An Efficient Algorithm for Computing High-Quality Paths amid Polygonal Obstacles

Pankaj K. Agarwal
Duke University

Kyle Fox
Duke University

Oren Salzman
Tel-Aviv University
An Efficient Algorithm for Computing High-Quality Paths amid Polygonal Obstacles

Pankaj K. Agarwal
Duke University

Kyle Fox
Duke University

Oren Salzman
Tel-Aviv University
Motion Planning

• Robot needs to go from $s$ to $t$
Motion Planning

• Robot needs to go from \( s \) to \( t \)

• What’s the best route?
• Take a shortest path?
• Take a shortest path?
• That seems dangerous…
• Maximize the minimum clearance?
• Maximize the minimum clearance?

• That seems exhausting…
• How about both?
• How about both?

• Optimize a metric that balances path length with clearance
• First, some notation
• First, some notation

\( \mathcal{O} \): set of polygonal obstacles in \( \mathbb{R}^2 \) with \( n \) vertices total
• First, some notation

\( \mathcal{O} \): set of polygonal obstacles in \( \mathbb{R}^2 \) with \( n \) vertices total

\( \gamma \): a path; formally \( \gamma : [0,1] \rightarrow \mathbb{R}^2 \)
First, some notation

\( \mathcal{O} \): set of polygonal obstacles in \( \mathbb{R}^2 \) with \( n \) vertices total

\( \gamma \): a path; formally \( \gamma : [0,1] \rightarrow \mathbb{R}^2 \)

\( \gamma[p,q] \): subpath of \( \gamma \) from \( p \) to \( q \)
First, some notation

\( \mathcal{O} \): set of polygonal obstacles in \( \mathbb{R}^2 \) with \( n \) vertices total

\( \gamma \): a path; formally \( \gamma : [0,1] \to \mathbb{R}^2 \)

\( \gamma[p,q] \): subpath of \( \gamma \) from \( p \) to \( q \)

\( \|p,q\| \): the Euclidean distance between \( p, q \in \mathbb{R}^2 \)
• First, some notation

\( \mathcal{O} \): set of polygonal obstacles in \( \mathbb{R}^2 \) with \( n \) vertices total

\( \gamma \): a path; formally \( \gamma : [0,1] \rightarrow \mathbb{R}^2 \)

\( \gamma[p,q] \): subpath of \( \gamma \) from \( p \) to \( q \)

\( ||p,q|| \): the Euclidean distance between \( p,q \in \mathbb{R}^2 \)

\( \text{cl}(p) \): the clearance of a point; \( \text{cl}(p) = \min_{o \in \mathcal{O}} ||p,o|| \)
Our Goal

• Minimize the cost $\mu(\gamma)$ of an $s,t$-path $\gamma$ where

$$\mu(\gamma) = \int_\gamma \frac{1}{\text{cl}(\gamma(\tau))} \, d\tau$$

[Wein, van den Berg, Halperin ’08]
Our Goal

- Minimize the cost $\mu(\gamma)$ of an $s,t$-path $\gamma$ where

$$\mu(\gamma) = \int_{\gamma} \frac{1}{\text{cl}(\gamma(\tau))} \, d\tau$$

[Wein, van den Berg, Halperin '08]
\( \mu(p, q) \): cost of cheapest path from \( p \) to \( q \)
$\mu(p,q)$: cost of cheapest path from $p$ to $q$

- Want an *approximately* cheapest $s,t$-path $y$ such that $\mu(y) \leq (1+\varepsilon) \mu(s,t)$ for any $\varepsilon > 0$ (a $(1+\varepsilon)$-approximation or *approximation scheme*)
\( \mu(p, q) \): cost of cheapest path from \( p \) to \( q \)

• Want an \textit{approximately} cheapest \( s, t \)-path \( y \) such that \( \mu(y) \leq (1+\varepsilon) \mu(s,t) \) for any \( \varepsilon > 0 \) (a \( (1+\varepsilon) \)-approximation or \textit{approximation scheme})

• Wein \textit{et al.} ['08] gave an approximation algorithm with running time polynomial in \( n, 1/\varepsilon \), and \( \Lambda \)
\( \mu(p, q) \): cost of cheapest path from \( p \) to \( q \)

- Want an *approximately* cheapest \( s, t \)-path \( y \) such that \( \mu(y) \leq (1+\varepsilon) \mu(s, t) \) for any \( \varepsilon > 0 \) (a \((1+\varepsilon)\)-approximation or *approximation scheme*)

- *Wein et al.* ['08] gave an approximation algorithm with running time polynomial in \( n, 1/\varepsilon \), and \( \Lambda \)

\( \Lambda \): essentially the total cost of edges in \( \mathcal{G}'s \) *Voronoi diagram* (not a polynomial in input size)
\( \mu(p,q) \): cost of cheapest path from \( p \) to \( q \)

- Want an *approximately* cheapest \( s,t \)-path \( y \) such that \( \mu(y) \leq (1+\varepsilon) \mu(s,t) \) for any \( \varepsilon > 0 \) (a \( (1+\varepsilon) \)-approximation or *approximation scheme*)

- Wein *et al.* [’08] gave an approximation algorithm with running time polynomial in \( n, 1/\varepsilon \), and \( \Lambda \)

\( \Lambda \): essentially the total cost of edges in \( \mathcal{G} \)'s *Voronoi diagram* (not a polynomial in input size)

- Error is actually *additive*
Our Contribution

• First polynomial time approximation scheme
Our Contribution

• First polynomial time approximation scheme

• Returns a path of cost at most \((1+\varepsilon) \mu(s,t)\) in time

\[
O\left(\frac{n^2}{\varepsilon^2 \log \frac{n}{\varepsilon}}\right)
\]
This Talk’s Path

• Describe Wein et al.’s approach and observations
This Talk’s Path

• Describe Wein et al.’s approach and observations

• Explain our new key tools, well behaved paths and anchor points
This Talk’s Path

• Describe Wein et al.’s approach and observations

• Explain our new key tools, **well behaved paths** and **anchor points**

• Describe our algorithm!
This Talk’s Path

• Describe Wein et al.’s approach and observations

• Explain our new key tools, well behaved paths and anchor points

• Describe our algorithm!

  1. An $O(n \log n)$ time $O(n)$-approximation =>
This Talk’s Path

• Describe Wein et al.’s approach and observations

• Explain our new key tools, well behaved paths and anchor points

• Describe our algorithm!
  1. An $O(n \log n)$ time $O(n)$-approximation =>
  2. An $O(n^2 \log n)$ time $O(1)$-approximation =>
This Talk’s Path

• Describe Wein et al.’s approach and observations

• Explain our new key tools, well behaved paths and anchor points

• Describe our algorithm!
  1. An $O(n \log n)$ time $O(n)$-approximation =>
  2. An $O(n^2 \log n)$ time $O(1)$-approximation =>
  3. The approximation scheme (time permitting)
Short Sited

- Clearance defined entirely on the closest obstacle
Short Sited

- Clearance defined entirely on the \textit{closest} obstacle

\( \mathcal{V} \): \textit{Voronoi diagram} of \( \mathcal{O} \); subdivides \( \mathbb{R}^2 \setminus \mathcal{O} \) into \textit{cells} of points sharing a closest obstacle feature
Short Sited

• Clearance defined entirely on the closest obstacle

\( \mathcal{V} \): Voronoi diagram of \( \mathcal{O} \); subdivides \( \mathbb{R}^2 \setminus \mathcal{O} \) into cells of points sharing a closest obstacle feature

\( O(n) \) complexity, computable in \( O(n \log n) \) time
• Wein et al. sample several points along edges of $\mathcal{V}$
• Wein et al. sample several points along edges of $\mathcal{V}$

• Near-optimal path enters cells at sample points
• Wein et al. sample several points along edges of $\mathcal{V}$

• Near-optimal path enters cells at sample points

• Build a graph over cheapest paths between sample points sharing a cell
• Let \( s = r_s \exp(i\theta_s) \), \( t = r_t \exp(i\theta_t) \)

• Let \( o \) be a solo point obstacle at the origin
• Let \( s = r_s \exp(i \theta_s) \), \( t = r_t \exp(i \theta_t) \)

• Let \( o \) be a solo point obstacle at the origin

• Optimal path is a \textit{logarithmic spiral} centered on \( o \) with cost

\[
\mu(s, t) = \sqrt{(\theta_t - \theta_s)^2 + (\ln r_t - \ln r_s)^2}
\]
• Let \( s = r_s \exp(i\theta_s), \ t = r_t \exp(i\theta_t) \)

• Let \( o \) be a solo point obstacle at the origin

• Optimal path is a \textit{logarithmic spiral} centered on \( o \) with cost

\[
\mu(s, t) = \sqrt{(\theta_t - \theta_s)^2 + (\ln r_t - \ln r_s)^2}
\]
• Let $s = r_s \exp(i\theta_s)$, $t = r_t \exp(i\theta_t)$

• Let $o$ be a horizontal line obstacle through origin
• Let \( s = r_s \exp(i \theta_s) \), \( t = r_t \exp(i \theta_t) \).

• Let \( o \) be a horizontal line obstacle through origin.

• Optimal path is a \textit{circular arc} centered on \( o \) with cost

\[
\mu(s, t) = \ln \tan \frac{\theta_t}{2} - \ln \tan \frac{\theta_s}{2}
\]
• Let $s = r_s \exp(i\theta_s)$, $t = r_t \exp(i\theta_t)$

• Let $o$ be a horizontal line obstacle through origin

• Optimal path is a \textit{circular arc} centered on $o$ with cost

$$\mu(s, t) = \ln \tan \frac{\theta_t}{2} - \ln \tan \frac{\theta_s}{2}$$

![Diagram showing the optimal path as a circular arc centered on a horizontal line obstacle through the origin.](image)
• Let $s = r_s \exp(i\theta_s)$, $t = r_t \exp(i\theta_t)$

• Let $o$ be a horizontal line obstacle through origin with line through $s$ and $t$ perpendicular to $o$
• Let $s = r_s \exp(i\theta_s)$, $t = r_t \exp(i\theta_t)$

• Let $o$ be a horizontal line obstacle through origin with line through $s$ and $t$ perpendicular to $o$

• Optimal path is a *line segment* with cost

$$
\mu(s, t) = \ln c_l(t) - \ln c_l(s)
$$
• Let \( s = r_s \exp(i \theta_s) \), \( t = r_t \exp(i \theta_t) \)

• Let \( o \) be a horizontal line obstacle through origin with line through \( s \) and \( t \) perpendicular to \( o \)

• Optimal path is a line segment with cost

\[
\mu(s, t) = \ln \text{cl}(t) - \ln \text{cl}(s)
\]

Cheapest way to change clearance in all cases
• Let $s$ and $t$ lie on a Voronoi edge $e$
• Let $s$ and $t$ lie on a Voronoi edge $e$

• Optimal path between $s$ and $t$ follows the edge (and we can compute its cost)
• Let \( s \) and \( t \) lie on a Voronoi edge \( e \)

• Optimal path between \( s \) and \( t \) follows the edge (and we can compute its cost)

• So in general, optimal paths are a sequence of \textit{logarithmic spirals, circular arcs, line segments,} and \textit{walks along Voronoi edges}
Difficulties

• How do we reduce the number of sample points?
Difficulties

• How do we reduce the number of sample points?
• Where do we place them?
Difficulties

• How do we reduce the number of sample points?
• Where do we place them?
• Can we keep the graph sparse?
$\tilde{\mathcal{V}}$: Voronoi diagram *refined* by adding:
\( \tilde{\mathcal{V}} \) : Voronoi diagram refined by adding:

- Shortest paths from obstacle features to Voronoi vertices
\( \tilde{\mathcal{V}} \): Voronoi diagram *refined* by adding:

- Shortest paths from obstacle features to Voronoi vertices
- Shortest paths from obstacle features to Voronoi edges
\( \tilde{V} \): Voronoi diagram *refined* by adding:

- Shortest paths from obstacle features to Voronoi vertices
- Shortest paths from obstacle features to Voronoi edges
- Extensions of shortest paths from obstacle features through \( s \) and \( t \)
\(\tilde{V}\): Voronoi diagram *refined* by adding:

- Shortest paths from obstacle features to Voronoi vertices
- Shortest paths from obstacle features to Voronoi edges
- Extensions of shortest paths from obstacle features through \(s\) and \(t\)
- Each new edge is *perpendicular* to its feature
• Refined Voronoi diagram $\tilde{\mathcal{V}}$ has complexity $O(n)$
• Each cell $T \in \mathcal{Y}$ is incident to one obstacle feature and has three additional edges:
• Each cell $T \in \tilde{\mathcal{Y}}$ is incident to one obstacle feature and has three additional edges:
  
  - One external edge $K_T$ which is a monotone clearance parabolic arc (possibly degenerate)
• Each cell $T \in \tilde{\mathcal{V}}$ is incident to one obstacle feature and has three additional edges:
  
  ◦ One *external edge* $K_T$ which is a monotone clearance parabolic arc (possibly degenerate)
  
  ◦ Two *internal edges* $\alpha_T$ (shorter) and $\beta_T$ (longer) which are perpendicular to obstacle feature
Well-behaved Paths

• A $p,q$-path $\gamma$ through cell $T$ is well-behaved if
Well-behaved Paths

- A $p,q$-path $\gamma$ through cell $T$ is well-behaved if
  
  i. $\lambda = \gamma \cap \text{int}(T)$ is a connected subpath
Well-behaved Paths

- A $p, q$-path $\gamma$ through cell $T$ is well-behaved if
  i. $\lambda = \gamma \cap \text{int}(T)$ is a connected subpath
  ii. if it exists, then $\lambda$ has constant clearance
Lemma: Let $p$ and $q$ be two points on the edges of cell $T$. There exists a well-behaved path $p,q$-path $\gamma$ where $\mu(\gamma) \leq 7 \mu(p,q)$. 
Proof:

Let $\text{cl}_{\text{max}}(p, q)$ be the maximum clearance achieved by the cheapest $p, q$-path $\gamma^*$.
Proof:

- Let $\text{cl}_{\max}(p,q)$ be the *maximum clearance* achieved by the cheapest $p,q$-path $\gamma^*$

- Let $T'$ be the subset of $T$ containing points of clearance at most $\text{cl}_{\max}(p,q)$
Proof:

- Let $\text{cl}_{\text{max}}(p, q)$ be the *maximum clearance* achieved by the cheapest $p, q$-path $\gamma^*$
- Let $T'$ be the subset of $T$ containing points of clearance at most $\text{cl}_{\text{max}}(p, q)$
- Path $\gamma$ walks along the boundary of $T'$
• As a warmup, suppose $p \in \alpha_T$, $q \in \kappa_T$. 
• As a warmup, suppose \( p \in \alpha_T, \ q \in K_T \):

○ Let \( u_T \) be the intersection of \( \alpha_T \) and \( K_T \)
• As a warmup, suppose $p \in \alpha_T$, $q \in \kappa_T$:
  - Let $u_T$ be the intersection of $\alpha_T$ and $\kappa_T$:
  - $\mu(y[p, u_T]) \leq \mu(p, q)$
• As a warmup, suppose $p \in \alpha_T$, $q \in K_T$.
  
    Let $u_T$ be the intersection of $\alpha_T$ and $K_T$

    $\mu(\gamma[p, u_T]) \leq \mu(p, q)$

    $\mu(\gamma[u_T, q]) \leq 2\mu(p, q)$
• As a warmup, suppose $p \in \alpha_T$, $q \in K_T$.
  
  ◦ Let $u_T$ be the intersection of $\alpha_T$ and $K_T$
  
  ◦ $\mu(y[p, u_T]) \leq \mu(p, q)$
  
  ◦ $\mu(y[u_T, q]) \leq 2\mu(p, q)$
  
  ◦ $\mu(y) \leq 3\mu(p, q)$
• Now, suppose \( p \in \beta_T, \ q \in K_T \).
• Now, suppose $p \in \beta_T$, $q \in \kappa_T$.
  
  Let $w, w'$ be the intersection of $\lambda$ and $\beta_T, \kappa_T$
• Now, suppose $p \in \beta_T$, $q \in \kappa_T$.
  - Let $w, w'$ be the intersection of $\lambda$ and $\beta_T, \kappa_T$.
  - As $w'$ moves up $\kappa_T$, we see $||w', \beta_T||$ decrease.
• Now, suppose $p \in \beta_T$, $q \in \kappa_T$:
  
  - Let $w, w'$ be the intersection of $\lambda$ and $\beta_T, \kappa_T$
  - As $w'$ moves up $\kappa_T$, we see $||w', \beta_T||$ decrease
  - $\lambda$ is the *shortest* low-clearance path from $\beta_T$ to $\kappa_T$
• Now, suppose $p \in \beta_T$, $q \in K_T$.
  
  ⊡ Let $w$, $w'$ be the intersection of $\lambda$ and $\beta_T$, $K_T$
  
  ⊡ As $w'$ moves up $K_T$, we see $||w', \beta_T||$ decrease
  
  ⊡ $\lambda$ is the shortest low-clearance path from $\beta_T$ to $K_T$
  
  ⊡ $\lambda$ has clearance $\text{cl}_{\text{max}}(p, q)$ across its entire length
• Now, suppose $p \in \beta_T$, $q \in \kappa_T$.

○ $\mu(\lambda) \leq \mu(p, q)$
• Now, suppose $p \in \beta_T$, $q \in K_T$:
  - $\mu(\lambda) \leq \mu(p, q)$
  - $\mu(\gamma[p, w]) \leq \mu(p, q)$
• Now, suppose $p \in \beta_T$, $q \in K_T$:
  
  - $\mu(\lambda) \leq \mu(p, q)$
  - $\mu(\nu[p, w]) \leq \mu(p, q)$
  - $\mu(\nu[w', q]) \leq 3\mu(p, q)$
Now, suppose \( p \in \beta_T, \ q \in K_T. \)

\[
\begin{align*}
\mu(\lambda) & \leq \mu(p, q) \\
\mu(\gamma[p, w]) & \leq \mu(p, q) \\
\mu(\gamma[w', q]) & \leq 3\mu(p, q) \\
\mu(\gamma) & \leq 5\mu(p, q)
\end{align*}
\]
• Suppose $p \in \beta_T$, $q \in \alpha_T$, and $w' \in K_T$.
• Suppose $p \in \beta_T$, $q \in \alpha_T$, and $w' \in K_T$:

  - $\mu(\gamma[p, w]) \leq \mu(p, q)$, $\mu(\lambda) \leq \mu(p, q)$
• Suppose $p \in \beta_T$, $q \in \alpha_T$, and $w' \in \kappa_T$:
  - $\mu(\gamma[p, w]) \leq \mu(p, q)$, $\mu(\lambda) \leq \mu(p, q)$
  - $\mu(\gamma[u_T, q]) \leq \mu(p, q)$
Suppose $p \in \beta_T$, $q \in \alpha_T$, and $w' \in K_T$:

- $\mu(\gamma[p, w]) \leq \mu(p, q)$, $\mu(\lambda) \leq \mu(p, q)$
- $\mu(\gamma[u_T, q]) \leq \mu(p, q)$
- $\mu(\gamma[w', u_T]) \leq 4\mu(p, q)$, so $\mu(\gamma) \leq 7\mu(p, q)$
• Suppose $p \in \beta_T$, $q \in \alpha_T$, and $w' \in K_T$:
  - $\mu(\gamma[p, w]) \leq \mu(p, q)$, $\mu(\lambda) \leq \mu(p, q)$
  - $\mu(\gamma[u_T, q]) \leq \mu(p, q)$
  - $\mu(\gamma[w', u_T]) \leq 4 \mu(p, q)$, so $\mu(\gamma) \leq 7 \mu(p, q)$
  - Case with $w' \in \alpha_T$ is same as last slide
• Proof required knowing $c_{\text{max}}(p,q)$ to pick $\lambda$
• Proof required knowing $cl_{\text{max}}(p,q)$ to pick $\lambda$

• Can we find a cheap well-behaved path without knowing $cl_{\text{max}}(p,q)$?
Anchor Points

**Lemma**: Let $T$ be a cell of $\tilde{\mathcal{V}}$. There exist constant-time computable *anchor points* $w_\alpha^*$ and $w_\kappa^*$ on long internal edge $\beta_T$ such that:
Anchor Points

**Lemma**: Let $T$ be a cell of $\tilde{\mathcal{V}}$. There exist constant-time computable *anchor points* $w_{\alpha}^*$ and $w_{\kappa}^*$ on long internal edge $\beta_T$ such that:

- For any $p$ and $q$ on the edges of $T$, there exists a well-behaved path $p,q$-path $\gamma$ where $\mu(\gamma) \leq 7 \mu(p,q)$. Moreover...
Anchor Points

**Lemma**: Let $T$ be a cell of $\tilde{\mathcal{Y}}$. There exist constant-time computable anchor points $w_{\alpha^*}$ and $w_{K^*}$ on long internal edge $\beta_T$ such that:

- For any $p$ and $q$ on the edges of $T$, there exists a well-behaved path $p,q$-path $\gamma$ where $\mu(\gamma) \leq 7\mu(p,q)$. Moreover...

- If neither $p$ nor $q$ lie on $\beta_T$, then $\gamma$ stays on $\alpha_T$ and $K_T$. 
Anchor Points

**Lemma**: Let $T$ be a cell of $\tilde{\mathcal{V}}$. There exist constant-time computable anchor points $w_\alpha^*$ and $w_K^*$ on long internal edge $\beta_T$ such that:

- For any $p$ and $q$ on the edges of $T$, there exists a well-behaved path $p,q$-path $\gamma$ where $\mu(\gamma) \leq 7\mu(p,q)$. Moreover…
- If neither $p$ nor $q$ lie on $\beta_T$, then $\gamma$ stays on $\alpha_T$ and $K_T$.
- Otherwise, $\lambda \cap \beta_T \in \{w_\alpha^*, w_K^*, p, q\}$
Proof sketch:

- Assume $p$ is on $\beta_T$
Proof sketch:

- Assume $p$ is on $\beta_T$
- Pick $w \in \beta_T$ that minimizes $\mu(p, w) + \mu(\lambda)$
Proof sketch:

- Assume $p$ is on $\beta_T$
- Pick $w \in \beta_T$ that minimizes $\mu(p, w) + \mu(\lambda)$
- Use proof from earlier that best $\mu(p, w) + \mu(\lambda) \leq 2\mu(p, q)$ along with triangle inequality
When $\text{cl}(w) \geq \text{cl}(\rho)$, expression $\mu(\rho, w) + \mu(\lambda)$ is equal to $\ln \text{cl}(w) - \ln \text{cl}(\rho) + \mu(\lambda)$.
- When $\text{cl}(w) \geq \text{cl}(\rho)$, expression $\mu(\rho, w) + \mu(\lambda)$ is equal to $\ln \text{cl}(w) - \ln \text{cl}(\rho) + \mu(\lambda)$.

- Expression has one minimum $w^*$ over $w \in \beta_T$ which is independent of $\rho$; dependent on $T$ and edge holding $q$. 
- When $\text{cl}(w) \geq \text{cl}(p)$, expression $\mu(p, w) + \mu(\lambda)$ is equal to $\ln \text{cl}(w) - \ln \text{cl}(p) + \mu(\lambda)$

- Expression has one minimum $w^*$ over $w \in \beta_T$ which is independent of $p$; dependent on $T$ and edge holding $\lambda$

- When $\text{cl}(w) \leq \text{cl}(p)$, the best $w$ is $p$ itself
When \( \text{cl}(w) \geq \text{cl}(p) \), expression \( \mu(p, w) + \mu(\lambda) \) is equal to \( \ln \text{cl}(w) - \ln \text{cl}(p) + \mu(\lambda) \)

Expression has one minimum \( w^* \) over \( w \in \beta_T \) which is independent of \( p \); dependent on \( T \) and edge holding \( q \)

When \( \text{cl}(w) \leq \text{cl}(p) \), the best \( w \) is \( p \) itself

Have \( \lambda \) intersect higher clearance point of \( \{w^*, p\} \)
Algorithms!

- Compute progressively better approximations
Algorithms!

• Compute progressively better approximations

  1. $O(n \log n)$ time $O(n)$-approximation =>
Algorithms!

- Compute progressively better approximations
  1. $O(n \log n)$ time $O(n)$-approximation $\Rightarrow$
  2. $O(n^2 \log n)$ time $O(1)$-approximation $\Rightarrow$
Algorithms!

• Compute progressively better approximations
  1. $O(n \log n)$ time $O(n)$-approximation =>
  2. $O(n^2 \log n)$ time $O(1)$-approximation =>
  3. $O(n^2 / \epsilon^2 \log (n / \epsilon))$ time $(1+\epsilon)$-approximation
Algorithms!

- Compute progressively better approximations
  1. $O(n \log n)$ time $O(n)$-approximation $\Rightarrow$
  2. $O(n^2 \log n)$ time $O(1)$-approximation $\Rightarrow$
  3. $O(n^2 / \epsilon^2 \log (n / \epsilon))$ time $(1+\epsilon)$-approximation
- Assume $cl(s) \leq cl(t)$
Algorithms!

• Compute progressively better approximations
  1. $O(n \log n)$ time $O(n)$-approximation $\Rightarrow$
  2. $O(n^2 \log n)$ time $O(1)$-approximation $\Rightarrow$
  3. $O(n^2 / \varepsilon^2 \log (n / \varepsilon))$ time $(1+\varepsilon)$-approximation
• Assume $\text{cl}(s) \leq \text{cl}(t)$
• Let $y^*$ be the cheapest $s,t$-path
Algorithms!

• Compute progressively better approximations
  1. $O(n \log n)$ time $O(n)$-approximation $\Rightarrow$
  2. $O(n^2 \log n)$ time $O(1)$-approximation $\Rightarrow$
  3. $O(n^2 / \epsilon^2 \log (n / \epsilon))$ time $(1+\epsilon)$-approximation

• Assume $cl(s) \leq cl(t)$

• Let $y^*$ be the cheapest $s,t$-path

• Let $d^* = \mu(y^*) = \mu(s,t)$
O(n)-approximation

• Build a geometric graph $G_1$ by adding $O(n)$ edges to $\tilde{V}$
O(n)-approximation

- Build a geometric graph $G_1$ by adding $O(n)$ edges to $\tilde{V}$

- Edges have a cost equal to the cost of their paths through the plane
O(n)-approximation

• Build a geometric graph $G_1$ by adding $O(n)$ edges to $\tilde{\mathcal{V}}$

• Edges have a cost equal to the cost of their paths through the plane

• Compute cheapest path between $s$ and $t$ in $G_1$ in $O(n \log n)$ time
• For each cell $T$ of $\tilde{\mathcal{Y}}$: 
• For each cell $T$ of $\mathcal{V}$:
  - Add vertex at point $w_s \in \beta_T$ where $\text{cl}(w_s) = \text{cl}(s)$
• For each cell $T$ of $\tilde{\mathcal{V}}$:
  
  - Add vertex at point $w_s \in \beta_T$ where $\text{cl}(w_s) = \text{cl}(s)$
  - Add vertices for anchor points $w_{\alpha^*}$ and $w_{K^*}$
For each cell $T$ of $\tilde{\mathcal{V}}$:

- Add vertex at point $w_s \in \beta_T$ where $\text{cl}(w_s) = \text{cl}(s)$
- Add vertices for anchor points $w_\alpha^*$ and $w_K^*$
- Add constant clearance edges from $w_s$, $w_\alpha^*$, and $w_K^*$

[Diagram of a cell $T$ with vertices and edges labeled $\alpha_T$, $\beta_T$, $\alpha_T^*$, $\beta_T^*$, $w_s$, $w_\alpha^*$, $w_K^*$, $\lambda_s$, $\lambda_{\alpha_T}$, $\lambda_{\kappa_T}$, $\kappa_T$, and $\nu_T$. The diagram shows the relationships between these points and the cell boundaries.]
Lemma: $G_1$ contains an $s,t$-path of cost $O(n) \cdot d^*$
Lemma: $G_1$ contains an $s,t$-path of cost $O(n) \cdot d^*$

Proof:
- Suppose $y^*$ is disjoint from $G_1$ in cell $T$
Lemma: $G_1$ contains an $s,t$-path of cost $O(n) \cdot d^*$

Proof:

- Suppose $\gamma^*$ is disjoint from $G_1$ in cell $T$
- Let $p$ and $q$ be first and last intersection of $\gamma^*$ with $T$
Lemma: $G_1$ contains an $s,t$-path of cost $O(n) \cdot d^*$

Proof:

- Suppose $\gamma^*$ is disjoint from $G_1$ in cell $T$
- Let $p$ and $q$ be first and last intersection of $\gamma^*$ with $T$
- Will show there exists a $p,q$-walk $\gamma$ through $G_1 \cap T$ with cost $O(d^*)$
Lemma: $G_1$ contains an $s,t$-path of cost $O(n) \cdot d^*$

Proof:

- Suppose $\gamma^*$ is disjoint from $G_1$ in cell $T$
- Let $p$ and $q$ be first and last intersection of $\gamma^*$ with $T$
- Will show there exists a $p,q$-walk $\gamma$ through $G_1 \cap T$ with cost $O(d^*)$
- Replace $\gamma^*[p,q]$ with $\gamma$
Lemma: $G_1$ contains an $s,t$-path of cost $O(n) \cdot d^*$

Proof:

- Suppose $\gamma^*$ is disjoint from $G_1$ in cell $T$
- Let $p$ and $q$ be first and last intersection of $\gamma^*$ with $T$
- Will show there exists a $p,q$-walk $\gamma$ through $G_1 \cap T$ with cost $O(d^*)$
- Replace $\gamma^*[p,q]$ with $\gamma$
- Total cost of replacement paths is $O(n) \cdot O(d^*)$
• Will show there exists a $p, q$-walk $\gamma$ through $G_1 \cap T$ with cost $O(d^*)$
• Will show there exists a \( p, q \)-walk \( y \) through \( G_1 \cap T \) with cost \( O(d^*) \)

○ Assume \( p \) is on \( \beta_T \)
Will show there exists a \( p, q \)-walk \( \gamma \) through \( G_1 \cap T \) with cost \( O(d^*) \)

- Assume \( p \) is on \( \beta_T \)
- \( \gamma \) walks from \( p \) to \( w_s \) (costs at most \( d^* \))
Will show there exists a $p,q$-walk $\gamma$ through $G_1 \cap T$ with cost $O(d^*)$

- Assume $p$ is on $\beta_T$
- $\gamma$ walks from $p$ to $w_s$ (costs at most $d^*$)
- Walk along well-behaved path from $w_s$ to $q$ (costs at most $7(\mu(p,q) + d^*) \leq 14(d^*)$)
• Will show there exists a $p,q$-walk $y$ through $G_1 \cap T$ with cost $O(d^*)$

- Assume $p$ is on $\beta_T$
- $y$ walks from $p$ to $w_s$ (costs at most $d^*$)
- Walk along well-behaved path from $w_s$ to $q$ (costs at most $7(\mu(p,q) + d^*) \leq 14(d^*)$)
• Will show there exists a $p,q$-walk $\gamma$ through $G_1 \cap T$ with cost $O(d^*)$

- Assume $p$ is on $\beta_T$
- $\gamma$ walks from $p$ to $w_s$ (costs at most $d^*$)
- Walk along well-behaved path from $w_s$ to $q$ (costs at most $7(\mu(p,q) + d^*) \leq 14(d^*)$)
O(1)-approximation

- Guess an estimate $d$ of $d^*$
O(1)-approximation

• Guess an estimate $d$ of $d^*$
• Build *planar* graph $G_2$ by adding $O(n^2)$ edges to $\tilde{\mathcal{V}}$
O(1)-approximation

- Guess an estimate $d$ of $d^*$
- Build planar graph $G_2$ by adding $O(n^2)$ edges to $\tilde{\mathcal{V}}$
- Compute cheapest path between $s$ and $t$ in $G_2$
• For each cell $T$ of $\tilde{\mathcal{V}}$: 

$$w_{\max}^{T} \leq u^{T} \leq w_{\min}^{T}$$
• For each cell $T$ of $\tilde{\mathcal{V}}$:

- Let $\hat{\beta}_T$ be the region of points on $\beta_T$ with clearance between $\text{cl}(t) / \exp(d)$ and $\text{cl}(s) \cdot \exp(d)$.
• For each cell $T$ of $\tilde{\mathcal{Y}}$:
  
  ⊘ Let $\hat{\beta}_T$ be the region of points on $\beta_T$ with clearance between $\text{cl}(t) / \exp(d)$ and $\text{cl}(s) \cdot \exp(d)$
  
  ⊘ Place vertices along $\hat{\beta}_T$
• For each cell $T$ of $\tilde{V}$:
  ✷ Let $\hat{\beta}_T$ be the region of points on $\beta_T$ with clearance between $\text{cl}(t) / \exp(d)$ and $\text{cl}(s) \cdot \exp(d)$
  ✷ Place vertices along $\hat{\beta}_T$
  ✷ The cost between consecutive vertices is $d / n$
• For each cell $T$ of $\tilde{\mathcal{V}}$:
  - Add vertices for $w_{\alpha^*}$ and $w_{\kappa^*}$ also.

\[ u_T \quad \kappa_T \quad \beta_T \]

\[ v_T \quad w_{\max} \quad w_{\min} \]

\[ \alpha_T \quad T \quad \beta_T \]
• For each cell $T$ of $\tilde{\mathcal{V}}$:
  
  – Add vertices for $w_{\alpha^*}$ and $w_{K^*}$ also
  
  – Add constant clearance edges from each new vertex
**Lemma**: If $d \geq d^*$, then points on $y^*$ have clearance at least $cl(t) / \exp(d)$ and at most $cl(s) \cdot \exp(d)$
Lemma: If \( d \geq d^* \), then points on \( y^* \) have clearance at least \( \frac{cl(t)}{\exp(d)} \) and at most \( cl(s) \cdot \exp(d) \)

Proof:

- Costs more than \( \ln cl(t) - \ln(\frac{cl(t)}{\exp(d)}) = d \) to go from point of clearance less than \( \frac{cl(t)}{\exp(d)} \) to \( t \)
Lemma: If $d \geq d^*$, then points on $\gamma^*$ have clearance at least $\text{cl}(t) / \exp(d)$ and at most $\text{cl}(s) \cdot \exp(d)$

Proof:

- Costs more than $\ln \text{cl}(t) - \ln(\text{cl}(t) / \exp(d)) = d$ to go from point of clearance less than $\text{cl}(t) / \exp(d)$ to $t$

- Costs more than $\ln(\text{cl}(s) \cdot \exp(d)) - \ln(s) = d$ to go from $s$ to point of clearance more than $\text{cl}(s) \cdot \exp(d)$
**Lemma**: If $d \geq d^*$, $G_2$ contains an $s,t$-path of cost $O(d)$
**Lemma**: If $d \geq d^*$, $G_2$ contains an $s,t$-path of cost $O(d)$

**Proof**: 

- Let $p$ and $q$ be first and last intersection of $\gamma^*$ with $T$
**Lemma:** If \( d \geq d^* \), \( G_2 \) contains an \( s,t \)-path of cost \( O(d) \)

**Proof:**

- Let \( p \) and \( q \) be first and last intersection of \( \gamma^* \) with \( T \)
- Will show there exists a \( p,q \)-walk \( \gamma \) through \( G_2 \cap T \) with cost \( O(\mu(p,q)) + O(d/n) \)
Lemma: If \( d \geq d^* \), \( G_2 \) contains an \( s,t \)-path of cost \( O(d) \)

Proof:

- Let \( p \) and \( q \) be first and last intersection of \( \gamma^* \) with \( T \)
- Will show there exists a \( p,q \)-walk \( \gamma \) through \( G_2 \cap T \) with cost \( O(\mu(p,q)) + O(d/n) \)
- Replace \( \gamma^*[p,q] \) with \( \gamma \)
Lemma: If \( d \geq d^* \), \( G_2 \) contains an \( s,t \)-path of cost \( O(d) \)

Proof:

- Let \( p \) and \( q \) be first and last intersection of \( y^* \) with \( T \)
- Will show there exists a \( p,q \)-walk \( y \) through \( G_2 \cap T \) with cost \( O(\mu(p,q)) + O(d/n) \)
- Replace \( y^*[p,q] \) with \( y \)
- Total cost of replacement paths is \( O(d^*) + O(n) \cdot O(d/n) = O(d) \)
• Will show there exists a $p,q$-walk $y$ through $G_2 \cap T$ with cost $O(\mu(p,q)) + O(d/n)$
• Will show there exists a \( p, q \)-walk \( y \) through \( G_2 \cap T \) with cost \( O(\mu(p, q)) + O(d / n) \)

Porno: Assume \( p \) is on \( \beta_T \)
Will show there exists a $p, q$-walk $y$ through $G_2 \cap T$ with cost $O(\mu(p,q)) + O(d / n)$

- Assume $p$ is on $\beta_T$
- Point $p$ must lie on $\hat{\beta}_T$
• Will show there exists a $p, q$-walk $y$ through $G_2 \cap T$ with cost $O(\mu(p, q)) + O(d / n)$

๏ Assume $p$ is on $\beta_T$

๏ Point $p$ must lie on $\hat{\beta}_T$

๏ Walk from $p$ to nearest vertex $p'$ on $\hat{\beta}_T$ (costs less than $d / n$)
• Will show there exists a $p,q$-walk $\gamma$ through $G_2 \cap T$ with cost $O(\mu(p,q)) + O(d/n)$

- Assume $p$ is on $\beta_T$
- Point $p$ must lie on $\hat{\beta}_T$
- Walk from $p$ to nearest vertex $p'$ on $\hat{\beta}_T$ (costs less than $d/n$)
- Walk along well-behaved path from $p'$ to $q$ (costs at most $7(\mu(p,q) + d/n)$)
Lemma: \(O(n)\) vertices are placed on \(\hat{\beta}_T\)
Lemma: \( O(n) \) vertices are placed on \( \hat{\beta}_T \)

Proof:

- The cost between consecutive vertices is \( d/n \)
Lemma: $O(n)$ vertices are placed on $\hat{\beta}_T$

Proof:

- The cost between consecutive vertices is $d/n$
- The total cost of walking along $\hat{\beta}_T$ is $\ln{(\text{cl}(s) \cdot \exp(d))} - \ln{(\text{cl}(t) / \exp(d))} \leq 2d$
• Added $O(n^2)$ vertices total to make $G_2$
• Graph $G_2$ remains planar
• Cheapest path in $O(n^2)$ time [Henzinger et al. ’97]
• If $d \geq d^*$, then cost of path in $G_2$ is $O(d)$
• If $d \geq d^*$, then cost of path in $G_2$ is $O(d)$
• $O(n)$-approximation yields $\tilde{d}$ with $d^* \leq \tilde{d} \leq O(n) \cdot d^*$
• If \( d \geq d^* \), then cost of path in \( G_2 \) is \( O(d) \)

• \( O(n) \)-approximation yields \( \tilde{d} \) with \( d^* \leq \tilde{d} \leq O(n) \cdot d^* \)

• Run graph search using \( O(\log n) \) values of \( d \) starting with \( \tilde{d} \)
• If $d \geq d^*$, then cost of path in $G_2$ is $O(d)$
• $O(n)$-approximation yields $\tilde{d}$ with $d^* \leq \tilde{d} \leq O(n) \cdot d^*$
• Run graph search using $O(\log n)$ values of $d$ starting with $\tilde{d}$
• Finds a good approximation when $d^* \leq d \leq 2d^*$
Approximation Scheme

• Compute $\hat{d}$ such that $d^* \leq \hat{d} \leq O(1) \cdot d^*$
Approximation Scheme

• Compute \( \hat{d} \) such that \( d^* \leq \hat{d} \leq O(1) \cdot d^* \)

• Build graph \( G_3 \) by adding \( O\left(\frac{n^2}{\epsilon}\right) \) vertices and \( O\left(\frac{n^2}{\epsilon^2 \log(n/\epsilon)}\right) \) edges to \( \tilde{\mathcal{V}}\).
Approximation Scheme

• Compute $\hat{d}$ such that $d^* \leq \hat{d} \leq O(1) \cdot d^*$

• Build graph $G_3$ by adding $O(n^2 / \epsilon)$ vertices and $O(n^2 / \epsilon^2 \log(n / \epsilon))$ edges to $\tilde{V}$

• Compute cheapest path between $s$ and $t$ in $G_3$
• For each cell $T$ of $\tilde{V}$:
• For each cell $T$ of $\tilde{\mathcal{V}}$:

- Let $\hat{\beta}_T$, $\hat{\alpha}_T$ be the regions of points on $\beta_T$, $\alpha_T$ with clearance between $\text{cl}(t) / \exp(\hat{d})$ and $\text{cl}(s) \cdot \exp(\hat{d})$. 
• For each cell $T$ of $\tilde{\mathcal{V}}$:
  
  - Let $\hat{\beta}_T, \hat{\alpha}_T$ be the regions of points on $\beta_T, \alpha_T$ with clearance between $\text{cl}(t) / \exp(\hat{d})$ and $\text{cl}(s) \cdot \exp(\hat{d})$
  
  - Region $\hat{\kappa}_T$ contains points of cost at most $2\hat{d}$ from $u_T$ and points of cost at most $4\hat{d}$ from point $v'$ with clearance $\text{cl}(s) \cdot \exp(\hat{d})$
• Vertices are placed within each region so cost between adjacent vertices is \( \epsilon \hat{d} / n \) and there are \( O(n / \epsilon) \) vertices per cell.
• Vertices are placed within each region so cost between adjacent vertices is $\epsilon \hat{d} / n$ and there are $O(n / \epsilon)$ vertices per cell

• Edges can be added as cheapest paths between all pairs of vertices on $T$, as done by Wein et al. ['08]
• Vertices are placed within each region so cost between adjacent vertices is $\epsilon \hat{d}/n$ and there are $O(n/\epsilon)$ vertices per cell

• Edges can be added as cheapest paths between all pairs of vertices on $T$, as done by Wein et al. [’08]

• But then we’d have $O(n^2/\epsilon^2)$ edges per cell, for $O(n^3/\epsilon^2)$ edges total $G_3$
• Instead, for each vertex $p$ on $\beta_T$, add edges to $O(1 / \epsilon \log(n / \epsilon))$ vertices on $\kappa_T$ and $\alpha_T$
• Instead, for each vertex $p$ on $\beta_T$, add edges to $O(1 / \epsilon \log(n / \epsilon))$ vertices on $\kappa_T$ and $\alpha_T$

• Consecutive neighbors of $p$ along $\kappa_T$ and $\alpha_T$ are spaced geometrically apart starting from locations with same clearance as anchor points and $p$
• Let $p$ and $q$ be first and last intersection of $\gamma^*$ with $T$
• Let \( p \) and \( q \) be first and last intersection of \( \gamma^* \) with \( T \)

• Use well-behaved paths to show existence of a \( p,q \)-walk \( \gamma \) through \( G_3 \cap T \) with cost \( (1+O(\epsilon))\mu(p,q) + O(\epsilon \hat{d} / n) \)
• Let $p$ and $q$ be first and last intersection of $\gamma^*$ with $T$

• Use well-behaved paths to show existence of a $p,q$-walk $\gamma$ through $G_3 \cap T$ with cost $(1+O(\epsilon))\mu(p,q) + O(\epsilon \hat{d} / n)$

• As before, perform $O(n)$ replacements
• $G_3$ has $O(n^2 / \epsilon^2)$ vertices and $O(n^2 / \epsilon^2 \log(n / \epsilon))$ edges
• $G_3$ has $O(n^2 / \epsilon^2)$ vertices and $O(n^2 / \epsilon^2 \log(n / \epsilon))$ edges

• Use Dijkstra’s algorithm with Fibonacci heaps to compute cheapest path [Fredman, Tarjan ’87]
• $G_3$ has $O(n^2 / \epsilon^2)$ vertices and $O(n^2 / \epsilon^2 \log(n / \epsilon))$ edges

• Use Dijkstra’s algorithm with Fibonacci heaps to compute cheapest path [Fredman, Tarjan ’87]

• That’s it!
Summary and Open Problems

• Gave $O(n^2 / \epsilon^2 \log (n / \epsilon))$ time $(1+\epsilon)$-approximation for high quality paths
Summary and Open Problems

• Gave $O(n^2 / \epsilon^2 \log (n / \epsilon))$ time $(1+\epsilon)$-approximation for high quality paths

• Near-linear time?
Summary and Open Problems

- Gave $O(n^2 / \epsilon^2 \log (n / \epsilon))$ time $(1+\epsilon)$-approximation for high quality paths
- Near-linear time?
- Problem complexity?
Summary and Open Problems

• Gave $O(n^2 / \varepsilon^2 \log (n / \varepsilon))$ time $(1+\varepsilon)$-approximation for high quality paths

• Near-linear time?

• Problem complexity?
  ◦ Algebraic complexity
Summary and Open Problems

• Gave $O(n^2 / \epsilon^2 \log (n / \epsilon))$ time $(1+\epsilon)$-approximation for high quality paths

• Near-linear time?

• Problem complexity?
  • Algebraic complexity
  • Combinatorial complexity (NP-hard?)
Summary and Open Problems

• Gave $O(n^2 / \epsilon^2 \log (n / \epsilon))$ time $(1+\epsilon)$-approximation for high quality paths

• Near-linear time?

• Problem complexity?
  - Algebraic complexity
  - Combinatorial complexity (NP-hard?)

• Higher dimensions?
Thank you!