Math 6390 (Numerical Linear Algebra) Midterm Exam
3/5/14
Dr. Minkoff

Name:

Instructions: You may not use notes or books on this exam. Don't spend too much time on any one problem. Show your work!

[16 pts] (1a) Prove that if the matrix A has the decomposition $A = M^t M$ with M nonsingular, then A is positive definite.

① Show $A = M^t M$ is symmetric:

Proof: $A^t = (M^t M)^t = M^t M^{tt} = M^t M = A$ ✓

② Show that $x^t A x > 0 \ \forall x \neq 0$

Proof: $x^t A x = x^t (M^t M) x$. Let $y = Mx$. Then

$x^t A x = y^t y = \sum_{i=1}^{n} y_i^2 \geq 0$

$\sum_{i=1}^{n} y_i^2 = 0$ iff $\vec{y} = \vec{0}$. But $y = Mx \neq 0$ because $x \neq 0$ and M is nonsingular.

$\Rightarrow A$ is spd. ∎

(b) Find the Cholesky factor M in the decomposition $A = M^t M$ for

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix}.$$

$$A = M^t M = \begin{bmatrix} 1 & 0 \\ 2 & \sqrt{5} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & \sqrt{5} \end{bmatrix}$$

[24 pts] (2) Consider the system $Ax = b$ where $\epsilon << 1$,

$$A = \begin{bmatrix} \epsilon & 1 \\ 1 & 1 \end{bmatrix}$$

and

$$b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

(a) Assuming your computer uses rounding to store floating point numbers, determine the computed solution gotten from applying **naive** Gaussian elimination to $Ax = b$. (Hint: do not round until the end.)

naive GE $\Rightarrow$ no pivoting!

$M_{21} = \frac{1}{\varepsilon}$ so $LU = \begin{bmatrix} 1 & 0 \\ 1/\varepsilon & 1 \end{bmatrix} \begin{bmatrix} \varepsilon & 1 \\ 0 & 1-\frac{1}{\varepsilon} \end{bmatrix}$

$c = \begin{bmatrix} 1 \\ 2-\frac{1}{\varepsilon} \end{bmatrix}$ $\quad Ux = c: \begin{bmatrix} \varepsilon & 1 \\ 0 & 1-\frac{1}{\varepsilon} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2-\frac{1}{\varepsilon} \end{bmatrix}$

$x_2 = \dfrac{2-\frac{1}{\varepsilon}}{1-\frac{1}{\varepsilon}} \approx \dfrac{-\frac{1}{\varepsilon}}{-\frac{1}{\varepsilon}} = 1$ for $\varepsilon << 1$

and $\varepsilon . x_1 + x_2 = 1 \Rightarrow \varepsilon x_1 + 1 = 1 \Rightarrow x_1 = 0$ $\quad \boxed{x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$

(b) What is the true solution to the problem $Ax = b$? (Hint: don't apply Gaussian Elimination. Just solve the system simultaneously.)

$\begin{bmatrix} \varepsilon & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow$ $\quad \begin{aligned} \varepsilon x_1 + x_2 &= 1 \\ x_1 + x_2 &= 2 \end{aligned}$

1st eqn:

$(\varepsilon - 1)x_1 = -1$

$x_1 \approx 1$

$\boxed{\text{for } \varepsilon << 1, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

2nd eqn:

$x_1 + x_2 = 2$

$\Rightarrow 1 + x_2 = 2 \Rightarrow x_2 = 1$

(c) Explain why your solution in part (a) differs from your solution in part (b). Is this error due to the matrix A or the algorithm you used?

Error is due to algorithm.
Matrix is fine, but you need to use pivoting to get well-scaled multiplier and stable algorithm.

(d) Approximately how many steps will elimination take to solve 10 systems with the same 50x50 coefficient matrix A?

For forward elimination cost to reduce

$A = LU$ is $O(n^3) \approx (50^3) = 125{,}000$ flops

Back substitution is $O(50^2) \approx 10(50^2)$

$\approx 25{,}000$

So total cost to solve 10 systems

is $125{,}000$
$+ \ 25{,}000$

$\approx 150{,}000$ flops.

[20 pts](3) Let

$$A = \begin{bmatrix} 1 & 1000 \\ 0 & 1 \end{bmatrix}.$$

(a) Calculate A^{-1} and the condition number $k_\infty(A)$.

$$A^{-1} = \begin{bmatrix} 1 & -1000 \\ 0 & 1 \end{bmatrix}$$

$\|A\|_\infty =$ max row sum (A)

so $\|A\|_\infty = 1001$

$\|A^{-1}\|_\infty = 1001$

$$K_\infty(A) = (1001)^2 = 1{,}002{,}001$$

so $K_\infty(A) \approx 1 \times 10^6$

(b) Find the solution to the system $Ax = b$ for the two right hand sides:

$$b = \begin{bmatrix} 1000 \\ 1 \end{bmatrix}$$

and

$$b + \delta b = \begin{bmatrix} 1000 \\ 0 \end{bmatrix}$$

$$x = \begin{bmatrix} 1 & -1000 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1000 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$x + \delta x = \begin{bmatrix} 1 & -1000 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1000 \\ 0 \end{bmatrix} = \begin{bmatrix} 1000 \\ 0 \end{bmatrix}$$

(c) Calculate $||\delta b||_\infty/||b||_\infty$ and $||\delta x||_\infty/||x||_\infty$ for the two right hand sides in Part (b) above. Carefully explain Part (b) in light of Part (a).

$$\|\delta b\|_\infty = \left\|\begin{bmatrix}0\\1\end{bmatrix}\right\|_\infty = 1$$

$$\|b\|_\infty = \left\|\begin{bmatrix}1000\\1\end{bmatrix}\right\|_\infty = 1000$$

$$\Rightarrow \frac{\|\delta b\|_\infty}{\|b\|_\infty} = \frac{1}{1000}$$

$$\|\delta x\|_\infty = \left\|\begin{bmatrix}1000\\-1\end{bmatrix}\right\|_\infty = 1000$$

$$\|x\|_\infty = \left\|\begin{bmatrix}0\\1\end{bmatrix}\right\|_\infty = 1$$

$$\Rightarrow \frac{\|\delta x\|_\infty}{\|x\|_\infty} = 1000$$

so $$\frac{\|\delta x\|}{\|x\|} \le K(A)\frac{\|\delta b\|}{\|b\|}$$

$$\Rightarrow 1000 \le 10^6(10^{-3}) = 10^3 \checkmark$$

(d) If we solve the system $Ax = b$ using Gaussian Elimination with partial pivoting on a machine with 8 digits of accuracy, what does the condition number in Part (a) tell us about the number of digits we can trust in the solution x?

$S = 8$ digits accuracy

$K(A) = 10^6 \Rightarrow 6 = t$

$S - t$ accurate digits

$\Rightarrow$ $\boxed{\text{we can trust 2 digits in soln}}$

[14 pts](4) Prove that the back substitution algorithm is backwards stable for matrices of size 2×2.

$$\begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

In real arithmetic:

$$r_{22} x_2 = b_2$$
$$\Rightarrow x_2 = \frac{b_2}{r_{22}}$$

floating pt arithmetic: $x_2 = b_2 \oplus\!\!\!\!\div\; r_{22} = \frac{b_2}{r_{22}}(1+\varepsilon_1)$ (*)

where $|\varepsilon_1| \le \varepsilon_{mach}$

Want data perturbations to occur on matrix entries of R.

Let $\varepsilon_1' = \frac{-\varepsilon_1}{1+\varepsilon_1}$, $\tilde{x} = \frac{b_2}{r_{22}(1+\varepsilon_1')} = \frac{b_2}{r_{22}\left(1 - \frac{\varepsilon_1}{1+\varepsilon_1}\right)} = \frac{b_2}{r_{22}\left(\frac{1}{1+\varepsilon_1}\right)}$

$= \frac{b_2(1+\varepsilon_1)}{r_{22}}$ (what we had in (*))

Taylor's Thm gives $\varepsilon_1' = -\varepsilon_1 + O(\varepsilon_1^2)$ so $x_2 = \frac{b_2}{r_{22}(1+\varepsilon_1')}$

$r_{11}x_1 + r_{12}x_2 = b_1 \Rightarrow r_{11}x_1 = b_1 - r_{12}\tilde{x}_2$

$\Rightarrow \tilde{x}_1 = (b_1 \ominus (r_{12} \otimes \tilde{x}_2)) \oplus\!\!\!\!\div\; r_{11}$

$\Rightarrow \tilde{x}_1 = (b_1 \ominus \underbrace{\tilde{x}_2 r_{12}(1+\varepsilon_2)}_{\text{from mult}}) \oplus\!\!\!\!\div\; r_{11}$ where $|\varepsilon_2| \le \varepsilon_{mach}$

$\Rightarrow \tilde{x}_1 = (b_1 - \tilde{x}_2 r_{12}(1+\varepsilon_2))\underbrace{(1+\varepsilon_3)}_{\text{for subtraction}} \oplus\!\!\!\!\div\; r_{11}$

$\Rightarrow \tilde{x}_1 = (b_1 - r_{12}\tilde{x}_2(1+\varepsilon_2))(1+\varepsilon_3)\underbrace{(1+\varepsilon_4)}_{\text{for division}}$ $|\varepsilon_3|, |\varepsilon_4| \le \varepsilon_{mach}$.

or

$$\tilde{x} = \frac{(b_1 - \tilde{x}_2 r_{12}(1+\varepsilon_2))}{r_{11}(1+\varepsilon_3')(1+\varepsilon_4')} \approx \frac{b_1 - \tilde{x}_2 r_{12}(1+\varepsilon_2)}{r_{11}(1+2\varepsilon_5)}$$

with $|\varepsilon_5| \le \varepsilon_{mach} + O(\varepsilon_{mach}^2)$

so $(R+\delta R)\tilde{x} = b$ where

$$\delta R = \begin{bmatrix} \frac{|\delta r_{11}|}{|r_{11}|} & \frac{|\delta r_{12}|}{|r_{12}|} \\ 0 & \frac{|\delta r_{22}|}{|r_{22}|} \end{bmatrix} \le \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \varepsilon_m + O(\varepsilon_{ma}$$

So 2x2 case of back substitution is backwards stable.

5a

[16 pts](~~6~~a) Using Gram-Schmidt orthogonalization write the matrix

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 1 & 2 \end{bmatrix}$$

as the product of an orthogonal matrix Q and an upper triangular matrix R.

By Gram-Schmidt $a = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ so $q_1 = \dfrac{a}{\|a\|}$

$\|a\|_2 = \sqrt{3^2+2^2+1^2} = \sqrt{14}$ so $\boxed{q_1 = \dfrac{1}{\sqrt{14}} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}}$

$b = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$ $\quad b' = b - (q_1^t b) q_1 \quad q_1^t b = \dfrac{1}{\sqrt{14}} [3\ 2\ 1] \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$

$= \dfrac{1}{\sqrt{14}} [6+6+2] = \sqrt{14}$

so $b' = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} - \dfrac{\sqrt{14}}{\sqrt{14}} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ $\quad \|b'\| = \sqrt{3}$

so $\boxed{q_2 = \dfrac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}}$ $\quad Q = \begin{bmatrix} \frac{3}{\sqrt{14}} & -\frac{1}{\sqrt{3}} \\ 2/\sqrt{14} & \frac{1}{\sqrt{3}} \\ 1/\sqrt{14} & \frac{1}{\sqrt{3}} \end{bmatrix}$

$R = \begin{bmatrix} q_1^t a & q_1^t b \\ 0 & q_2^t b \end{bmatrix}$ $\quad q_1^t a = \dfrac{1}{\sqrt{14}} [3\ 2\ 1] \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \sqrt{14}$

$q_2^t b = \dfrac{1}{\sqrt{3}} [-1\ 1\ 1] \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} = \dfrac{1}{\sqrt{3}} [-2+3+2] = \sqrt{3}$

so $\boxed{R = \begin{bmatrix} \sqrt{14} & \sqrt{14} \\ 0 & \sqrt{3} \end{bmatrix}}$ $\quad \boxed{A = QR}$

(b) Now use your QR decomposition of A to find the least squares solution for the system

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$$

Least Squares: $Rx = Q^t b$

$$Q^t b = \begin{bmatrix} 3/\sqrt{14} & 2/\sqrt{14} & 1/\sqrt{14} \\ -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 10/\sqrt{14} \\ -2/\sqrt{3} \end{bmatrix}$$

$$Rx = Q^t b: \begin{bmatrix} \sqrt{14} & \sqrt{14} \\ 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10/\sqrt{14} \\ -2/\sqrt{3} \end{bmatrix}$$

$$\Rightarrow x_2 = \frac{-2/\sqrt{3}}{\sqrt{3}} = \boxed{-2/3 = x_2}$$

$$\sqrt{14}\, x_1 - \frac{2}{3}\sqrt{14} = \frac{10}{\sqrt{14}}$$

$$\Rightarrow x_1 - 2/3 = \frac{10}{14}$$

$$\Rightarrow \boxed{x_1 = \frac{29}{21}}$$

Please sign the following honor statement: *On my honor, I pledge that I have neither given nor received any aid on this exam.*

$$\boxed{\bar{x} = \begin{bmatrix} 29/21 \\ -2/3 \end{bmatrix}}$$

(6)[EXTRA CREDIT, 8 pts] Prove that

$$||A||_1 = max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|$$

Proof:

$$A = \left[a_1 \middle| a_2 \middle| \cdots \middle| a_n \right]$$

where each a_j is a column vector of length n.

Unit ball for one norm is set $\{x \in \mathbb{R}^n : \sum_{j=1}^{n} |x_j| \le 1\}$

Any vector Ax in image of this set

satisfies $\underbrace{||Ax||_1}_{\text{linear combination of columns of } A} = || \sum_{j=1}^{n} x_j a_j ||_1 \le \sum_{j=1}^{n} \underbrace{|x_j|}_{\text{scalar}} \underbrace{||a_j||_1}_{\text{vector}}$

$$\le \max ||a_j||_1 \sum_{j=1}^{n} |x_j| \le \max_{1 \le j \le n} ||a_j||_1$$

If $x = e_j$ where j maximizes $||a_j||_1$, then bound is attained and $||A||_1 = \max_{1 \le j \le n} ||a_j||_1$

$\boxed{\equiv}$