CS 4349.400 Homework 8

Due Wednesday November 14th, in class

Please answer each of the following questions.

1. Let $G = (V, E, w)$ be a directed graph with weighted edges $w : E \rightarrow \mathbb{R}$; edge weights could be positive, negative, or zero. We’re going to design another algorithm for computing all-pairs shortest paths. For simplicity, you may assume $G$ is complete, meaning $E = V \times V$.

(a) Let $v$ be an arbitrary vertex of $G$. Describe an algorithm that constructs a directed graph $G' = (V', E', w')$ with edges weights $w' : E' \rightarrow \mathbb{R}$, where $V' = V \setminus \{v\}$, and the shortest-path distance between any two nodes in $G'$ is equal to the shortest-path distance between the same two nodes in $G$. The algorithm should run in $O(V^2)$ time. [Hint: When should $w'(u \rightarrow w) \neq w(u \rightarrow w)$?]

(b) Now suppose we have already computed all shortest-path distances in $G'$. Describe an algorithm to compute the shortest-path distances in the original graph $G$ from $v$ to every other vertex, and from every vertex to $v$, all in $O(V^2)$ time. [Hint: Guess the first/last edge on each path.]

(c) Combine parts (a) and (b) into another all-pairs shortest path algorithm. Your algorithm should run in $O(V^3)$ time. [Hint: Recursion.]

2. Consider the directed graph $G = (V, E)$ below with non-negative capacities $c : E \rightarrow \mathbb{R}_{\geq 0}$ and an $(s, t)$-flow $f : E \rightarrow \mathbb{R}_{\geq 0}$ that is feasible with respect to $c$. Each edge is labeled with its flow/capacity.

An $(s, t)$-flow $f$. Each edge is labeled with its flow/capacity.
(a) Draw the residual graph $G_f = (V, E_f)$ for flow $f$. Be sure to label every edge of $G_f$ with its residual capacity.

(b) Describe an augmenting path $s = v_0 \to v_1 \to \ldots \to v_r = t$ in $G_f$ by either drawing the path in your residual graph or listing the path’s vertices in order.

(c) Let $F = \min_i c_f(v_i \rightarrow v_{i+1})$ and let $f' : E \to \mathbb{R} \geq 0$ be the flow obtained from $f$ by pushing $F$ units through your augmenting path. Draw a new copy of $G$, and label its edges with the flow values for $f'$.

(d) Draw the residual graph $G_{f'} = (V, E_{f'})$ for flow $f'$.

(e) There shouldn’t be any augmenting paths in $G_{f'}$, implying $f'$ is a maximum flow. Draw or list the vertices in $S$ for some minimum $(s, t)$-cut $(S, T)$.

(f) What is the value of the maximum flow/capacity of the minimum cut?

3. Suppose we are given a flow network $G = (V, E)$ in which every edge has capacity 1, together with an integer $k \geq 0$. Describe an algorithm to identify $k$ edges in $G$ such that after deleting those $k$ edges, the value of the maximum $(s, t)$-flow in the remaining graph is as small as possible. [Hint: The value of the maximum $(s, t)$-flow is equal to the capacity of the minimum $(s, t)$-cut.]