CS 4349.501 Homework 9

Due Wednesday November 8th, in class

November 1, 2017

Please answer both questions (Problem 2 appears on the second page). Each student must write their solutions in their own words and submit their solutions on paper at the beginning of class. Include your name and/or Net ID at the top of each page. And please staple your papers together if you can.

1. Let $G = (V, E)$ be a directed graph with edge weights $w : E \to \mathbb{R}$ (which may be positive, negative, or zero). Assume there are no negative length cycles in $G$.

   (a) How could we delete an arbitrary vertex $v$ from graph $G$, without changing the shortest-path distance between any other pair of vertices? Describe and analyze an algorithm that constructs a directed graph $G' = (V', E')$ with edge weights $w' : E' \to \mathbb{R}$, where $V' = V \setminus \{v\}$, and the shortest-path distance between any two nodes in $G'$ is equal to the shortest-path distance between two nodes in $G$. The algorithm should run in $O(V^2)$ time.

   (b) Now suppose we have already computed all shortest-path distances in $G'$. Describe and analyze an algorithm to compute the shortest-path distances from $v$ to every other vertex, and from every other vertex to $v$, in the original graph $G$. Again, the algorithm should run in $O(V^2)$ time.

   (c) Finally, combine parts (a) and (b) to describe and analyze another all-pairs shortest path algorithm. This algorithm should run in $O(V^3)$ time. (The resulting algorithm is not the same as Floyd-Warshall!)
2. Consider the directed graph $G = (V, E)$ below with non-negative capacities $c : E \rightarrow \mathbb{R}_{\geq 0}$ and an $(s, t)$-flow $f : E \rightarrow \mathbb{R}_{\geq 0}$ that is feasible with respect to $c$. Each edge is labeled with its flow/capacity.

![Diagram of the directed graph $G$]

An $(s, t)$-flow $f$. Each edge is labeled with its flow/capacity.

(a) Draw the residual graph $G_f = (V, E_f)$ for flow $f$. Be sure to label every edge of $G_f$ with its residual capacity.

(b) Describe an augmenting path $s = v_0 \rightarrow v_1 \rightarrow \ldots \rightarrow v_r = t$ in $G_f$ by either drawing the path in your residual graph or listing the path's vertices in order.

(c) Let $F = \min_i c_f(v_i \rightarrow v_{i+1})$ and let $f' : E \rightarrow \mathbb{R}_{\geq 0}$ be the flow obtained from $f$ by pushing $F$ units through your augmenting path. Draw a new copy of $G$, and label its edges with the flow values for $f'$. Is your new flow a maximum flow in $G$?

(d) For this last part, let $G = (V, E)$ be an arbitrary directed graph (not necessarily the one given above) with non-negative capacities $c : E \rightarrow \mathbb{R}_{\geq 0}$ on the edges and two special vertices $s$ and $t$. Suppose we assign a non-negative limit $\ell : V \setminus \{s, t\} \rightarrow \mathbb{R}_{\geq 0}$ for the amount of flow that can pass through each vertex other than $s$ or $t$. Formally, a flow $f : E \rightarrow \mathbb{R}_{\geq 0}$ is feasible with respect to both $c$ and $\ell$ if for all edges $e \in E$ we have $f(e) \leq c(e)$ and for all vertices $v \in V \setminus \{s, t\}$ we have $\sum_{u \rightarrow v} f(u \rightarrow v) \leq \ell(v)$.

Describe and analyze an algorithm to compute a graph $G' = (V', E')$ with non-negative edge capacities $c' : E' \rightarrow \mathbb{R}_{\geq 0}$ but no vertex limits so that the value of the maximum feasible flow in $G'$ with respect to $c'$ is equal to the value of the maximum feasible flow in $G$ with respect to both $c$ and $\ell$. 
