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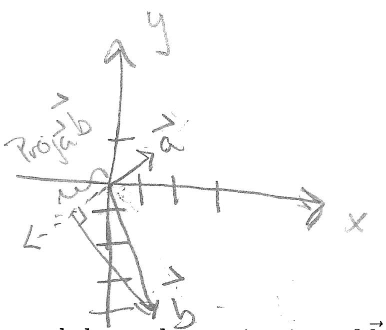
| 1 /13 | 2 /12 | 3 /10 | 4 /16 |
| 5 /16 | 6 /8 | T /75 | |

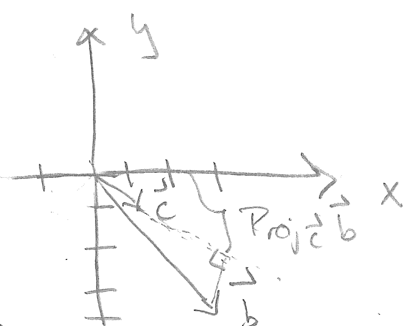
Math 2415 (Spring 2014) Exam 1 2/20/14
Dr. Minkoff

No calculators, books or notes! Show all work and give **complete explanations**. Don't spend too much time on any one problem.

[13 pts](1a) Let $\vec{a} = \vec{i} + \vec{j}$, $\vec{c} = \vec{i} - \vec{j}$, and $\vec{b} = 3\vec{i} - 4\vec{j}$. Show that $\vec{a}$ and $\vec{c}$ are perpendicular.

$\vec{a} \cdot \vec{c} = 1 \cdot 1 + 1 \cdot (-1) = 0$ ✓ so $\vec{a}$ and $\vec{c}$ are $\perp$.




(b) Calculate and draw the projection of $\vec{b}$ onto $\vec{a}$ and $\vec{c}$.

$\text{Proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a} = \frac{(1,1)\cdot(3,-4)}{(\sqrt{1^2+1^2})^2}(1,1) = \frac{3-4}{(\sqrt{2})^2}(\vec{i}+\vec{j}) = \frac{-1}{2}(\vec{i}+\vec{j})$

so $\boxed{\text{Proj}_{\vec{a}}\vec{b} = \frac{-1}{2}\vec{i} - \frac{1}{2}\vec{j}}$

Since $\vec{a}$ and $\vec{c}$ are $\perp$,

$\text{Proj}_{\vec{c}}\vec{b} = \vec{b} - \text{Proj}_{\vec{a}}\vec{b} = 3\vec{i} - 4\vec{j} - (-\frac{1}{2}\vec{i} - \frac{1}{2}\vec{j})$

$= 3\frac{1}{2}\vec{i} - 3\frac{1}{2}\vec{j}$

$\boxed{\text{Proj}_{\vec{c}}\vec{b} = \frac{7}{2}\vec{i} - \frac{7}{2}\vec{j}}$

[12 pts](2a) Find parametric equations for the line through the points $(2, -1, -3)$ and $(0, 2, -3)$.

need vector and a pt on line
(direction)

direction given by $(2-0, -1-2, -3+3) = (2, -3, 0)$

$$\boxed{\text{so line is} \quad \begin{aligned} x &= 2+2t \\ y &= -1-3t \\ z &= -3 \end{aligned}}$$

(b) Find the vector equation of the line that contains the point $(-1, 3, 0)$ and is parallel to $2\vec{\imath} - 3\vec{\jmath} - \vec{k}$.

line is $\boxed{l(t) = (-1, 3, 0) + t(2, -3, -1)}$

or

$\boxed{l(t) = (2t-1)\vec{\imath} + (3-3t)\vec{\jmath} - t\vec{k}}$

[10 pts](3) Show that the points $(2, 3, 2)$, $(1, -1, -3)$, $(1, 0, -1)$, and $(5, 9, 5)$ all lie on the same plane. (Hint: find the equation of the plane.)

two vectors spanning plane:

① $(2-1, 3+1, 2+3) = (1, 4, 5)$
② $(2-1, 3-0, 2+1) = (1, 3, 3)$

normal to plane is

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 4 & 5 \\ 1 & 3 & 3 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} 4 & 5 \\ 3 & 3 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 5 \\ 1 & 3 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix}$$

$$= (12-15)\vec{i} - (3-5)\vec{j} + (3-4)\vec{k} = -3\vec{i} + 2\vec{j} - \vec{k}$$

Plane eqn: $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$ or

$(-3, 2, -1) \cdot ((x-1), (y-0), (z+1)) = 0$

$\Rightarrow -3(x-1) + 2y - (z+1) = 0$

$\Rightarrow \boxed{-3x + 2y - z = -2}$ eqn of plane.

Check pts satisfy this eqn:

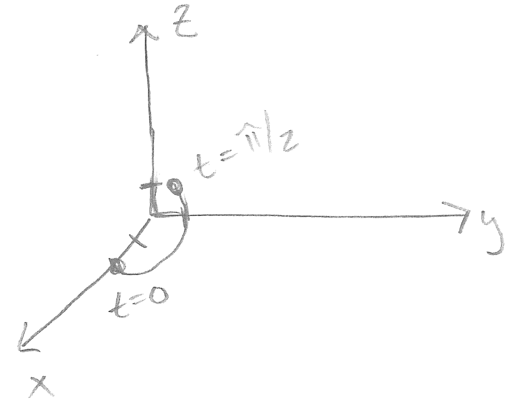
① $-3(2) + 2(3) - 2 \stackrel{?}{=} -2 \Rightarrow -6 + 6 - 2 \stackrel{?}{=} -2$ ✓

② $-3(1) + 2(-1) - (-3) = -3 - 2 + 3 = -2$ ✓

③ $-3(1) + 2(0) - (-1) = -2$ ✓

$-3(5) + 2(9) - 5 = -15 + 18 - 5 = -2$ ✓

[16 pts](4) (a) Sketch the curve $\mathbf{r}(t) = (2\cos t, \sin t, t)$ for $0 \le t \le \pi/2$.




$t=0$, $\vec{r}(t) = (2,0,0)$
$t=\pi/2$, $\vec{r}(t) = (0,1,\pi/2)$

(b) Find parametric equations for the tangent line to the curve in Part (a) at the point $t = \pi/2$.

$\vec{r}(t) = (0,1,\pi/2)$
$\vec{r}\,'(t) = (-2\sin t, \cos t, 1)$
$\vec{r}\,'(\pi/2) = (-2(1), 0, 1) = (-2,0,1)$

tangent line: $\ell(t) = (0,1,\pi/2) + (t-0)(-2,0,1)$

Parametric eqns of tangent line: $x = -2t$, $y = 1$, $z = \pi/2 + t$

(c) Write down the integral to calculate the arclength of the curve in Part (a) from $0 \le t \le \pi/2$. (Note: you do not need to evaluate this integral!)

$L = \int_a^b |\vec{r}\,'(t)|\,dt$

arclength $L = \int_0^{\pi/2} \sqrt{(-2\sin t)^2 + \cos^2 t + 1}\; dt$

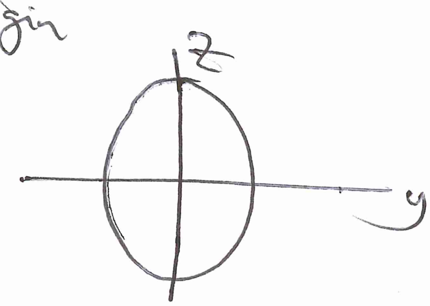
[16 pts](5) Find the traces of the surface $x^2 - 9y^2 - 4z^2 = 36$ in the planes $x = k$, $y = k$, and $z = k$. Identify the type of surface and graph it.

$\boxed{x=k}$ $9y^2 + 4z^2 = k^2 - 36.$

IF $|k| < 6$ no sol$^{ns}$

IF $k = 6$ get ~~2 points~~ origin

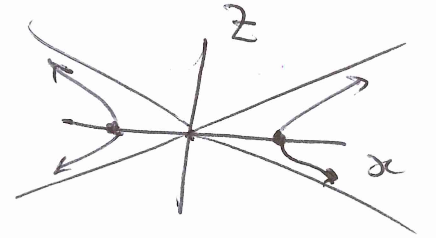
IF $|k| > 6$ get ellipse




$\boxed{y=k}$ $x^2 - 4z^2 = 36 + 9k^2.$

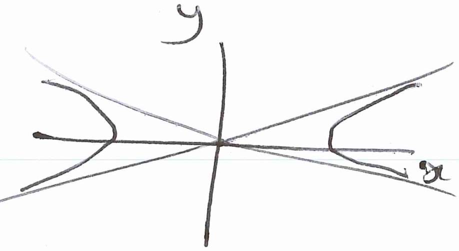
Asymptotes: $z = \pm x/2$

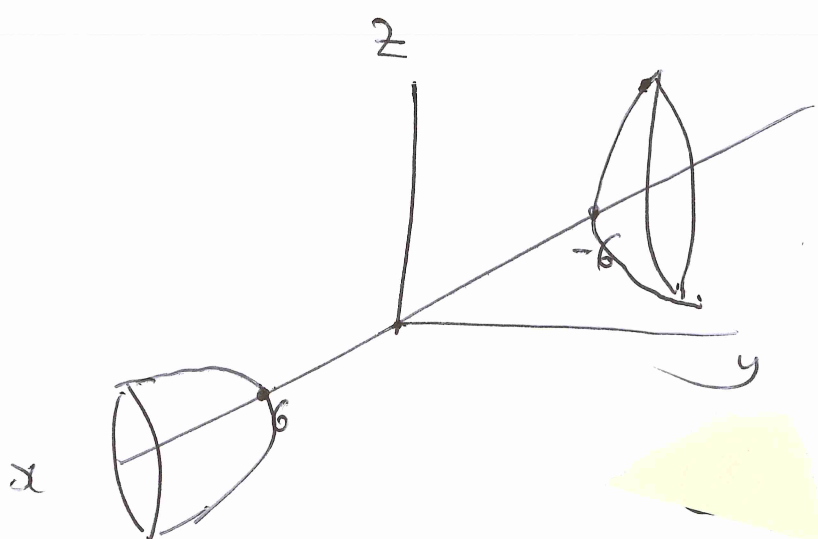
$z = 0$, $x \in \mathbb{R}$




$\boxed{z=k}$ $x^2 - 9y^2 = 36 + 4k^2$

Asymptotes $y = \pm x/3$




Hyperbolic Paraboloid of 2 sheets

[8 pts](6) Show that if $|\vec{r}(t)| = c$ (where $c$ is a constant), then $\vec{r}'(t)$ is orthogonal to $\vec{r}(t)$ for all $t$.

Proof: $|\vec{r}(t)| = c$ by assumption so squaring both sides gives

$$|\vec{r}(t)|^2 = c^2$$

$$\Rightarrow \vec{r}(t) \circ \vec{r}(t) = c^2$$

$$\frac{d}{dt}\left[\vec{r}(t) \cdot \vec{r}(t)\right] = 0$$

$$\Rightarrow \vec{r}\,'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}\,'(t) = 0$$

$$\Rightarrow 2\vec{r}\,'(t) \circ \vec{r}(t) = 0$$

$$\Rightarrow \vec{r}\,'(t) \perp \vec{r}(t).$$

Please sign the following honor statement:

*On my honor, I pledge that I have neither given nor received any aid on this exam.*

Signature: ____________________