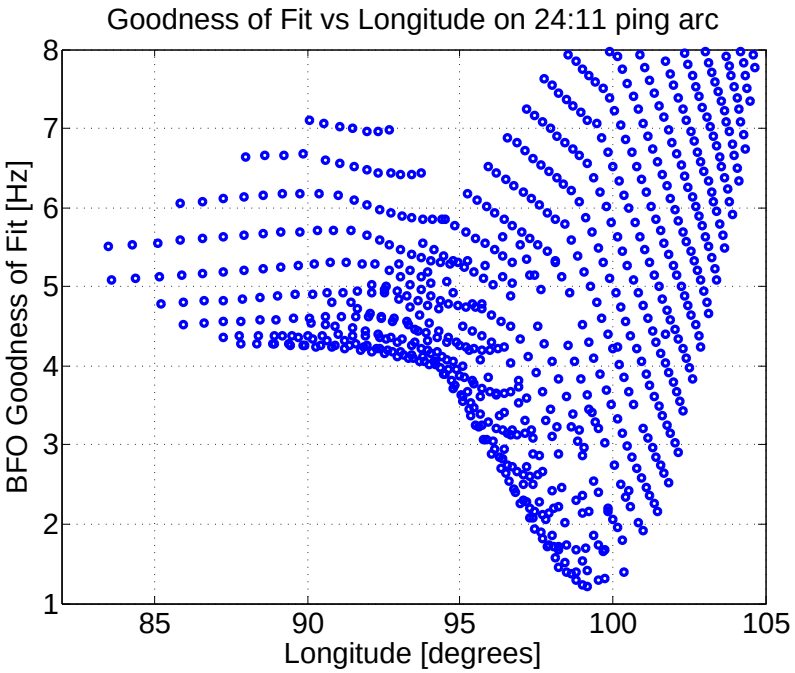
Simulation Results





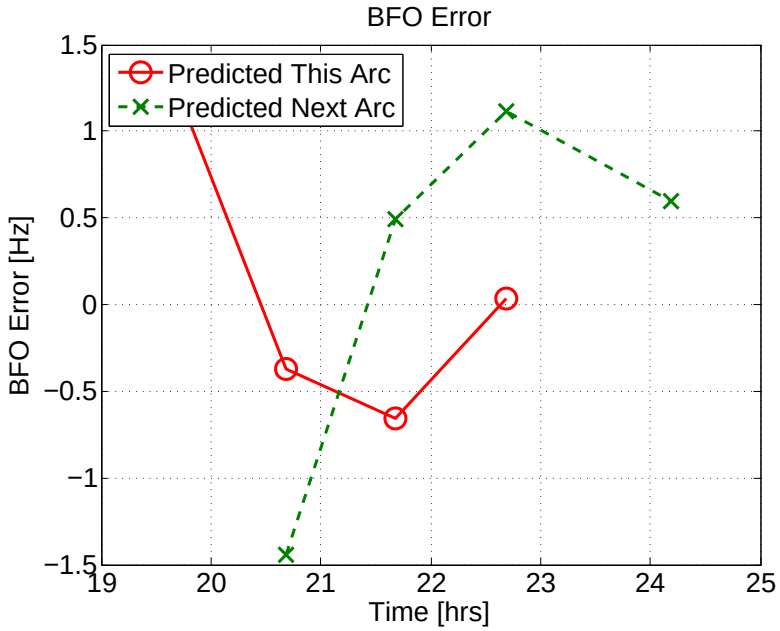
MH370 Results [JZ, June 26 Data]

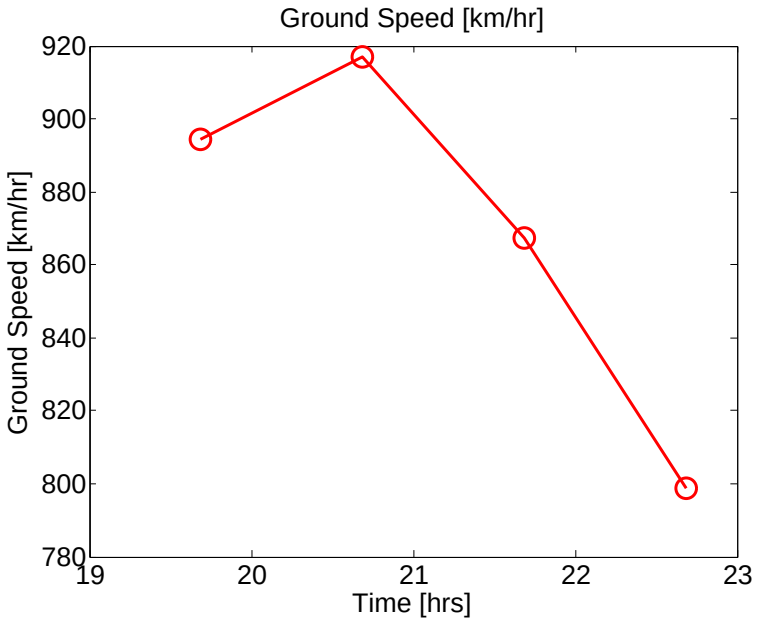




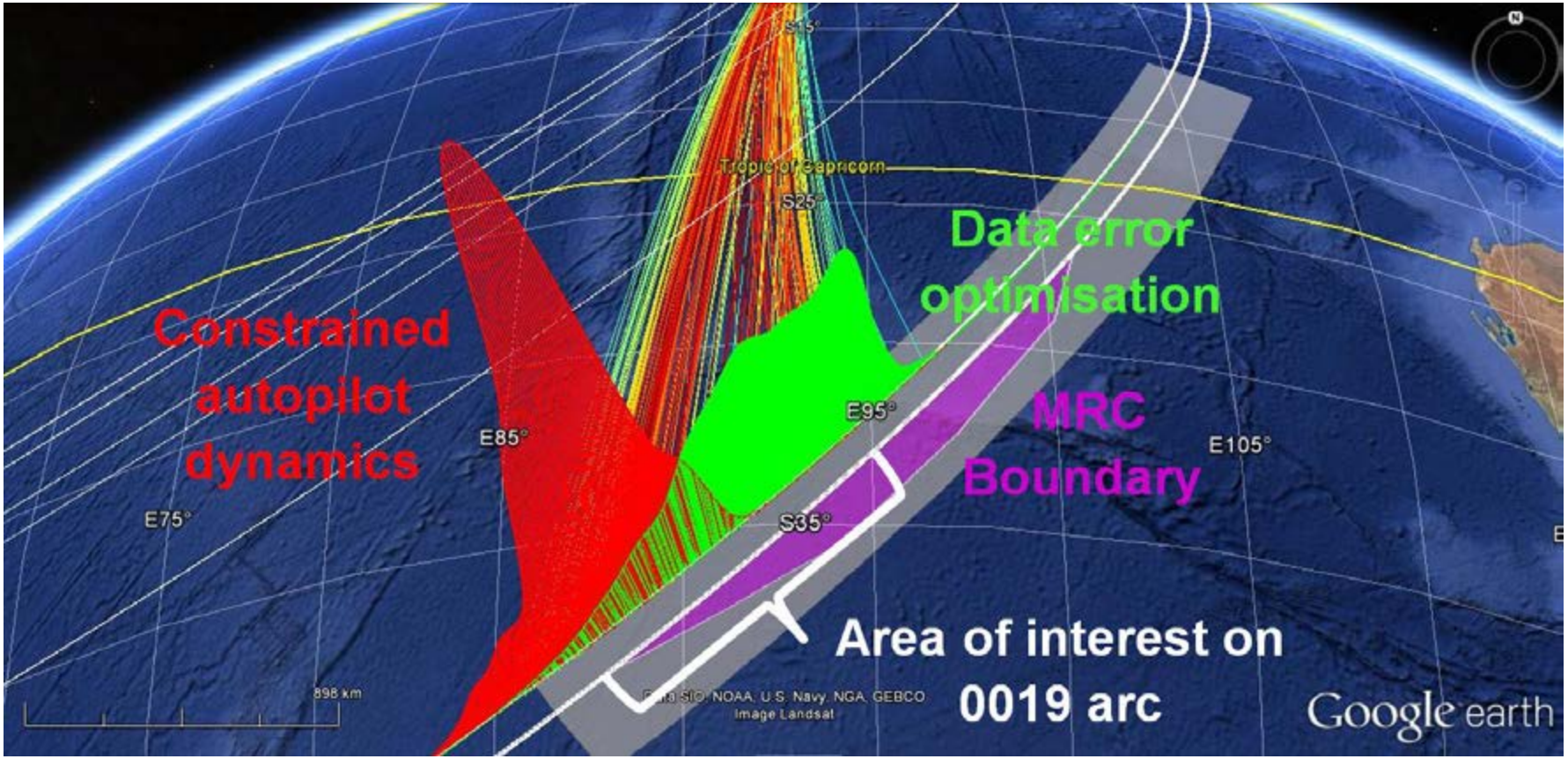
Very Good Agreement with June 26 ATSB Report

MH370 Results [JZ, June 26 Data]





MH370 Results [ATSB, Oct 8 Data]



Source: Google Earth/ Flight path reconstruction group

On Oct 8, search area was shifted 800 km SW.

Explanation for Oct 8 Shift

- Shift based on two unanswered ground-to-aircraft phone calls between 1st and 2nd ping
- These suggest aircraft turned south earlier than previously thought

“The sensitivity of the reconstructed flightpath to frequency errors is such that there remains significant uncertainty in the final location”. [5]

“Ongoing work may result in changes to the prioritization and location of search activity within the current search area.". [6]

[5] Inmarsat, *The Journal of Navigation*
[6] ATSB, Oct 8. as reported in *Aviation Week*

Historical Context

- Engineers at JHU-APL tracked the flight path of Sputnik in 1957.
- They realized that, knowing the path of Sputnik, they could use the Doppler effect to determine their location on earth.
- In 1964, JHU-APL engineers developed TRANSIT, the first satellite navigation system.
- TRANSIT was used by the US Navy to provide accurate location information to Polaris submarines.
- TRANSIT was superceded by GPS in 1996.
- Doppler tracking is still widely used for many applications.

Acknowledgements

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