Buyout Price Optimization in the Rent-to-Own Business

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Problem definition: We study the multidimensional price optimization problem faced by a rent-to-own (RTO) firm, which rents a product for a periodic fee and during the rental repeatedly offers it for sale to the renter at a sequence of prices forming the *buyout price path*. **Methodology/results:** We first employ calculus of variations to obtain optimal buyout prices in closed form for a special case. Next, to overcome the nonconcavity of the profit in the general problem, we formulate an equivalent bilevel optimization over the resource utilization and price path. We then transform the inner pricing problem into a deterministic dynamic program (DP) with a one-dimensional state. Leveraging this transformation, we develop an efficient algorithm to find the optimal price path. We also apply our methodology to jointly optimize buyout prices and inventory levels. **Managerial implications:** Standard practice in the RTO industry is to use a price path that decreases steeply early in the agreement and gradually later. However, we prove for the special case that the optimal prices are in contrast concave decreasing; prices should optimally decrease gradually early in the agreement and steeply later. In the joint inventory and pricing case, our results reveal, perhaps counterintuitively, that higher inventory levels tend to entail higher optimal prices. Applying our algorithm in a case study with parameters calibrated from our discussions with an RTO firm, we again find that prices should optimally decrease gradually early in the agreement and steeply later, validating our insights from the special case. Moreover, our methodology yields approximately a 22% increase in profit relative to industry prices.

Key words: rent-to-own, price path and inventory optimization, calculus of variations, dynamic programming, practical algorithms

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1. Introduction

A rent-to-own (RTO) firm offers products such as appliances, furniture and televisions to its renters under rental agreements with a specified term. If a renter continues paying the periodic rental fee—usually paid weekly or monthly—for the length of the term, then he owns the product without additional payments. This is referred to as a *payoff*. On the other hand, he can terminate the agreement and return the product without penalty, in which case the firm can re-rent it to someone else. The RTO firm also typically offers the renter repeated opportunities for a *buyout*, i.e., to purchase the product early before completion of the term, according to a sequence of buyout prices forming the *buyout price path*.

The RTO business has operated for almost half a century (APRO 2015), and it has grown into an \$8.2 Billion industry in the United States (May 2017). Annual revenues for well-known firms Aaron's and Rent-A-Center easily reach into the billions on their own (Aaron's 2022, Rent-A-Center 2022).

RTO represents a unique blend of renting and selling, and it is attractive to prospective renters who desire flexibility, such as individuals occupying short-term housing while renovating a house, those temporarily living away from home for work or school, or parents whose child wants to take up a musical instrument (RTO for musical instruments brings in approximately $2 Billion per year by itself; see APRO 2015). It is also popular with those who can afford periodic payments for an item but have less-than-perfect credit, since RTO firms do not always require a credit check (APRO 2015). Such individuals may find it desirable to eventually achieve ownership of the item through gradual payments, similar to a mortgage on a house. An RTO agreement offers such a route toward ownership, and moreover "customers can return the merchandise at any time for any reason without penalty" (APRO 2015). So, if a renter's financial situation or need for the item changes, he can easily return it, or, on the other hand, he can accept one of the early buyout offers.

The present work sprang from our discussions with a large RTO firm. Managing an RTO business poses unique challenges not faced by standard retailers (APRO 2015). Because renters can initiate and terminate their agreements at will and purchase rented items at any time, the flow of items to and from renters is highly stochastic, and items also leave the system at random times due to these early buyouts (and payoffs) and must be replaced to serve future demand. Additionally, RTO renters pay rental fees between successive buyout price offers for the same product, so RTO firms must trade off two sources of revenue: rental fees and buyouts. A daunting problem facing RTO firms is how to determine the buyout price path in the face of these as well as other, more familiar uncertainties like random demand for new rentals. In practice, buyout prices in an RTO agreement typically reduce from a period to the next. These reductions are usually steeper at the start of a renter's agreement and more gradual later in the agreement. An RTO firm generally posts a verbal description of the price path as opposed to detailed price schedules, keeping the price path opaque (see Appendix A.1 for details). In our discussions, the RTO firm has shared with us that its current pricing policy is based on cost, and a chief aim is to achieve a sufficient amount of both rental and buyout revenues from an item. The firm is currently experimenting with different pricing policies, but the experimentation is complicated by the high dimension of the price path: an agreement's term is often a year or longer, and a buyout price is offered in each period of rental (week or month).

Despite the size and history of the RTO business, the determination of buyout price paths has received negligible attention in the academic literature. Hence, open questions abound, in particular: (i) how should the random flow of a product through RTO agreements be modeled? (ii) does an optimal buyout price path possess a particular structure? (iii) can we find an efficient algorithm to optimize buyout price paths, ideally one that also supports joint pricing and inventory optimization? and (iv) what is the magnitude of potential profit improvement from such optimization relative to current practice? A sizable knowledge gap thus exists, and we address this gap in the pages to come.

Specifically, we develop a faithful but parsimonious model of the RTO system; we use this model to formulate the buyout price optimization problem; we derive the optimal price path in closed form (as well as its shape) for a special case; in general, we devise an efficient price optimization algorithm; and finally, we benchmark this algorithm's performance against industry prices in a case study, which reveals that the associated profit improvements are significant. We now survey the operations and revenue management literatures related to renting and/or selling, before detailing the approach and contributions of our work.

Related Literature: The present work is related to a few streams of literature: pricing and inventory management for rental products, pricing for reusable resources, the sell-vs.-rent (or concurrent selling and renting) decision and related pricing, and research on the operations of RTO businesses.

Some past work has considered pricing for rental units. Examples include Huang et al. (2001), Gans and Savin (2007) and Gilbert et al. (2014). Huang et al. (2001) studies a durable-goods setting with a secondary market, focusing on the case where goods last exactly two periods; the firm chooses the sales and rental prices for new goods, while the sales prices for used goods (which can also be resold by consumers) are determined by a market-clearing condition. Contrasting with the RTO business and with our work, in their model rentals last only a single period and "lease contracts do not contain an option" to purchase. Gans and Savin (2007) uses a continuous-time framework to study a rental firm that must jointly determine admission controls for fixed-price contract customers and dynamically optimize prices for walk-in customers; importantly, and in direct contrast to our work, payments are for rental *access* to the product and there is no buyout price for customers to obtain ownership of it. Gilbert et al. (2014) studies a firm that rents and/or sells a digital product. Although it maintains a constant sales price, e.g., independent of the elapsed time in the rentals, it provides insights on when a firm should offer both rentals and sales, how the optimal rental (sales) price is affected by the sales (rental) option.

More recent work has considered capacity/inventory management for rental units. Slaugh et al. (2016), motivated by fashion rental firms like Rent-the-Runway, studies a setting with a one-time procurement opportunity before the renting season, with products that are subject to "usage-based loss" due, for example, to wear and tear. By contrast, our focus is on durable goods like washer/dryers and refrigerators which offer multiple procurement opportunities: if these items are sold, then the firm can easily replace them at any time for another unit of the same or a very similar model. Importantly, although buyout opportunities are mentioned in passing in Slaugh et al. (2016) as a possible reason for lost units, their study does not involve pricing: the prices and costs are exogenously given (and fixed) in their model, and they optimize instead over inventory management and "recirculation" policies, which designate whether to allocate units with higher or lower likelihood of loss to new rentals. They show that incorporating product loss into these decisions can lead to significant profit improvements. Slaugh et al. (2016) also provides an excellent review of prior work on rental capacity management going all the way back to Tainiter (1964); almost all of this work adopts a continuous-time queueing-theoretical framework, and none of it, to the best of our knowledge, includes a buyout option. Most recently, Firouz et al. (2022) studies a fleet-sizing problem for equipment rental, using a similar modeling approach to ours, but they do not optimize prices. They allow breakdown for a rented unit, in which case it undergoes a repair process with random duration to satisfy demand for new rentals. On the other hand, several studies perform joint inventory and pricing optimization (see, e.g., Govindarajan et al. 2020 and references therein) in a retail (i.e., non-rental) setting.

Our work is closely related to research that focuses on the sell-vs.-rent decision. Jalili and Pangburn (2020) solves the rental fee and price optimization problem of a firm that can make available post-rental sales to its customers. They characterize when it is optimal to offer this opportunity and find the corresponding optimal outright purchase price, rental fee and post-rental purchase price. The main factor that contrasts our work with theirs is that in our setting renters have multiple purchase opportunities after rental initiation which entails optimizing many prices (forming a price path) rather than a single price. Altug and Ceryan (2022) considers a firm that concurrently sells and rents its fixed inventory of a perishable product and optimizes its revenue by dynamically determining rental and sales inventory allocation for fixed rental duration, rental fee and purchase price. Finally, Zhang et al. (2022) constructs game-theoretic models between a manufacturer and a rental company and studies the effect of power structure in the supply chain on the pricing decisions of both firms; in their setup, neither party concurrently engages in both renting and selling.

Other work has studied pricing with reusable resources; see, e.g., Lei and Jasin (2020), Rusmevichientong et al. (2020), Jia et al. (2022), Besbes et al. (2022), and references therein. Pricing with reusable resources is similar to rental fee optimization, and typically there are no buyouts or payoffs. In most of these studies usage duration is exogenous, e.g., realized according to a distribution. In contrast, rental duration in our model is an endogenous factor affected by the prices offered during rental. For challenging problems like these, the optimization is frequently intractable, requiring approximations and heuristics. A different but somewhat related setting is that of pricing and return policy decisions when customers either keep the purchased product or return it to the firm within a specified return window: a product return in exchange for a (partial) refund or credit can be viewed as a short-term rental. In Liu and Zhang (2023), a customer may have repeated need to buy the same type of product, and his valuation for the product is realized independently in each period. If a customer buys and then returns the product in one period, he receives a partial credit to spend on a future purchase with the firm, but the credit expires after some time. It derives the optimal price, credit amount and the length of time until credit expiration. Other recent work on this topic includes Wagner and Martínez-de Albéniz (2020) and Nageswaran et al. (2020). Unlike RTO, in this context there is a single purchase price rather than a price path, and the price is paid upfront in full, with no option for gradual payments.

We briefly shed light on other research conducted regarding the RTO business. Anderson and Jaggia (2012) estimates RTO agreement outcomes (return, buyout and payoff). Anderson and Sibdari (2012) studies investment decision of an RTO firm. Jaggia et al. (2019) empirically studies the profit margin of an RTO firm but does not optimize buyout prices. Guajardo (2019) investigates the relationship between rental payment behavior and usage frequency of the rented product in a developing country. Most recently, Armaghan et al. (2023) performs a counterfactual analysis on buyout prices but does not optimize them, leaving RTO buyout price optimization as an open question.

Approach and Contributions: Substantial research has been performed on topics related in various ways to rental operations, along with limited work on the RTO business itself. Some of this research even studies the determination of rental fees (although not for the RTO business). However, to the best of our knowledge, no systematic study has yet been conducted on the optimization of RTO buyout price paths. Motivated by this real-world problem that also represents a notable gap in the literature, in this work, we introduce, formulate, and solve the *RTO buyout pricing problem*.

The high dimension and opacity of price paths make it difficult for an individual renter to engage in strategic purchase timing. Accordingly, we consider renters that are myopic in purchase decisions. In addition, several empirical studies document the prevalence of myopic customers. Both Moon et al. (2018) and Li et al. (2014) find that at least 80% of customers decide myopically in the online retailing and airline industries, respectively. Yılmaz et al. (2022) also finds a high fraction of myopic customers in the hotel industry. In the same spirit, Armaghan et al. (2023) performs structural estimation using various renter decision-making models—including those with strategic renters—and assesses renters' utilities for products. It finds that RTO renters are myopic or nearly so in their purchase decisions, aligning with our model of renter purchase decisions.

We first develop a new, discrete-time Markov chain model of RTO agreements, using the notion of an inventory *slot*. An inventory slot begins in the idle state, from which it initiates a new agreement with a specified probability. After an agreement is initiated, the firm offers buyout prices to the renter in successive periods according to its chosen buyout price path. The probability of the renter accepting a buyout price offer depends on the price and his valuation for the item (whose probability distribution may vary over time). If he accepts, then he takes ownership, the agreement ends, and the firm procures a new item to occupy the inventory slot, which resets to the idle state. If he declines the price and continues the rental by paying the rental fee, then the state, which measures the number of periods remaining until payoff, decrements by one. If he terminates the agreement, then he returns the item to the firm, and the slot resets to the idle state, but no replacement is required. In the last period (i.e., when the state reaches one), paying the rental fee entails paying off the item, and in this case (like a buyout) the renter takes ownership of the item, a new one is procured to replace it, and the slot resets to the idle state. We derive the stationary probabilities for this Markov chain as functions of the price path and the probability distribution of the renter's valuation for the item. This allows us to formulate the profit optimization problem over the price path, namely the RTO buyout pricing problem.

The profit in the RTO buyout pricing problem is in general nonconcave and also high-dimensional, and a customized approach is thus required. In the simplest version of the problem when only a single buyout price is optimized, we show that the optimal price is decreasing in the agreement initiation probability, i.e., the probability that an available item becomes rented in a given period. If this probability is low, then after selling the item and replacing it, a long period of idleness may occur before the next rental, during which this item does not produce revenue for the firm. For low agreement initiation probabilities, the firm thus prefers to set the price high so that either the rental will continue for another period or the firm will be compensated for the expected idleness with a substantial lump sum.

By considering a fixed utilization (fraction of on-rent time) of an inventory slot, we make two key observations: we can control the expected rental revenue via the utilization, and we can anchor the stationary probabilities to the utilization. This anchoring decouples stationary probabilities of higher states (those closer to agreement initiation) from prices of lower states. We then reformulate the problem as a bilevel, constrained optimization: an outer optimization over the achievable range of utilizations and an inner optimization over the price path.

To obtain additional structural insights, we then consider a continuous-time analog of the inner pricing problem (for fixed utilization), with the specifications that (i) renter valuation is uniformly distributed and stationary and (ii) renters do not return the item but own it either through buyout or payoff. We refer to this problem as the special case. We determine the exact optimal price path in closed form. Consistent with the above finding, these prices also shift up as the agreement initiation probability decreases. Moreover, the optimal price path is *concave decreasing* over the course of a renter's agreement; that is, the price should decrease gradually early in the agreement and more steeply towards the end. Intuitively, the firm should try to earn substantial rental revenue before selling the item and incurring the replacement cost. So, it needs to keep the price relatively high in the early stages of an agreement so that the renter is unlikely to purchase and even if he does, then the high price brings a substantial sales revenue for the firm. On the other hand, towards the end of an agreement, the firm wishes to avoid "giving up" the item through payoff without a lump sum payment, so it optimally drops the price more steeply at this stage to entice the renter to buy. This finding contrasts with industry conventions, which as mentioned entail steep price drops early in the agreement and gradual price drops later on.

Armed with these insights, we then move on to develop an approach to solve the general problem in discrete time and without the above specifications (of the special case). First, we use the decoupling to transform the inner pricing problem for fixed utilization—a static optimization problem over the entire price path—into an equivalent, single-state deterministic dynamic program (DP) that determines the prices sequentially. This DP can be solved efficiently by discretizing its state space. Since the utilization is bounded in a narrow interval, the outer optimization requires a limited number of iterations, and we thus arrive at an efficient optimization algorithm (subject only to discretization error) to solve the RTO buyout pricing problem. Our algorithm solves a pressing problem in the RTO business, and as (to the best of our knowledge) the first to make a detailed study of this problem, our contribution is significantly bolstered by the fact that we traverse the entire path from initial formulation to an efficient optimization algorithm.

After solving the pricing problem, we then consider the joint buyout pricing and inventory optimization problem. In the joint problem, an RTO firm must choose not only the buyout price path but also the inventory level to maintain. This can be thought of as a base-stock inventory level maintained by the firm, such that when items leave the system through sales (buyouts or payoffs), they are replaced to maintain the base-stock level (i.e., the target inventory level). Indeed, the RTO firm that inspired this work has a target inventory level. The inventory units, associated with inventory slots in our Markov chain, compete for rental demand, leading to dependence among the states of the slots. Crucially, this dependence is confined to the idle state, and based on this observation, we devise a simple and highly accurate approximation for the evolution of the multi-slot system. Using this approximation, we then apply our methodology from the single-slot system to solve the joint pricing and inventory optimization problem for the multi-slot system.

Finally, to demonstrate the power of our methodology, we conduct a case study using parameters calibrated from our discussions with the RTO firm and price paths (for benchmarking) computed with their price formula. The optimal prices from our algorithm tend to be decreasing over the course of an agreement; this decreasing property is in line with industry practice as described above, but there the resemblance ends. Consistent with our structural results from the simplified continuous-time setting, we find in the realistic setting of the case study that the optimal price path from our algorithm decreases gradually early in the agreement and steeply later in the agreement. Since the convention in the industry is to reduce prices steeply early in the agreement and gradually later in the agreement, our findings suggest that RTO firms may be dropping prices too quickly at the start of an agreement. Our findings counsel RTO firms to refrain from substantial price drops early in an agreement; with higher prices early on, the agreement is more likely to generate substantial rental revenue before a buyout, and if an early buyout does occur, it will be at a high enough price to justify the replacement cost and the opportunity cost of future idleness. We also quantify the profit impact of our methodology.

We find that applying our methodology to jointly optimize prices and inventory yields approximately a 22% increase in profit compared to inventory optimization alone with industry prices. We also find that there is value in price-only optimization for given inventory: across different fixed inventory levels, we observe an average of 20% higher profit with the optimal prices versus the industry prices. Finally, the character of the optimal solutions reveals the importance of cooperation between the supply chain and revenue management groups within an RTO firm. Not only is profit left on the table if the inventory (price path) is optimized assuming fixed price path (inventory), but also the optimal decisions are interdependent; the optimal price path is substantially different for different inventory levels, and the optimal inventory level is different for different price paths. In particular, since inventory units compete for demand, higher inventory levels lead to lower agreement initiation probabilities. As we showed analytically in continuous time, lower agreement initiation probabilities lead to higher optimal prices, and accordingly in the case study we find that higher inventory levels tend to yield higher optimal prices.

The rest of the paper is organized as follows. In §2, we introduce our Markov chain model of RTO agreements and derive its stationary distribution. §3 formulates the RTO buyout pricing problem based on the stationary distribution, then derives the equivalent bilevel price-utilization formulation. In §4, we solve a simplified version of the problem in continuous time and derive structural insights about the shape of the price path. In §5, we present our dynamic programming approach to optimize buyout price paths in the general setting. §6 studies the joint buyout pricing and inventory optimization problem, and it also presents the benchmarked case study. §7 concludes.

2. Markov Chain Model of RTO Agreements

RTO firms offer ownership opportunities to customers during their rentals. A renter can own the rented item either through *payoff* or *buyout*. Payoff requires renting the item for a specified duration, referred to as the *term*, while buyout occurs earlier by accepting one of the periodic (monthly or weekly) *buyout price* offers made within each agreement according to the firm's chosen *buyout price path*.

An RTO renter who initiates an agreement judges the use of the item to be worth the rental fee. The status quo is continuing to rent, unless either the buyout is sufficiently attractive or circumstances change such that the item is no longer needed or the rental is not affordable. As discussed in §1, we consider myopic renters, who purchase the rented item if their valuation is greater than the offered buyout price. An attempt to consider strategic renters would lead to a high dimensional stochastic dynamic game requiring the firm's determination of an entire price path as well as renters' rational expectation of the path (in contrast to the single-price setting in, e.g., Liu and Zhang 2023), whose analysis is almost certainly intractable and hence avoided in the absence of strong empirical evidence for strategic behavior among RTO renters. Moreover, a considerable portion of renters are individuals who might not be able to commit to a specified duration of rental. Such a lack of commitment could emanate from short-term needs or other life circumstances like relocation, unemployment, etc. A renter may terminate his agreement and return the item in any period during the rental without penalty. We account for this possibility through the probability ρ, with which a renter sustains his rental agreement in each period, independent of other periods. Accordingly, with probability (wp) $1-\rho$, the renter terminates his agreement and returns the item. In each period that the renter sustains his agreement, he receives a buyout price and compares it against his valuation for the product, which is realized independently in each period similar to Gilbert et al. (2014) and Liu and Zhang (2023) and whose distribution may vary during the rental. If he rejects the offer, then he pays the *rental fee* and continues to the next period. If he accepts the offer, then he pays the buyout price and owns the product. The benefit from renting can be expected to be very similar to the rental fee. Differing values of these quantities can be easily accommodated by appropriate shifts of the valuation distribution, so without loss of generality (wlog) we normalize the rental benefit to zero.

A renter's valuation is $V_k = \nu_k + \xi_k$, where k is the number of periods remaining in the term, ν_k is its expected value and ξ_k is the mean-zero random component (e.g., Özer and Phillips 2012, §18.3). This specification is versatile enough to accommodate various settings, such as valuation with decreasing expected value due perhaps to diminishing novelty of the product. Moreover, variability of the random component ξ_k may shrink over time, e.g., due to decreasing valuation. The cumulative distribution function of V_k is denoted by F_k.

We consider an RTO firm that owns several items of inventory for a focal product (e.g., a particular model of refrigerator). At each point in time, some of the items are out on rent, while others (if any) are *idle* waiting to be rented. Consistent with other works on durable goods (e.g., Coase 1972, Bulow 1982), items do not depreciate due to use. Moreover, within the planning horizon for pricing (a few years), depreciation due to obsolescence is minor for commonly rented items such as refrigerators or washer/dryers with stable design and functionality. The firm replaces each sold item (via payoff or buyout) with a similar one at *replacement cost* c. The firm forms the same agreement for each rented item, i.e., all rental agreements for the focal product have term $T \geq 2$, rental fee $s < c$ and identical buyout price path. We translate the inventory policy of the firm to maintaining a fixed number of *inventory slots* for the focal product. Although items are sold, inventory slots are perpetual and suitable for long-run profit calculations. We first take the perspective of a single inventory slot and later in §6 extend our analysis to multiple slots.

To represent the progression of the slot's rental agreements, we use a discrete-time Markov chain (MC) with state space $[1:T]$, where $[a:b]$ is the set containing a, b and the integers between them. We refer to the MC as the *RTO MC* and to its state as the *rental state*. Rental state $k < T$ signifies that the slot's current rental has k periods remaining until the renter owns the item through payoff, while the rental state is T when the slot is holding an idle item waiting to initiate a new rental. In each period, an idle item initiates a rental wp α and remains idle wp $1-\alpha$, i.e, the slot transitions from rental state T to $T-1$ wp α and remains in T wp $1-\alpha$. We refer to α as the agreement initiation probability and to rental state T as the idle state. The probability α depends on the demand distribution for new rentals as well as the total number of slots (see §6), but for the purposes of this section, it is given as a primitive.

The slot transitions from a non-idle state $k \in [1:T-1]$ to the idle state if the renter either returns the item or purchases it. The return event occurs wp $1-\rho$, and the buyout event occurs when the renter sustains the agreement and his valuation is no less than the buyout price p_k in rental state k. Therefore, the buyout event occurs wp $\rho\bar{F}_k(p_k)$, where $\bar{F}_k(p_k) = 1 - F_k(p_k)$. Accordingly, the slot transitions from rental state k to $k-1$ wp $\rho F_k(p_k)$ if neither the return nor the buyout occurs. In contrast to other rental states, sustaining the agreement in rental state 1 leads to item ownership through payoff, so the slot transitions wp 1 from this rental state to the idle state.

With each transition of the slot from rental state k to $k-1$, the firm earns the rental fee s. In transitions from rental state k to the idle state that occur via a buyout, the firm earns the revenue p_k and incurs replacement cost c. Finally, in transitions from rental state 1 to the idle state via payoff, the firm earns s and incurs the cost c. Any other transition is to the idle state via return of the item and thus involves no monetary transaction. As a side note, an RTO firm might incur a restocking cost for returned items, e.g., to refurbish, repair, or (if the item is damaged beyond repair) replace them. For parsimony, we do not include such a cost, but our model could easily be extended to include it by adding terms with this cost to the profit function for the transitions associated with returns.

Figure 1 illustrates the RTO MC and rental state transitions as well as their probabilities and the associated profits. Note that the idle state (rental state T) is accessible from all other states (including itself), but any state $k \in [1:T-1]$ is only accessible from state $k+1$. We seek to determine the stationary probabilities of the different rental states. These probabilities most importantly depend on the agreement initiation probability α and the buyout price path $\boldsymbol{p} := (p_{T-1}, \cdots, p_1) > 0$. Note that there is no buyout in rental state T since a renter who initiates a rental agreement is by definition not seeking to purchase immediately. The RTO MC is both aperiodic and irreducible. Therefore, its stationary probabilities are all positive. The following lemma provides formulas for the stationary probability $\pi_k(\boldsymbol{p};\alpha)$ for each $k \in [1:T]$. Hereafter, when the lower index of a sum $\sum$ is larger than its upper index, the sum becomes zero, and when the lower index of a product $\prod$ is larger than its upper index, the product becomes 1. Table 1 summarizes our important notations.

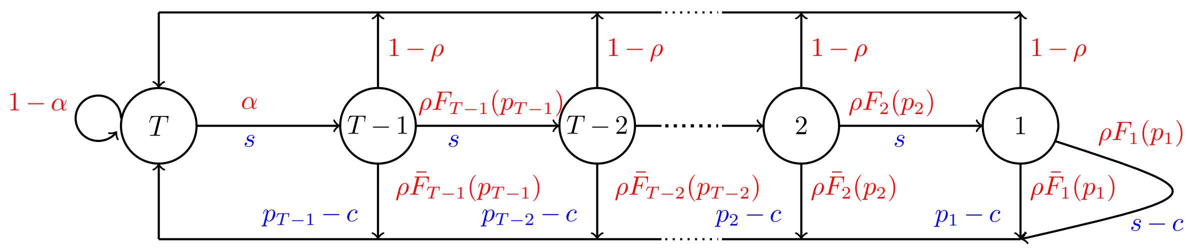


Figure 1 The RTO MC with transition probabilities and monetary transactions.

LEMMA 1. *The stationary probabilities of the RTO MC in Figure 1 are given by*

$$\pi_T(\boldsymbol{p};\alpha)=\left[1+\alpha+\alpha\sum_{k=1}^{T-2}\rho^{T-k-1}\prod_{j=k+1}^{T-1}F_j(p_j)\right]^{-1},\quad \pi_k(\boldsymbol{p};\alpha)=\alpha\pi_T(\boldsymbol{p};\alpha)\rho^{T-k-1}\prod_{j=k+1}^{T-1}F_j(p_j)\ \textit{for}\ k\le T-1. \tag{1}$$

We observe in Lemma 1 that $\pi_T(\boldsymbol{p};\alpha)\ge\cdots\ge\pi_1(\boldsymbol{p};\alpha)$; rental state $k\in[1:T-1]$ is only accessible from state $k+1$. Therefore, each visit to state k warrants at least a visit to the prior state $k+1$, which makes $\pi_{k+1}(\boldsymbol{p};\alpha)\ge\pi_k(\boldsymbol{p};\alpha)$. As a result, the slot spends more time in the idle state than any other single state over the long run.

3. The RTO Buyout Pricing Problem

We use the stationary probabilities in Lemma 1 to derive the expected per-period profit in the RTO MC, which we refer to as the *profit rate*. For $k\in[2:T-1]$, a transition from state k to state $k-1$ occurs at the rate $\pi_k(\boldsymbol{p};\alpha)\rho F_k(p_k)$ and generates the profit s. A transition from state k to state T, if involving a monetary transaction, is only possible through a buyout, occurs at the rate $\pi_k(\boldsymbol{p};\alpha)\rho\bar{F}_k(p_k)$ and generates the profit p_k-c. A transition from state 1 to state T, if not due to a return, either occurs through a buyout at the rate $\pi_1(\boldsymbol{p};\alpha)\rho\bar{F}_1(p_1)$ or through a payoff at the rate $\pi_1(\boldsymbol{p};\alpha)\rho F_1(p_1)$ with respective profits of p_1-c and $s-c$. Finally, a transition from state T to $T-1$ occurs at the rate $\alpha\pi_T(\boldsymbol{p};\alpha)$ and generates the profit s. Figure 1 depicts these transition probabilities and monetary transactions. All other possible transitions are due to the item being returned; these do not involve a monetary transaction and thus do not affect the profit rate. The discussed transition rates and the associated profits yield the (expected) profit rate as

$$R_T(\boldsymbol{p};\alpha)=\alpha\pi_T(\boldsymbol{p};\alpha)s+\sum_{k=2}^{T-1}\rho\pi_k(\boldsymbol{p};\alpha)\Big(F_k(p_k)s+\bar{F}_k(p_k)(p_k-c)\Big)+\rho\pi_1(\boldsymbol{p};\alpha)\Big(F_1(p_1)(s-c)+\bar{F}_1(p_1)(p_1-c)\Big). \tag{2}$$

Since sold items are replaced to maintain a target inventory level, the inventory holding cost is sunk and can thus be safely ignored when optimizing the buyout price path. However, holding cost will be important in §6 when we optimize both the price path and the inventory level.

In setting the buyout price path $\boldsymbol{p}$, we aim to maximize the profit rate given in (2). We can further restrict the price space with appropriate constraints, which we now describe. First, in any non-idle rental state k, it would be sub-optimal to set a buyout price $p_k<s$ because selling at such price

Table 1 Important notations.

	Inputs:
α and ρ	Probabilities of initiating an agreement and Sustaining an agreement
s and c	Rental fee and Replacement cost
T	Agreement term
F_k	Cumulative distribution function (cdf) of a renter's valuation in rental state $k \in [1:T-1]$
	Abbreviations:
ϕ_i^k and Φ_i^k	$\prod_{j=i}^k F_j(s)$ and $\prod_{j=i}^k F_j(js)$
	Outputs:
γ and $1-\gamma$	Utilization and Idleness
π_k	Stationary probability for rental state $k \in [1:T]$
R_T and R_T^a	Expected profit rate and Expected profit per agreement
$\boldsymbol{p}$ and I	Buyout price path and Number of inventory slots

(and thus incurring the replacement cost) would be strictly worse for the firm than receiving s for continuation of the rental (without incurring the replacement cost). Second, a renter in state k would reject any buyout price $p_k > ks$ because it would be cheaper to instead pre-pay (or pre-allocate) the remaining rental fees to own the item through payoff. Thus, we can restrict our attention to buyout prices p_k such that $s \le p_k \le ks$. Note that the realized total rental payments are equal to ks only if the renter continues the rental agreement until the end of the term without purchasing or returning the item. Thus, the expected total rental payments will be less than ks, and $p_k \le ks$ does not imply that buying in rental state k is preferred to continuing to rent. The price constraint $s \le p_k \le ks$ and (2) together yield the *RTO buyout pricing problem*

$$\max_{\boldsymbol{p}} \{R_T(\boldsymbol{p};\alpha) : p_k \in [s, ks] \quad \text{for} \quad k \in [1:T-1]\}. \tag{3}$$

The optimization in (3) is to determine a single buyout price path that is common across renters. The price offered to a renter is determined by the state of his agreement; for instance, given the price path $\boldsymbol{p}$, a renter whose agreement is in state k is offered the price p_k, while another renter whose agreement is in state $k' \neq k$ is offered $p_{k'}$. Now, we show that the buyout price path optimization in (3) is in general nonconcave.

Remark 1. The profit rate function $R_T(\boldsymbol{p};\alpha)$ in (3) is not in general concave in $\boldsymbol{p}$.

The nonconcavity of $R_T(\boldsymbol{p}:\alpha)$ implies that traditional gradient-based methods, even if they could identify a critical point efficiently in our high-dimensional price space, could get trapped at local optima or saddle points. The special case of $T=3$ has a single buyout price p_2 to determine, as $p_1 = s$ and rental state $T=3$ has no buyout price. For this case with exponentially distributed valuations, we can characterize the relationship between the initiation probability and the optimal buyout price.

Lemma 2. *For exponentially distributed valuations with mean ν and a given ρ, the optimal buyout price is decreasing in $\alpha \in (0,1)$ when $T=3$.*

Lemma 2 features a structural property for $T=3$ and exponential valuations. Although a buyout generates more momentary revenue than the rent obtained if the renter continues the agreement, it starts a duration of idleness with no revenue. The lower the agreement initiation probability is, the longer the expected duration of idleness (with no revenue), and thus the higher the opportunity cost of stopping the recurring rental payments. Thus, for a low agreement initiation probability, the firm is only willing to give up the recurring rental revenue in exchange for a high price. The general problem with $T \geq 4$ is more complex and requires optimization of multiple prices. For this problem, due to nonconcavity of the profit rate, an alternative and customized approach is required, which we proceed to develop.

In the forthcoming analysis, we will leverage another profit-related quantity, namely the expected profit from a single rental agreement, which we denote by $R_T^a(\boldsymbol{p})$. An important distinction between $R_T^a(\boldsymbol{p})$ and the profit rate $R_T(\boldsymbol{p};\alpha)$ is that the former ignores the idle time between agreements when no revenue is generated; unlike the profit rate, its calculation is contingent on agreement initiation and thus does not depend on α. The profit per agreement is calculated similarly to the profit rate by computing the probability of the agreement ending in each possible rental state with each outcome that is possible in that state, as well as the associated profit. The resulting expression for the (expected) profit per agreement is

$$\begin{aligned} R_T^a(\boldsymbol{p}) = &\sum_{k=2}^{T-1}\Big(\rho^{T-k-1}\prod_{j=k+1}^{T-1}F_j(p_j)\Big)\Big(\rho\bar{F}_k(p_k)((T-k)s+p_k-c)+(1-\rho)(T-k)s\Big) \\ &+\Big(\rho^{T-2}\prod_{j=2}^{T-1}F_j(p_j)\Big)\Big(\rho\bar{F}_1(p_1)((T-1)s+p_1-c)+(1-\rho)(T-1)s+\rho F_1(p_1)(Ts-c)\Big), \end{aligned} \tag{4}$$

where $p_1 = s$ because of the constraint in (3). This profit can be grouped into three terms: those proportional to rent s, to replacement cost c and the rest, which respectively are (expected) rental revenue, incurred replacement cost and sales revenue (all per agreement). The last period's payment to the firm is interpreted as sales revenue.

The next lemma relates the profit rate $R_T(\boldsymbol{p};\alpha)$ with the profit $R_T^a(\boldsymbol{p})$ per agreement. This relationship will later prove useful in our algorithm for buyout price path optimization.

Proposition 1 **(Profit Rate vs. Profit per Agreement)**. *For any price vector $\boldsymbol{p}$,*

$$R_T(\boldsymbol{p};\alpha) = \alpha\pi_T(\boldsymbol{p};\alpha)R_T^a(\boldsymbol{p}). \tag{5}$$

Proposition 1 delineates the components of the profit rate $R_T(\boldsymbol{p};\alpha)$ and reveals the tradeoffs in maximizing it. Unsurprisingly, the profit rate is increasing in the agreement initiation probability α because a higher α implies a smaller fraction of time spent idle (as discussed, $R_T^a(\boldsymbol{p})$ does not depend on α, and it is straightforward to show that $\alpha\pi_T(\boldsymbol{p};\alpha)$ is increasing in α). The agreement initiation probability is treated as exogenous in this section, but we will later endogenize it in §6. For a given α, the firm's choice of price path $\boldsymbol{p}$ affects both the profit per agreement $R_T^a(\boldsymbol{p})$ and the idleness rate $\pi_T(\boldsymbol{p};\alpha)$. Intuitively, increasing the prices in $\boldsymbol{p}$ leads to fewer buyouts and thus longer agreement durations, i.e., less profit from buyouts and more from rental fees. The net impact on the profit per agreement depends on the relative magnitudes of these changes. Additionally, the longer agreement durations lead to a decreased proportion of time spent idle, i.e., a smaller $\pi_T(\boldsymbol{p};\alpha)$. Hence, to maximize the profit rate, the firm must optimally balance three interdependent quantities (which are also all affected by the given probability ρ of sustaining the agreement): profit per agreement from buyouts, profit per agreement from rental payments, and idleness rate. This interdependence in the multidimensional price space leads to the nonconcavity identified in Remark 1, and it makes the price path optimization in (3) a highly challenging problem to solve. To tackle this challenge, our approach is to decompose the problem into a bilevel optimization, which we describe below, but first we discuss the important concept of *utilization* as it relates to our problem.

The utilization of an RTO inventory slot is the proportion of time its item is rented (i.e., is not idle). In our discussions, the RTO firm has emphasized that utilization (or equivalently, idleness) is a key metric for evaluating its operational performance, and moreover that it sets a target idleness (rate) for its inventory. It is possible that the target utilization will be provided exogenously by upper management, in which case methodology is required that provides optimal prices for a given utilization. This idea also suggests a decomposition of the general problem into two stages: determining the target utilization, and then determining the optimal buyout price path for that utilization.

As a proportion, utilization is bounded between 0 and 1, but the parameters of the system (e.g., ρ, α) may further limit what utilizations can be achieved. Recall that $\pi_T(\boldsymbol{p};\alpha)$ is the proportion of time the slot is in the idle state and hence is equal to the slot's idleness rate. As such, $1-\pi_T(\boldsymbol{p};\alpha)$ is the slot's utilization. For $\gamma \in [0,1]$, to achieve the utilization $1-\gamma$, the firm must set a buyout price path $\boldsymbol{p}$ such that $\pi_T(\boldsymbol{p};\alpha)=\gamma$. A utilization $1-\gamma$ is *achievable* for a given α if there exists $\boldsymbol{p}$ with $p_k \in [s, ks]$ for $k \in [1:T-1]$ such that $\pi_T(\boldsymbol{p};\alpha)=\gamma$. With the next lemma, we explicitly characterize the interval of achievable utilizations in terms of the problem parameters. We define $\phi_i^k := \prod_{j=i}^k F_j(s)$ and $\Phi_i^k := \prod_{j=i}^k F_j(js)$ for tidier expressions. Note that $\phi_i^k \le \Phi_j^k$.

LEMMA 3. *For a given α, the utilization $1-\gamma$ is achievable if and only if $\gamma \in [\gamma_T^l(\alpha), \gamma_T^u(\alpha)]$, where*

$$\gamma_T^l(\alpha) := \left[1+\alpha+\alpha\sum_{i=1}^{T-2}\rho^{T-i-1}\Phi_{i+1}^{T-1}\right]^{-1} \quad \textit{and} \quad \gamma_T^u(\alpha) := \left[1+\alpha+\alpha\sum_{i=1}^{T-2}\rho^{T-i-1}\phi_{i+1}^{T-1}\right]^{-1}.$$

Using Lemma 3 to identify the range of achievable utilizations, we can now express the buyout pricing problem problem (3) as a bilevel optimization. The outer problem chooses a utilization from the achievable interval, and for each such utilization, the inner problem determines the optimal price path among those leading to that utilization. Formally, we cast (3) alternatively as

$$\max_{\gamma\in[\gamma^l(\alpha),\gamma^u(\alpha)]}\Big\{R_T^*(\gamma;\alpha):=\max_{\boldsymbol{p}}\{R_T(\boldsymbol{p};\alpha):p_k\in[s,ks]\ \text{ for }\ k\in[1:T-1],\pi_T(\boldsymbol{p};\alpha)=\gamma\}\Big\} \tag{6}$$

(Appendix C.1 formally establishes equivalence of (3) and (6)). Note that the expression for $\pi_T(\boldsymbol{p};\alpha)$ in the inner maximization above is obtained from Lemma 1.

In addition to its role in profit rate maximization, as hinted earlier, the inner problem of (6) is of particular interest to an RTO firm that has an exogenously given target utilization. For example, the utilization may be benchmarked by the industry as a performance metric of inventory management.

We now take a key step to enable our approach for optimizing the buyout price path. Recall that we have $R_T(\boldsymbol{p};\alpha)=\alpha\pi_T(\boldsymbol{p};\alpha)R_T^a(\boldsymbol{p})$ by Proposition 1. Since $\pi_T(\boldsymbol{p};\alpha)=\gamma$ in the inner problem of (6), we make a substitution and arrive at an equivalent problem with a greatly simplified objective function, namely $R_T^a(\boldsymbol{p})$ (which does not depend on α or the stationary probabilities) instead of $R_T(\boldsymbol{p};\alpha)$.

Lemma 4. *The solution to the inner maximization problem of* (6) *for* $\gamma\in[\gamma_T^l(\alpha),\gamma_T^u(\alpha)]$ *can be obtained by solving:*

$$R_T^*(\gamma;\alpha)=\alpha\gamma\max_{\boldsymbol{p}}\left\{R_T^a(\boldsymbol{p}):p_k\in[s,ks]\ \textit{for}\ k\in[1:T-1],\ \sum_{k=1}^{T-2}\rho^{T-2-k}\prod_{j=k+1}^{T-1}F_j(p_j)=\frac{1-\gamma-\alpha\gamma}{\alpha\gamma\rho}\right\}. \tag{7}$$

We refer to (7) as the *inner pricing problem.* The equality constraint in (7) guarantees the utilization $1-\gamma$. We refer to this as the *utilization constraint.* We have obtained the inner pricing problem in (7) by deriving the stationary distribution of the RTO MC and using it to construct the profit rate function in Proposition 1. Alternatively, it is also possible to obtain $R_T^a(\boldsymbol{p})$ and the utilization constraint via renewal theory (see Appendix C.2).

Towards simplifying the objective $R_T^a(\boldsymbol{p})$, we reconsider the rental and sales revenues, and the incurred replacement cost. The rental revenue is proportional to the duration of the rental. To represent rental and sales revenues, we let $\tau(\boldsymbol{p})$ and $\lambda_{\tau(\boldsymbol{p})}$ respectively be the rental duration random variable and its failure rate under price path $\boldsymbol{p}$. This random variable has the range $[1:T-1]$ and the tail probability (see Appendix C.2 for details),

$$\mathrm{P}(\tau(\boldsymbol{p})\geq t)=\rho^{t-1}\prod_{j=T-t+1}^{T-1}F_j(p_j)\qquad\text{for } t\in[1:T-1]. \tag{8}$$

The replacement cost is incurred when the rented item is sold during the rental. To represent this cost, we let $S(\boldsymbol{p})$ be a binary random variable taking the value of 1 when the rented item is sold. So,

$$\mathrm{P}(S(\boldsymbol{p})=1)=\sum_{t=1}^{T-2}\mathrm{P}(\tau(\boldsymbol{p}))\geq t)\rho\bar{F}_{T-t}(p_{T-t})+\mathrm{P}(\tau(\boldsymbol{p}))\geq T-1)\rho.$$

Equipped with these random variables, we can reinterpret the utilization constraint and drop the expected rental revenue and replacement cost from $R^a_T(\boldsymbol{p})$ in (7) to obtain a reduced yet equivalent formulation presented next.

Proposition 2 **(Reduction of Inner Pricing Problem)**. *The expected rental duration is* $\mathbb{E}(\tau(\boldsymbol{p})) = (1-\gamma)/(\alpha\gamma) \le 1/(1-\rho)$ *and the probability of selling the rented item is* $\mathrm{P}(S(\boldsymbol{p})=1) = 1-(1-\rho)\mathbb{E}(\tau(\boldsymbol{p})) \ge 0$. *Given* $\mathbb{E}(\tau(\boldsymbol{p}))$, *the expected rental revenue and incurred replacement cost are constant in buyout prices. The relevant objective to maximize is the expected sales revenue* $R^{a\setminus}_T(\boldsymbol{p})$, *where*

$$R^{a\setminus}_T(\boldsymbol{p}) = \sum_{t=1}^{T-2} \mathrm{P}(\tau(\boldsymbol{p}) \ge t)\rho \bar{F}_{T-t}(p_{T-t})p_{T-t} + \mathrm{P}(\tau(\boldsymbol{p}) \ge T-1)\rho p_1.$$

Then instead of (7), *the following can alternatively be solved to obtain the optimal prices:*

$$\max_{\boldsymbol{p}} \left\{ \mathbb{E}(p_{T-\tau(\boldsymbol{p})}|S(\boldsymbol{p})=1): \quad p_t \in [s, st] \text{ for } t \in [1:T-1], \quad \mathbb{E}(\tau(\boldsymbol{p})) = \frac{1-\gamma}{\alpha\gamma} \right\}. \tag{9}$$

Under $\rho=1$, $R^{a\setminus}_T(\boldsymbol{p}) = \mathbb{E}(p_{T-\tau(\boldsymbol{p})}) = \mathbb{E}\big(\bar{F}^{-1}_{T-\tau(\boldsymbol{p})}(1-\lambda_{\tau(\boldsymbol{p})}(\tau(\boldsymbol{p})))\big)$ *and so with this objective,* (9) *reduces to*

$$\max_{\boldsymbol{p}} \left\{ R^{a\setminus}_T(\boldsymbol{p}): \quad 1-\lambda_{\tau(\boldsymbol{p})}(t) \in [F_{T-t}(s), F_{T-t}((T-t)s)] \text{ for } t \in [1:T-1], \quad \mathbb{E}(\tau(\boldsymbol{p})) = \frac{1-\gamma}{\alpha\gamma} \right\}. \tag{10}$$

Interestingly, the optimal solution to the inner pricing problem is independent of the rental revenue and the incurred replacement cost. The former independence follows from noting that the utilization constraint fixes the expected rental duration. The latter independence follows from two observations. First, the replacement cost is incurred in the event of sales, whose rate of occurrence is the difference between the agreement initiation rate $\alpha\pi_T(\boldsymbol{p};\alpha)$ and the item return rate $(1-\pi_T(\boldsymbol{p};\alpha))(1-\rho)$. Second, as these rates are dependent on prices only through $\pi_T(\boldsymbol{p};\alpha)$, the utilization constraint $\pi_T(\boldsymbol{p};\alpha)=\gamma$ fixes them and in turn fixes the incurred replacement cost.

Prices are related to the failure rate $\lambda_{\tau(\boldsymbol{p})}(t)$ of the rental duration. Failure (termination) of a rental can be due to a buyout or a return. The possibility of a return makes it impossible to directly connect the termination to the price at a particular time. For such a connection, at the end of Proposition 2, we consider the special case of no returns, i.e., $\rho=1$, and show that the failure rate at a time is the buyout probability with the price at that time, i.e., $\lambda_{\tau(\boldsymbol{p})}(t) = \bar{F}_{T-t}(p_{T-t})$.

4. Calculus of Variations for Optimal Price Path

The problem in (10) is complicated to solve given the high-dimensionality of the price space. To gain insight into the structure of the solution, we next consider a continuous-time analog of (10) with a stationary distribution F. We assume that a renter's valuation for the rented product is at least s, so

$F(s)=0$. In this section, t represents the amount of time elapsed since the beginning of the rental, so we let $\boldsymbol{p}(t)$ represent the buyout price at time $t \in [0,T]$. For brevity, we let $\delta = (1-\gamma)/(\alpha\gamma)$ and drop t as argument of $\boldsymbol{p}(t)$ when $\boldsymbol{p}$ itself is an argument of a functional.

With $g_{\tau(\boldsymbol{p})}$ and $\bar{G}_{\tau(\boldsymbol{p})}$ respectively denoting the density and tail probability functions of $\tau(\boldsymbol{p})$, the continuous-time analog of the expected sales revenue is

$$R_T^{a\setminus}(\boldsymbol{p}) = \mathbb{E}\Big(F^{-1}(1-\lambda_{\tau(\boldsymbol{p})}(t))\Big) = \int_0^T g_{\tau(\boldsymbol{p})}(t)\bar{F}^{-1}\Big(1-g_{\tau(\boldsymbol{p})}(t)/\bar{G}_{\tau(\boldsymbol{p})}(t)\Big)dt,$$

where the last equality follows from $\lambda_{\tau(\boldsymbol{p})}(t) = g_{\tau(\boldsymbol{p})}(t)/\bar{G}_{\tau(\boldsymbol{p})}(t)$. In addition, $\mathbb{E}(\tau(\boldsymbol{p}))$ in continuous time is $\int_0^T \bar{G}_{\tau(\boldsymbol{p})}(t)dt$. So, the continuous-time analog of (10) is

$$\max_{\boldsymbol{p}} \left\{ R_T^{a\setminus}(\boldsymbol{p}) : \quad \frac{g_{\tau(\boldsymbol{p})}(t)}{\bar{G}_{\tau(\boldsymbol{p})}(t)} \in [1-\bar{F}((T-t)s),1] \text{ for } t\in[0,T], \quad \int_0^T \bar{G}_{\tau(\boldsymbol{p})}(t)dt = \delta \right\}. \tag{11}$$

Instead of solving (11) in the space of prices, we solve its equivalent in the space of tail probabilities $\bar{G}:[0,T]\to[0,1]$ that have continuous second derivatives. With this restriction, the derivative $\dot{\bar{G}}$ is defined and is continuous over $[0,T]$. This tail probability is for the rental duration random variable τ whose density and failure rate are respectively $\dot{\bar{G}}$ and $-\dot{\bar{G}}(t)/\bar{G}(t)$. Thus, with some abuse of notation, we write the expected sales revenue as

$$R_T^{a\smallsetminus}(\bar{G}) = -\int_0^T \dot{\bar{G}}(t)\Big(\bar{F}^{-1}(1+\dot{\bar{G}}(t)/\bar{G}(t))\Big)dt.$$

We allow for $\bar{G}(T)>0$; this mass $\mathrm{P}(\tau=T)=\bar{G}(T)$ is the payoff probability. If the renter does not buy the rented item by T, he owns it wp $\bar{G}(T)$ and ends the rental with duration T. For mathematical convenience, we drop the lower bound on the failure rate in the forthcoming reformulation of (11). This corresponds to dropping the upper bounds $s(T-t)$ from prices in (7). After we find a solution $\bar{G}$, we can compute $p(t)=\bar{F}^{-1}(1+\dot{\bar{G}}(t)/\bar{G}(t))$ and then implement $\min\{p(t), s(T-t)\}$ to make the prices feasible. Hence, we arrive at

$$\max_{\bar{G}\in\mathcal{G}} R_T^{a\smallsetminus s}(\bar{G}), \text{ where } \mathcal{G} = \left\{\bar{G}: \ \bar{G}(0)=1, \ \bar{G}(t)\geq 0, \ -\dot{\bar{G}}(t)/\bar{G}(t)\in[0,1] \text{ for } t\in[0,T], \ \int_0^T \bar{G}(t)dt=\delta\right\},$$

which we refer to as the reformulated pricing problem. Note that constraints $\bar{G}(0)=1$ and $\bar{G}(t)\geq 0$ make $\bar{G}(t)$ a valid tail probability.

Inspecting $R_T^{a\smallsetminus s}(\bar{G})$, we observe

$$\text{Sales Revenue Linearity: } \ R_T^{a\smallsetminus}(\kappa\bar{G}) = \kappa R_T^{a\smallsetminus}(\bar{G}) \ \text{ for } \kappa \geq 0.$$

This allows us to easily evaluate $R_T^{a\smallsetminus}(\kappa\bar{G}_1)$ and compare with $R_T^{a\smallsetminus}(\kappa\bar{G}_2)$ by using only $R_T^{a\smallsetminus}(\bar{G}_1)$ and $R_T^{a\smallsetminus}(\bar{G}_2)$. By dropping $\int_0^T \bar{G}(t)dt=\delta$ and adjusting the equality and the last inequality in $\mathcal{G}$, we arrive at a version of the feasible set:

$$\mathcal{G}^r = \left\{\bar{G}: \ \bar{G}(0)=\frac{1}{\delta}\int_0^T \bar{G}(t)dt, \ \bar{G}(t)\geq 0, \ -\dot{\bar{G}}(t)/\bar{G}(t)\in[0,1] \text{ for } t\in[0,T], \ \int_0^T \bar{G}(t)dt \leq T\right\}.$$

Except for the last inequality in $\mathcal{G}^r$, other constraints remain to hold with $\kappa\bar{G}$ if they hold with $\bar{G}$. Equipped with this observation and the sales revenue linearity, we next connect the problem $\max\{R_T^{a\searrow}(\bar{G}) : \bar{G} \in \mathcal{G}\}$ to $\max\{R_T^{a\searrow}(\bar{G}) : \bar{G} \in \mathcal{G}^r\}$.

LEMMA 5. *If $\bar{G}^r$ solves $\max\{R_T^{a\searrow}(\bar{G}) : \bar{G} \in \mathcal{G}^r\}$ and yields $\int_0^T \bar{G}^r(t)dt = \delta^r > 0$, then $(\delta/\delta^r)\bar{G}^r$ solves $\max\{R_T^{a\searrow}(\bar{G}) : \bar{G} \in \mathcal{G}\}$.*

By Lemma 5, we can first solve $\max\{R_T^{a\searrow}(\bar{G}) : \bar{G} \in \mathcal{G}^r\}$ and then adjust the solution with a positive multiplier to construct a solution to $\max\{R_T^{a\searrow}(\bar{G}) : \bar{G} \in \mathcal{G}\}$. Although the constant $\bar{G}(t)$ at 0 for every t is trivially in $\mathcal{G}^r$, it yields $\delta^r = 0$. We next show that $\mathcal{G}^r$ includes a nontrivial $\bar{G}$ that yields $\delta^r > 0$ when we seek a solution under a specific distribution F.

We consider uniform valuations and deploy calculus of variations to solve $\max\{R_T^{a\searrow}(\bar{G}) : \bar{G} \in \mathcal{G}^r\}$; wlog, we set the support of F to $[s, s+1]$, i.e., $\bar{F}^{-1}(x) = s + x$ for $x \in [0,1]$.

PROPOSITION 3 **(Optimal Price Path with Fixed Utilization)**. *The optimal price path for an instance of the reformulated pricing problem with a uniformly distributed valuation over $[s, s+1]$ and $\delta/T \in [(T^2 + 6T + 12)/(3(T+2)^2), 1]$ is*

$$p(t) = 1 + s - \frac{2}{c_\lambda(T,\delta) - t} \quad \textit{for } t \in [0,T], \quad \textit{where } c_\lambda(T,\delta) = \frac{1 + \sqrt{1 - (4/3)(1 - \delta/T)}}{2(1 - \delta/T)} T.$$

This path is concave decreasing in time. It shifts down when the initiation probability α or the idleness rate γ increases, as larger α or γ yields smaller δ by $\delta = (1-\gamma)/(\alpha\gamma)$.

The concave decreasing price curve implies that optimally, prices will decrease gradually early in the agreement and more steeply later in the agreement. See Figure 2 in Section 5.1 for an example. By contrast, in practice, RTO firms usually decrease prices steeply early in the agreement and more gradually later in the agreement (see Appendix A.1). Thus, we find that the optimal price curve has a qualitatively different shape from that applied in practice. Additionally, when the idleness rate increases under a fixed initiation probability, the rate of return to the idle state must rise. To achieve such a higher rate of return, we need to induce shorter rental durations by motivating renters to buy with a lower price path. When the initiation probability increases, the rate of return to the idle state must rise to maintain a fixed idleness rate. Then, once again, we need to induce shorter durations with a lower price path. The same directional relationship is observed in Lemma 2 for a small problem without a fixed idleness rate as the initiation probability varies.

In this section, we have derived a closed-form function for the optimal price path and also characterized the curvature of this function and its monotonicity in the agreement initiation probability and utilization. However, obtaining this complete characterization of the optimal buyout prices required some assumptions—no returns, uniformly distributed valuations, and continuous time—that may fall short of representing reality faithfully enough for practical implementation.

5. Dynamic Programming Approach for the RTO Buyout Pricing Problem

For real-world RTO buyout price path optimization, a general-purpose algorithm is required that (i) incorporates the probability of items being returned, (ii) is flexible enough to accommodate a wide variety of valuation distributions and (iii) aligns with the discrete-time nature of practical RTO agreements. In this section, we develop such an algorithm using a dynamic programming (DP) approach. The DP allows us to sequentially determine the optimal buyout price for each rental state instead of determining the optimal price path $\boldsymbol{p}$ in one shot. We then deploy our algorithm over all feasible γ values to solve the RTO buyout pricing problem.

We denote the vector of buyout prices in states prior to state k by $\boldsymbol{p}_{>k} = (p_{T-1}, p_{T-2}, \ldots, p_{k+1})$. Also $\boldsymbol{p}_{\leq k} = (p_k, p_{k-1}, \cdots, p_1)$ and $\boldsymbol{p}_{>T-1} = \emptyset$. Based on the right-hand side (RHS) of the utilization constraint in (7), we set $a_{T-1} = a_{T-1}(\emptyset) = (1-\gamma-\alpha\gamma)/(\alpha\gamma\rho)$. For the forthcoming derivations, we define functions $\{a_k(\boldsymbol{p}_{>k}) : k \in [1:T-1]\}$ as

$$a_k(\boldsymbol{p}_{>k}) = \frac{a_{T-1}(\emptyset) - \sum_{i=k}^{T-2} \rho^{T-2-i} \prod_{j=i+1}^{T-1} F_j(p_j)}{\rho^{T-1-k} \prod_{j=k+1}^{T-1} F_j(p_j)} = \frac{1-\gamma-\alpha\gamma \sum_{i=k}^{T-1} \rho^{T-1-i} \prod_{j=i+1}^{T-1} F_j(p_j)}{\alpha\gamma\rho^{T-k} \prod_{j=k+1}^{T-1} F_j(p_j)}, \quad (12)$$

which we refer to as the *DP state* for a slot in rental state k. For a given $\boldsymbol{p}$, values of $a_k(\boldsymbol{p}_{>k})$ evolve according to the recursion

$$a_{k-1}(\boldsymbol{p}_{>k-1}) = (a_k(\boldsymbol{p}_{>k})/F_k(p_k) - 1)/\rho \quad \text{for } k \in [2:T-1]. \quad (13)$$

For any $k \in [1:T-1]$, the utilization constraint is equivalent to $\sum_{i=1}^{k-1} \rho^{k-1-i} \prod_{j=i+1}^{k} F_j(p_j) = a_k(\boldsymbol{p}_{>k})$ (see Appendix C.3 for details). Specializing this expression for $k = T-1$ and inserting in (12), we get $a_1(\boldsymbol{p}_{>1}) = 0$. That is, prices that are feasible to the inner pricing problem of (7) yield $a_1(\boldsymbol{p}_{>1}) = 0$. Next, we consider relevant states that produce feasible prices for (7).

In the sequentially solved DP formulation, when in rental state k, the buyout prices $\boldsymbol{p}_{>k}$ in states prior to k are already determined, and we are to determine p_k. As $\pi_T = \gamma$, by Lemma 1, we have $\pi_{T-1} = \alpha\gamma$, and the stationary probabilities for other rental states are

$$\pi_k(\boldsymbol{p}_{>k}) = \alpha\gamma\rho^{T-i-1} \prod_{j=k+1}^{T-1} F_j(p_j), \quad k \in [1:T-2]. \quad (14)$$

Note that π_k depends on $\boldsymbol{p}_{>k}$ and is independent of $\boldsymbol{p}_{\leq k}$. In essence, $\pi_T(\boldsymbol{p};\alpha)$ is a normalization factor that relates any other stationary probability to $\boldsymbol{p}$ in Lemma 1; when $\pi_T(\boldsymbol{p};\alpha)$ is fixed at γ, π_k is expressible in terms of α, γ and $\boldsymbol{p}_{>k}$ for $k \in [1:T-1]$. Consequently, when setting p_k, the value of π_i is known for $i > k$, and for any p_k, we can determine the resulting π_{k-1}. These probabilities must naturally satisfy $\sum_{i=k-1}^{T} \pi_i \leq 1$. However, this condition is insufficient to guarantee existence of $\{p_j \in [s, js] : j \leq k-1\}$ that satisfy $\sum_{i=1}^{T} \pi_i = 1$. So, p_k must be such that the equation

$$\sum_{i=1}^{k-1} \pi_i(\boldsymbol{p}_{>i}) = 1 - \sum_{i=k}^{T} \pi_i(\boldsymbol{p}_{>i}) \quad (15)$$

can be satisfied with $p_j \in [s, js]$ for $j \leq k-1$. The RHS of (15) is known and fixed when setting p_k as it depends only on $\boldsymbol{p}_{>k}$.

By (14), the left-hand side (LHS) of (15) is increasing in p_j for $j < k$. As such, for any fixed p_k, the maximum value of the LHS must be greater than the RHS and its minimum value must be lower than the RHS; otherwise, (15) is not satisfiable with $p_j \in [s, js]$ for $j < k$. For (15) to be satisfiable with $p_j \in [s, js]$ for $j < k$, two conditions must hold for any p_k; first, the RHS of (15) must be no less than the minimum of its LHS, achieved with $p_j = s$ for $j < k$, which is equivalent to

$$F_k(p_k) \leq \frac{a_k(\boldsymbol{p}_{\geq k})}{1 + \sum_{i=1}^{k-2} \rho^i \phi_{k-i}^{k-1}}. \tag{16}$$

Second, the RHS must be no larger than the maximum of the LHS, achieved with $p_j = js$ for $j < k$:

$$F_k(p_k) \geq \frac{a_k(\boldsymbol{p}_{\geq k})}{1 + \sum_{i=1}^{k-2} \rho^i \Phi_{k-i}^{k-1}}, \tag{17}$$

where ϕ and Φ were defined in Table 1. Constraints (16) and (17) are obtained algebraically (see Appendix C.4). Checking the constraints requires knowledge of only a_k, without the need to know individual prices of $\boldsymbol{p}_{>k}$.

Using (16) and (17) in addition to the constraint on prices in (7), we present a DP formulation to solve the inner pricing problem with DP state $\{a_k : k \geq 1\}$, specializing the formulation for $a_{T-1} = (1-\gamma-\alpha\gamma)/(\alpha\gamma\rho)$.

Proposition 4 **(DP for Optimal Prices with Fixed Utilization)**. *Let $v_1 = v_1(0) = \rho(s-c)$, and for $k \geq 2$*

$$\begin{aligned} v_k(a_k) = \max_{p_k \in [s, ks]} \; & \rho \bar{F}_k(p_k)(p_k - c) + \rho F_k(p_k)\Big(s + v_{k-1}((a_k/F_k(p_k) - 1)/\rho)\Big) \\ \text{s.t.} \quad & \frac{a_k}{1 + \sum_{i=1}^{k-2} \rho^i \Phi_{k-i}^{k-1}} \leq F_k(p_k) \leq \frac{a_k}{1 + \sum_{i=1}^{k-2} \rho^i \phi_{k-i}^{k-1}}. \end{aligned} \tag{18}$$

Obtaining $v_{T-1}((1-\gamma-\alpha\gamma)/(\alpha\gamma\rho))$ yields the optimal prices for (7) and $R_T^*(\gamma;\alpha) = \alpha\gamma(v_{T-1}((1-\gamma-\alpha\gamma)/(\alpha\gamma\rho)) + s)$ yields the optimal profit rate.

Proposition 4 states and validates the DP formulation (18) to solve the inner pricing problem (7). Our DP has a single, one-dimensional state variable that evolves deterministically according to recursion (13). Hence, it is simple and efficient to solve. It transforms a one-shot multidimensional optimization problem (whose prices are set all at once) into a sequential optimization problem. The simplicity of the DP facilitates the implementation of our pricing methodology in practice.

From specialization of (18) for $k = 2$, we infer $a_2 = F_2(p_2)$ and then $a_1 = a_2/F_2(p_2) - 1 = 0$ by (13). Hence, the ending state a_1 is fixed at 0. The DP is a fixed endpoint optimal control problem (Sethi 2018). Prices across different periods are coupled to achieve the fixed endpoint of $a_1 = 0$. Inequality constraints on $F_k(p_k)$ in (18) collectively ensure the equality constraint in (7), which keeps prices consistent with the fixed utilization. This consistency leads to fixing the endpoint at $a_1 = 0$.

LEMMA 6. *Construction of state intervals forward in time: Starting at* $a_{T-1} = (1-\gamma-\alpha\gamma)/(\alpha\gamma\rho)$ *and using recursion* (13) *and* $p_j \in [s, js]$*, we have* $a_k \in [a_k^l, a_k^u]$ *for* $k \geq 2$*, where* $0 \leq a_k^l \leq a_k^u$ *and*

$$a_k^l = \frac{(1-\gamma)/(\alpha\gamma)}{\rho^{T-k}\Phi_{k+1}^{T-1}} - \sum_{i=k+1}^{T} \frac{1}{\rho^{i-k}\Phi_{k+1}^{i-1}} \quad \text{and} \quad a_k^u = \frac{(1-\gamma)/(\alpha\gamma)}{\rho^{T-k}\phi_{k+1}^{T-1}} - \sum_{i=k+1}^{T} \frac{1}{\rho^{i-k}\phi_{k+1}^{i-1}}. \tag{19}$$

Construction of state intervals backward in time: Consider a price vector $\boldsymbol{p}$ *with* $p_j \in [s, js]$ *for* $j \geq 2$ *and numbers* $a_1, a_2, \ldots$ *that satisfy the recursion* (13)*. The constraint on* $F_k(p_k)$ *in* (18) *holds for all* $k \geq 2$ *if and only if* $p_2 = F_2^{-1}(a_2)$ *and* $a_k \in [\tilde{a}_k^l, \tilde{a}_k^u]$ *for* $k \geq 2$*, where* $\tilde{a}_k^l \leq \tilde{a}_k^u$ *and*

$$\tilde{a}_k^l = \sum_{i=1}^{k-1} \rho^{i-1}\phi_{k-i+1}^{k} \quad \text{and} \quad \tilde{a}_k^u = \sum_{i=1}^{k-1} \rho^{i-1}\Phi_{k-i+1}^{k}. \tag{20}$$

Lemma 6, for each rental state $k \in [2 : T-2]$, limits the range of the DP state a_k. The intervals in the lemma are constructed using the same underlying logic but starting from different fixed points of the DP state and moving in opposing directions in time. To further prune the values of the DP state, we use the intersection of the intervals, i.e., $a_k \in [a_k^l, a_k^u] \cap [\tilde{a}_k^l, \tilde{a}_k^u]$. The results in this section lead to the Dynamic Program for Pricing (DPP) algorithm for the inner pricing problem, provided in Appendix A.2.

5.1. Price Optimization with Target Utilization Rate

We have so far presented two methodologies for the inner pricing problem; one in continuous time in Proposition 3 and another in discrete time in Proposition 4. Proposition 3 explicitly presents the optimal price path, whereas the DP methodology of Proposition 4 is implemented via the DPP algorithm; here, we examine the solutions that these methodologies provide.

First, we compare the prices produced by the continuous-time and discrete-time methods under no returns, initiation probability of 80%, utilization of 87% and valuation in each rental state is (stationary and) uniformly distributed between 0.2 and 1.2. If the range of the underlying valuation is not 1, it can be made 1 via rescaling of the monetary parameters. We use agreement parameters of term $T = 20$ and rent $s = 0.2$. This problem instance satisfies the conditions in Proposition 3, which yields the optimal price path $p(t)$. Figure 2 shows the continuous-time $(\min\{p(t), s(T-t)\})$ and discrete-time (DP solution) price paths. The paths are very close, both in price magnitudes and curvature. In our numerical setting of identically distributed valuations, the price paths are concave decreasing with moderate price drops that are s or less; however, we note that if valuations had, for example, convex decreasing means, then the shape of the price path could change. That is, the DP solution mirrors the structural properties of the optimal price path obtained under the restrictions of no returns and uniformly distributed valuations in Proposition 3. The DP methodology, however, is advantageous in that it is efficiently solvable for any valuation distribution and with the possibility of returns, making it an appropriate tool for practitioners in determining buyout prices.

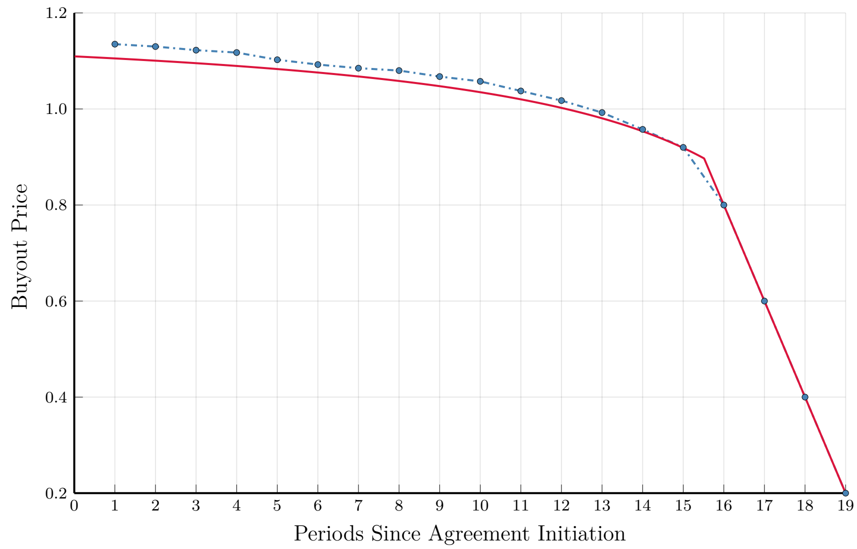


Figure 2 **Continuous-time (solid) and discrete-time (dashed) buyout price paths.**

Next, to observe directional changes in optimal price path with respect to (wrt) target utilization, we examine the DP solutions for different γ values. Consistent with empirical findings in Armaghan et al. (2023), we set $\rho = 0.92$ and consider exponentially distributed valuation with mean \$120 in each rental state, i.e., $F_k(p) = F(p) = 1 - e^{-p/120}$ for $k \geq 1$ (in particular, $\nu_k = 120$ and ξ_k has the cdf $1 - e^{-(1+x/120)}$ over $[-120, \infty]$). We consider a yearlong agreement with monthly rental fee of \$100, i.e., $T = 12$ and $s = \$100$. Finally, we take agreement initiation probability $\alpha = 80\%$, and find optimal buyout price paths for target utilizations of 81%, 82.4% and 84% which correspond to interior γ values of the interval specified in Lemma 3. Figure 3 shows the optimal price paths, which shift down as the target utilization decreases. This observation in discrete time agrees with its continuous-time analog in Proposition 3.

Recall from Proposition 2 that for each given utilization, the replacement cost does not influence the optimal prices. However, since different utilizations imply different rates of replacement, the replacement cost will have a nonhomogeneous effect on the profit rates for different utilizations. Thus, the optimal utilization (and hence also the optimal prices) will be different for different values of the replacement cost, even though the optimal prices for each given utilization will be unaffected.

5.2. Solving the RTO Buyout Pricing Problem

We now consider different utilizations (and the corresponding prices) to obtain the solution to the RTO buyout pricing problem in (6). In addition to the parameters in the previous subsection, we set the replacement cost $c = \$150$. The interval of achievable utilizations, from Lemma 3, is $[\gamma^l_{12}(0.8), \gamma^u_{12}(0.8)]$, where $\gamma^l_{12}(0.8) \approxeq 0.15$, and $\gamma^u_{12}(0.8) \approxeq 0.38$. To find the pair of utilization and buyout price path that together yield the highest profit rate, we discretize the achievable utilization interval and solve the

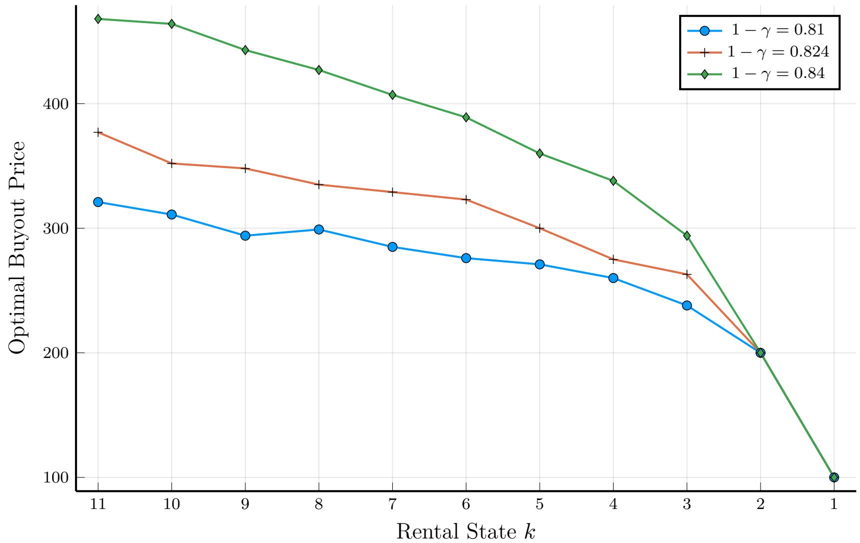


Figure 3 **Optimal buyout paths in dollars for different utilizations.**

DP (18) for each point to achieve the optimal buyout price path and the corresponding profit rate (see the DPP algorithm in Appendix A.2).

Figure 4 illustrates the optimal profit rate as a function of the utilization. As expected, higher utilizations do not in general yield higher profit rates; we observe that the profit rate is indeed concave in utilization. The optimal utilization is 82.4%, whose buyout price path is presented in Figure 3. These prices yield the profit rate of \$90.5 per month. An RTO firm can determine the optimal utilization by plotting profit rate, as in Figure 4, and retrieving the corresponding price path.

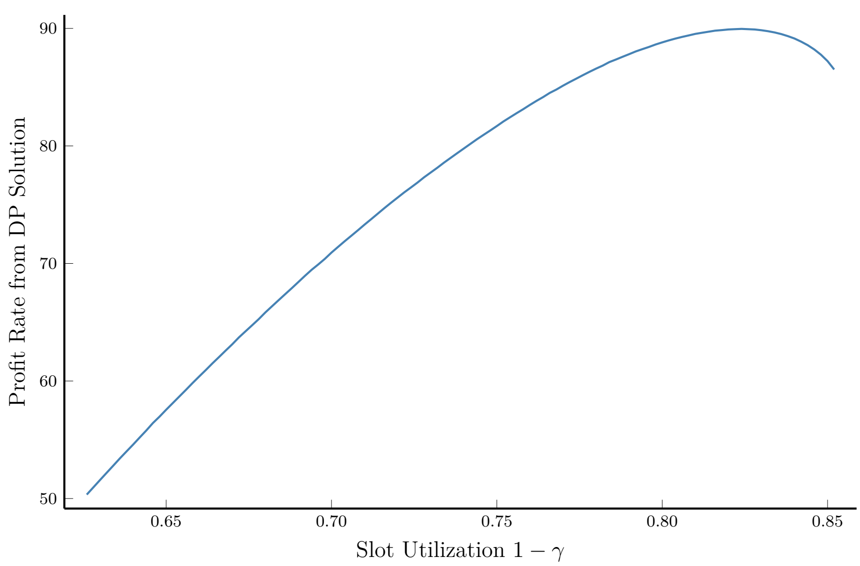


Figure 4 **Profit rates for different target utilizations.**

6. Joint Buyout Pricing and Inventory Optimization

To this point, we have considered a single inventory slot in isolation, and we have taken the agreement initiation probability α as a given parameter representing the probability that demand arrives for the particular inventory slot when it is idle. However, in practice, RTO firms maintain multiple inventory slots for the same type of item. Consequently, the probability of agreement initiation for a given idle slot depends on the total number of idle slots, as well as on the demand distribution.

To reflect this feature of real-world RTO operations, in this section, we extend our analysis to a system of $I \geq 1$ identical inventory slots, each with the same term T, rental fee s, replacement cost c and buyout price path $\boldsymbol{p}$. To each slot, we associate a *rental state*; if the slot has an active rental agreement, then this state belongs to $[1:T-1]$ and indicates the number of periods remaining until payoff; otherwise, the rental state is T, indicating that the slot is idle and available to be rented. The number of customers wishing to initiate a rental agreement in a period is iid across periods and denoted by integer-valued random variable D. If the demand is at least the number of idle slots, then all idle slots initiate a new rental agreement, and excess demand is lost. If instead there are strictly more idle slots than the demand, then each of the idle slots is equally likely to initiate an agreement.

The system of I inventory slots evolves as a complicated Markov chain with a multidimensional state space. We refer to this system as the *Grand Markov Chain (GMC)*. The GMC suffers from the curse of dimensionality in that the state space explodes combinatorially with the term T and the number I of inventory slots. For problems of realistic size, it thus becomes intractable even to compute the stationary distribution, much less perform price optimization (see Appendix D for the GMC and the state space explosion).

To overcome this difficulty, we propose an intuitive approximation motivated by the observation that each slot's rental state evolves independently while on an active rental agreement; that is, the dependence among inventory slots is confined to the states where these slots are idle. Our approach approximates the agreement initiation probability in a tractable way, and it is highly accurate (Appendix E.2). Conveniently, the approximation also allows us to optimize prices using our existing results for a single-slot system.

6.1. Independence Approximation for Agreement Initiation Probability

Our approximation assumes that the stationary probabilities for each slot's rental state are given by those found in Lemma 1 for some suitable agreement initiation probability to be determined, which we denote by $\alpha^{\mathcal{I}}(I,\gamma)$. We again deploy a fixed-utilization approach in inner maximization step, so when we optimize, we will consider a range of γ values, for each one restricting our attention to prices that will achieve $\pi_T(\boldsymbol{p};\alpha^{\mathcal{I}}(I,\gamma)) = \gamma$.

By symmetry, $\alpha^{\mathcal{I}}(I,\gamma)$ is the same for all the I slots, so wlog, we take the perspective of an arbitrary inventory slot, say slot 1. Suppose in a period that there are $m \geq 1$ idle inventory slots (one of which is slot 1) and that the demand is realized as d. Conditional on this event, the probability that slot 1 will be rented in this period is $\min\{d/m, 1\}$. To determine $\alpha^{\mathcal{I}}(I,\gamma)$, we take expectation over the demand and the number of idle slots.

Supposing that each inventory slot's state is realized independently, each slot is idle wp γ independent of the other slots. Then, the number of other idle slots (in addition to slot 1) is binomial with $I-1$ trials and success probability γ. Since $\alpha^{\mathcal{I}}(I,\gamma)$ is the probability that slot 1 will initiate an agreement if idle, and taking expectation of $\min\{d/m, 1\}$ over the demand distribution, we have

$$\begin{aligned}
\alpha^{\mathcal{I}}(I,\gamma) &= \mathrm{P}(\text{slot 1 initiates agreement} \mid \text{slot 1 is idle}) \\
&= \sum_{m=1}^{I} \mathrm{P}(\text{slot 1 initiates agreement} \mid \text{slot 1 is idle, } m \text{ idle slots})\mathrm{P}(m \text{ idle slots} \mid \text{slot 1 is idle}) \\
&= \sum_{m=1}^{I} \Big(\sum_{d=1}^{m-1} (d/m)\mathrm{P}(D=d) + \mathrm{P}(D \geq m) \Big) \binom{I-1}{m-1} \gamma^{m-1}(1-\gamma)^{I-m}. \qquad (21)
\end{aligned}$$

The long-run agreement initiation probability is approximated by (21), which can be easily calculated for a given demand distribution.

One important question remains, namely whether the idleness γ is *consistent* with the agreement initiation probability $\alpha^{\mathcal{I}}(I,\gamma)$ for a given number I of inventory slots. In words, do feasible prices exist that will lead to an idle probability of γ, given an agreement initiation probability of $\alpha^{\mathcal{I}}(I,\gamma)$ calculated from (21)? We already have the tools to answer this question; for $\alpha^{\mathcal{I}}(I,\gamma)$, we can use Lemma 3 to determine the interval $[\gamma_T^l(\alpha^{\mathcal{I}}(I,\gamma)), \gamma_T^u(\alpha^{\mathcal{I}}(I,\gamma))]$ of achievable γ values. If $\gamma \in [\gamma_T^l(\alpha^{\mathcal{I}}(I,\gamma)), \gamma_T^u(\alpha^{\mathcal{I}}(I,\gamma))]$, then we say that γ is *consistent* with $\alpha^{\mathcal{I}}(I,\gamma)$ (and also with I); otherwise, the idle probability γ cannot be achieved with I inventory slots. Finally, note that we derived (21) under a fixed utilization γ. In Appendix E.1, we provide the corresponding derivation under an arbitrary price path. In Appendix E.2, we validate our approximation on intermediate-size problems permitting the calculation of GMC's stationary distribution, and we show that it is highly accurate.

6.2. The Approximate Joint Buyout Pricing and Inventory Problem

In addition to the profit rate for an individual slot, an RTO firm must also consider inventory holding costs, since the firm retains ownership of the inventory until and unless a renter accepts the buyout price or owns the item through payoff. We denote by w the holding cost per period per inventory slot; so, for a system with I inventory slots, the total inventory holding cost rate is wI.

The firm faces a familiar tradeoff: more inventory slots will lead to fewer lost demand but more inventory holding cost. Under the independence approximation, an RTO firm faces the following *joint buyout pricing and inventory problem*:

$$\max_{I,\gamma,\boldsymbol{p}:p_k\in[s,ks]} \left\{ IR_T(\boldsymbol{p};\alpha^{\mathcal{I}}(I,\gamma)) - wI : \gamma \in [\gamma_T^l(\alpha^{\mathcal{I}}(I,\gamma)), \gamma_T^u(\alpha^{\mathcal{I}}(I,\gamma))], \pi_T(\boldsymbol{p};\alpha^{\mathcal{I}}(I,\gamma)) = \gamma \right\}. \qquad (22)$$

The objective function above is the firm's approximate aggregate profit rate, net of inventory holding cost rate. Despite the benefits of the independence approximation, (22) is still a challenging problem. Thankfully, it admits a nested structure to which we can apply our DP approach. Specifically, by nesting and Proposition 1, (22) is equivalent to

$$\max_I \left\{ I \max_\gamma \left\{ \alpha^{\mathcal{I}}(I,\gamma)\gamma \max_{\boldsymbol{p}: p_k\in[s,ks]} \{R_T^a(\boldsymbol{p})\colon \pi_T(\boldsymbol{p};\alpha^{\mathcal{I}}(I,\gamma))=\gamma\}\colon \gamma\in[\gamma_T^l(\alpha^{\mathcal{I}}(I,\gamma)),\gamma_T^u(\alpha^{\mathcal{I}}(I,\gamma))]\right\} - wI\right\}. \tag{23}$$

The innermost maximization above can be solved efficiently with the DP developed in §5, in particular via the DPP algorithm presented in Appendix A.2. A suitable upper bound can be found on I, and γ is bounded between 0 and 1, so the outer optimizations can be performed with a manageable number of iterations. This suggests a tractable approach for solving the joint buyout pricing and inventory problem to optimality (subject only to discretization error for γ and $\boldsymbol{p}$); we provide full implementation details in Appendix A.2. A huge advantage of the approximation is that the computation time does not explode with the number of inventory slots. This allows us to solve realistically-sized problems efficiently.

We briefly discuss the computational efficiency of our DPP algorithm for solving the RTO buyout pricing problem in (6) (i.e., joint pricing and utilization optimization) and joint pricing and inventory problem in (23) (see Appendix A.2 for details). The worst-case running time for the DPP algorithm (corresponding to a single slot with a fixed utilization) increases proportional to T^2; the running time for the RTO buyout pricing problem optimization increases proportional to T^2 and the number of γ values considered; the running time for joint pricing and inventory optimization increases proportional to T^2 and to the numbers of γ and I values considered. Moreover, our approach is highly conducive to parallelization. That is, the DPs for different pairs of γ and I can be solved in parallel on different processors on a multi-core machine. Indeed, we implemented most of the computations in this paper to leverage this parallelism, solving many DPs at once. This potential for parallel processing also implies that our approach could scale well; for instance, for a large problem where many different inventory levels must be considered, it is possible to merely assign more processors (or multiple cloud servers) to the task of getting a solution in comparable time to a smaller problem with fewer processors. Alternatively, for large problems, a coarser grid could be used for the inventory slots (e.g., increments of 5, 10, etc.).

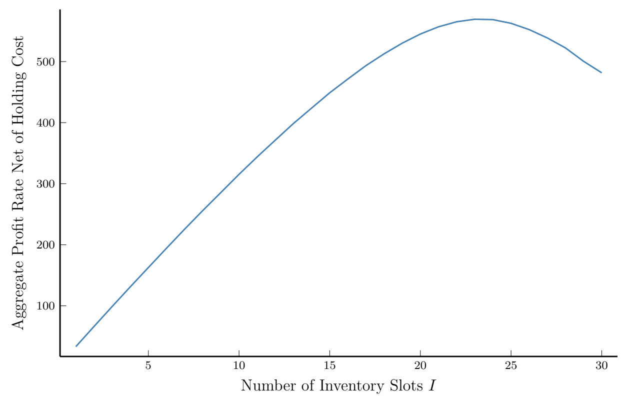
6.3. Case Study

In this subsection, using parameters calibrated to real-life RTO agreements (provided to us by our collaborating RTO firm), we present the solution to the joint buyout pricing and inventory problem. In addition, we shed light on two related reduced problems; an *inventory-only* optimization where inventory is optimized for a fixed price path and its counterpart *price-only* optimization in which inventory is fixed and prices are optimized. Across all our calculations, $c = \$300$, and $\rho = 0.92$. We consider iid Poisson distribution of mean μ for D, i.e., $\mathrm{P}(D = d) = \mu^d e^{-\mu}/(d!)$, and as in §5.1, exponentially distributed valuations with mean \$120.

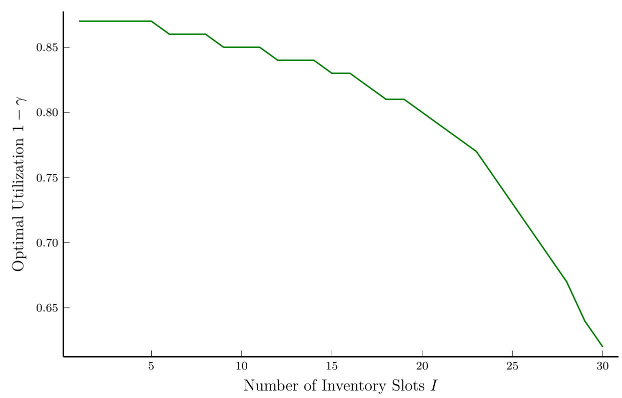
For the joint optimization problem (23), we consider an RTO agreement of term $T = 20$ with monthly rental fee payment schedule. Figure 5(a) plots the optimal aggregate profit rate net of holding cost rate wrt the number of slots; the optimal number of slots is 23. Starting from a single slot, as I increases, initially, the increase in aggregate profit rate made from satisfying more demand for rental agreements outweighs the increase in holding cost rate, and the firm would benefit from increasing the number of slots. However, the marginal benefit of an additional inventory slot is decreasing; more inventory means more slots competing for the same demand, which leads to lower utilization (e.g., Figure 5(b)) and thus less incremental revenue per period from the additional slot. Eventually, the marginal benefit drops below the (constant) marginal holding cost rate w, and at this point the inventory is optimized; see Figure 5(a).

The optimal buyout price path corresponding to $I = 23$ appears in Figure 5(c) in addition to the optimal price paths for values of $I \in \{1, 10, 20\}$. Across all values of I, the optimal paths tend to decrease over time. Moreover, the optimal prices tend to increase in I; as I increases, utilization decreases and a buyout would more likely lead to idleness. For this reason, when I is large, the firm is willing to tolerate the idleness risk only at high buyout prices; otherwise, instead of selling, it would rather collect rental profits.

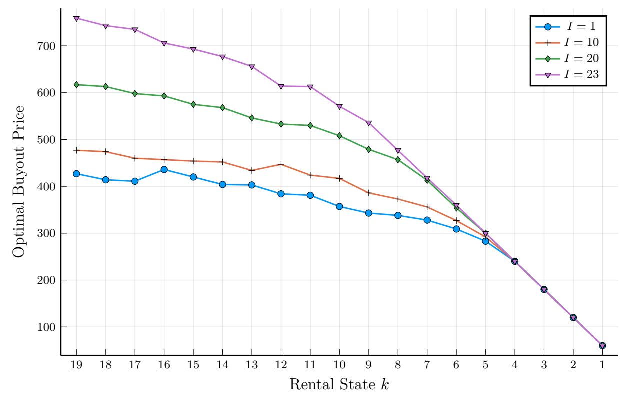
In practice, inventory decisions are often made by the operations group whereas the revenue management group is tasked with determining the buyout prices. Due to different goals and lack of communication, these groups may make decisions in silos without cooperating with each other. For example, the revenue manager might only be interested in optimizing prices for the current inventory level. To illustrate the value of price-only optimization, we study an RTO agreement with term $T = 60$ (corresponding to weekly rental fee payment schedule) and compare the aggregate profit rate net of holding cost made under our prices with that obtained using industry buyout prices, which are determined from the collaborating firm's price formula. Unfortunately, we do not have permission to share the exact industry prices. However, it is a common practice in the RTO business to reduce the rate of price reduction as the rental progresses. That is, in practice prices reduce more steeply early in the agreement than later (see Appendix A.1 for more). By contrast, the optimal prices from our algorithm exhibit steeper drops later in the agreement, similar to the plots in Figure 5(c). Figure 6 illustrates aggregate profit rates under optimal and the industry buyout prices for a range of I values. Overall, the improvements are sizable with an average of 20% increase in the aggregate profit rate across all values of $I \in [1 : 40]$, pointing to the value of the price-only optimization.



(a) Profit rate net of holding cost



(b) Utilization



(c) Optimal buyout prices

Figure 5 Joint pricing and inventory optimization for $T = 20, s = 60, w = 20$ and $\mu = 2$.

On the other hand, an inventory manager without authority over prices might consider optimizing only inventory for the existing price path (see Appendix E.3 for technical details). We observe in Figure 6 that when prices and inventory are jointly optimized, the optimal inventory level is 17, higher than 15 corresponding to when only inventory is optimized. Moreover, the aggregate profit rate when prices and inventory are jointly optimized is 22% higher than when only inventory is optimized. This is the value of joint optimization over and above inventory-only optimization, and its magnitude underscores the importance of cooperation between the operations and revenue management groups in determining inventory and buyout prices.

7. Conclusion

In this paper, we study the *RTO buyout pricing problem*, in which an RTO firm must determine a *path* of buyout prices to offer to its renters. In addition, we determine the optimal inventory level of rental items to maximize the firm's profit. The firm collects a periodic rental fee from a renter until he either returns the product or owns it outright, in the latter case by accepting one of the buyout price offers or by completing the designated term of the rental agreement. This problem underlies an \$8B+ industry, but it has received negligible attention to date in the operations literature.

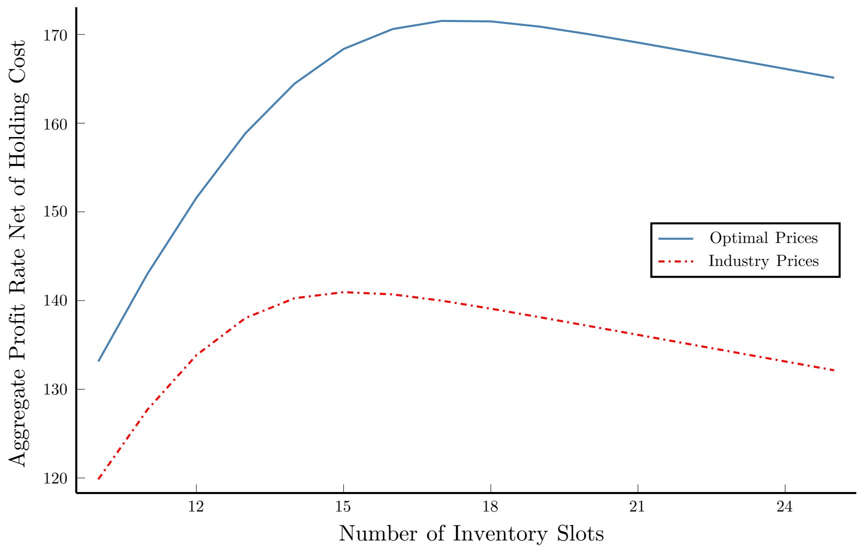


Figure 6 **Aggregate profit rates for optimal and industry prices.** $T = 60, s = 15, w = 1$ **and** $\mu = 1$**.**

We comprehensively address the RTO buyout pricing problem, also incorporating the closely related inventory optimization. Our work both enhances the academic understanding of the RTO business and has immediate practical value for RTO firms. Specifically, (i) we propose a novel Markov chain model of RTO agreements, (ii) we formulate the profit optimization problem under this model, (iii) we derive the exact solution for a special case and (iv) despite the general problem's high dimension and the profit's nonconcavity, we devise an efficient optimization algorithm to solve this problem.

We prove for the special case and observe in the general problem that the optimal price path is concave decreasing, exhibiting gradual price drops early in the agreement and steep price drops later in the agreement. Our results show that to optimally balance rental and sales revenues, RTO firms should instead keep prices relatively steady early in agreements; this way, any early buyout occurs at a high enough price to justify giving up the recurring rental revenue. In addition, the optimal price path shifts upward as the firm's target utilization increases or the agreement initiation probability—a proxy for the relative magnitude of the rental demand wrt inventory level—decreases.

We then extend our formulation to include optimization of rental inventory in the RTO system, and we show that our algorithm can be easily applied to jointly optimize the buyout price path and inventory level. The optimal prices and inventory levels exhibit substantial interdependence. Specifically, we observe that increasing the rental inventory shifts the optimal price path upward. Higher inventory levels lead to increased idleness, so as inventory increases, the firm optimally should only forgo an agreement's recurring rental revenue in exchange for a higher price.

Finally, in a realistic case study guided by our interactions with an RTO firm, we demonstrate the magnitude of profit improvements that our optimized prices provide relative to industry prices. Across different fixed inventory levels, our prices achieve a 20% profit improvement on average. Furthermore, due to the aforementioned interdependence between optimal prices and inventory levels, we find that jointly optimizing prices and inventory results in a substantial profit improvement of 22% over solely optimizing inventory under industry prices. The magnitude of this improvement highlights the importance of cooperation between the revenue management and operations groups of RTO firms in making pricing and inventory decisions.

To illustrate the practical implementability of our approach, we have also created a software tool with a graphical user interface to our algorithm and procedures. This software tool enables price-only, inventory-only, and joint pricing and inventory optimizations. Because our methodology (and hence the tool) requires only a small number of parameters, a manager can collect or estimate these parameters (e.g., agreement term, demand per period, etc.), input them directly into the tool, and obtain a solution at the click of a button without needing to implement a data interchange with an ERP system. For a realistically sized problem, the tool returns a solution in a few minutes (see Appendix A.2 for details and a screenshot).

Our work and its potential profit impact reveal the scale of opportunity for improving the academic understanding and practical performance of RTO firms, and the context is ripe for additional study. For instance, while our work addresses the RTO buyout pricing problem for a single product, RTO firms must also determine the set of products to offer. Jointly optimizing the assortment and pricing for different RTO products represents an exciting direction for future research. However, such a study could prove quite difficult since it would combine the combinatorial nature of assortment optimization with the already challenging and multidimensional problem of determining RTO buyout price paths. Another direction would be to consider a pricing game played by multiple RTO firms; this would highlight the role of competition but would likely also be difficult to analyze. Overall, we hope that our approach and findings will draw attention to the RTO business and inspire further research.

References

Aaron's (2022) *Aaron's Reports Fourth Quarter and Full Year 2021 Financial Results; Provides 2022 Outlook; and Announces Agreement to Acquire BrandsMart U.S.A.* https://investor.aarons.com/news-and-events/press-releases/press-release-details/2022/Aarons-Reports-Fourth-Quarter-and-Full-Year-2021-Financial-Results-Provides-2022-Outlook-and-Announces-Agreement-to-Acquire-BrandsMart-U.S.A/default.aspx, Accessed Jan. 2023.

Aaron's (2023) *Do I have the option of buying the product before my lease term ends?* https://www.aarons.com/FAQ, Accessed Apr. 2023.

Altug MS, Ceryan O (2022) Optimal dynamic allocation of rental and sales inventory for fashion apparel products. *IISE Transactions* 54(6):603–617.

Anderson MH, Jaggia S (2012) Return, purchase, or skip? Outcome, duration, and consumer behavior in the rent-to-own market. *Empirical Economics* 43(1):313–334.

Anderson MH, Sibdari S (2012) Investment decisions in the rent-to-own industry in the absence of inventory. *Journal of the Operational Research Society* 63(1):89–106.

APRO (2015) *APRO Annual Industry Statistical Survey and Trend Analysis.* https://www.rtohq.org/wp-content/uploads/2019/06/APRO-Flipbook-About-Us.pdf, Accessed Jan. 2023.

Armaghan M, Çakanyıldırım M, Frazelle AE, Glasky D, Rajamani D (2023) Buy now or keep renting? A modular estimation framework for renter decisions in the rent-to-own business. *Working Paper, Available on SSRN 4360629* .

Besbes O, Elmachtoub AN, Sun Y (2022) Technical Note—Static pricing: Universal guarantees for reusable resources. *Operations Research* 70(2):1143–1152.

Bulow JI (1982) Durable-goods monopolists. *Journal of Political Economy* 90(2):314–332.

Coase RH (1972) Durability and monopoly. *The Journal of Law and Economics* 15(1):143–149.

Firouz M, Li L, Keskin BB (2022) Managing equipment rentals: Unreliable fleet, impatient customers, and finite commitment capacity. *Production and Operations Management* 31(11):3963–3981.

Gans N, Savin S (2007) Pricing and capacity rationing for rentals with uncertain durations. *Management Science* 53(3):390–407.

Gilbert SM, Randhawa RS, Sun H (2014) Optimal per-use rentals and sales of durable products and their distinct roles in price discrimination. *Production and Operations Management* 23(3):393–404.

Govindarajan A, Jasin S, Wei L, Zhang Y (2020) Lead time independent joint inventory and pricing controls for lost sales system with probabilistic service level constraints. *Working Paper, University of Michigan* .

Guajardo JA (2019) How do usage and payment behavior interact in rent-to-own business models? Evidence from developing economies. *Production and Operations Management* 28(11):2808–2822.

Huang S, Yang Y, Anderson K (2001) A theory of finitely durable goods monopoly with used-goods market and transaction costs. *Management Science* 47(11):1515–1532.

Jaggia S, Roche H, Anderson MH (2019) Rent-to-own pricing: Theory and empirical evidence. *Journal of Consumer Affairs* 53(3):1025–1055.

Jalili M, Pangburn MS (2020) Pricing joint sales and rentals: When are purchase conversion discounts optimal? *Production and Operations Management* 29(12):2679–2695.

Jia H, Shi C, Shen S (2022) Online learning and pricing for service systems with reusable resources. *Operations Research* (Articles in Advance).

Lei Y, Jasin S (2020) Real-time dynamic pricing for revenue management with reusable resources, advance reservation, and deterministic service time requirements. *Operations Research* 68(3):676–685.

Li J, Granados N, Netessine S (2014) Are consumers strategic? Structural estimation from the air-travel industry. *Management Science* 60(9):2114–2137.

Liu Y, Zhang D (2023) Intra-consumer price discrimination with credit refund policies. *Management Science* .

May R (2017) The shape of rent-to-own by the numbers. *RTOHQ: The Magazine* 37–40.

Moon K, Bimpikis K, Mendelson H (2018) Randomized markdowns and online monitoring. *Management Science* 64(3):1271–1290.

Nageswaran L, Cho SH, Scheller-Wolf A (2020) Consumer return policies in omnichannel operations. *Management Science* 66(12):5558–5575.

Özer Ö, Phillips R (2012) *The Oxford handbook of pricing management* (OUP Oxford).

Progressive Leasing (2023) *What are my early purchase options?* https://help.progleasing.com/hc/en-us/articles/360051343391-What-are-my-early-purchase-options-, Accessed May 2023.

Rent-A-Center (2022) *Rent-A-Center, Inc. Reports Fourth Quarter and Full Year 2021 Results*. https://investor.rentacenter.com/news-releases/news-release-details/rent-center-inc-reports-fourth-quarter-and-full-year-2021, Accessed Jan. 2023.

Rent-A-Center (2023) *What are my ownership options? Can I own an item early?* https://www.rentacenter.com/en/faqs/ownership, Accessed Apr. 2023.

Rusmevichientong P, Sumida M, Topaloglu H (2020) Dynamic assortment optimization for reusable products with random usage durations. *Management Science* 66(7):2820–2844.

Sethi S (2018) *Optimal Control Theory: Applications to Management Science and Economics* (Springer International Publishing), ISBN 9783319982366.

Slaugh VW, Biller B, Tayur SR (2016) Managing rentals with usage-based loss. *Manufacturing & Service Operations Management* 18(3):429–444.

Sundaram RK (1996) *A First Course in Optimization Theory* (Cambridge University Press).

Tainiter M (1964) Some stochastic inventory models for rental situations. *Management Science* 11(2):316–326.

van Brunt B (2003) *The Calculus of Variations* (Springer New York), ISBN 9780387402475.

Wagner L, Martínez-de Albéniz V (2020) Pricing and assortment strategies with product exchanges. *Operations Research* 68(2):453–466.

Yılmaz Ö, Ferguson M, Pekgün P, Shang G (2022) Strategic behavior for hotel standby upgrade programs: Empirical evidence and pricing implications. *Journal of Operations Management* 68(6-7):675–701.

Zhang L, Zhang Y, Chutani A (2022) Riding the wave of fashion rental: The role of power structures and green advertising. *Transportation Research Part E: Logistics and Transportation Review* 168:102946.

Appendix

A. Real-life Buyout Prices and Computation of Optimal Prices

We first provide more details regarding current buyout prices in practice and then provide the details of our algorithm and procedures for buyout price optimization.

A.1. RTO Buyout Prices in Practice

Most RTO firms offer a same-as-cash interval early on in the rental. During this interval, the buyout price in a given period is determined by subtracting the cumulative rental fees paid from a known starting point. Thus, the buyout price reduces by s in each period during the same-as-cash interval. After this interval, as APRO (2015) states, "[t]he early-purchase option can reduce the total rent-to-own price by 50% of remaining payments," implying a reduction in the price of $s/2$ each period. In Figure 7, we show pricing descriptions from the websites of two $1B+ RTO firms, Aaron's (Aaron's 2023) and Rent-A-Center (Rent-A-Center 2023), as well as Progressive Leasing (Progressive Leasing 2023) who partners with Best Buy, Big Lots!, and Lowe's.

Do I have the option of buying the product before my lease term ends?

Yes. You can purchase the product at any time. If your ownership plan is longer than 6 months, you can take advantage of Aaron's same as cash option. For those new agreements with a payment option longer than 6 months, if you payout your merchandise within the applicable same as cash period, you will pay the cash price, plus tax and applicable fees (if any). The same as cash period varies by location but is generally 120 days. **For California residents** the same as cash option is 90 days for all rental purchase agreements.

In addition, after the same as cash option expires, you can purchase the merchandise for more than the cash price but less than the total of remaining lease payments, as described in your lease agreement. This **early purchase option** amount varies by state and is explained in the lease agreement.

To help you save money, Progressive Leasing offers two different early purchase options with each lease-to-own agreement. Our standard lease-to-own agreement is 12-months and offers easy-to-budget payments that align with the dates you receive your paycheck. Paying off your lease-to-own agreement early will help reduce your 12-month lease-to-own total and allow you to own your merchandise sooner.

90-Day (3 months in CA)

You can buy out your lease-to-own agreement within the first 90-days. This amount includes the cash price, plus the lease-to-own cost for the first 90-days. Taking advantage of the 90-day purchase option will save you the most money! You will n.ed to call us to exercise this option. Once this option is set up with one of our service representatives, you may make payments online in MyAccount.

Early Buyout

The Early Buyout option offers you the most flexibility and is available to you throughout the 12-month term of your lease-to-own agreement. The Early Buyout is a percentage of the unpaid 12-month lease-to-own total. Taking advantage of this purchase option will save you money on your 12-month lease-to-own total.

Ownership Options

What are my ownership options? Can I own an item early? –

YES. At Rent-A-Center, you always have an early purchase option that will save you money compared to paying the total cost to own in your lease. You have the freedom to choose the option that works best for your budget. When you're ready to own, simply pay the early purchase price and it's yours! *NOTE: Early Purchase Option requires a payment in addition to regular rental payments.

- 6 Months Same as Cash**: Own it in the first 6 months and you'll pay only the cash price. 6 Months Same as Cash not available in HI, NJ, NY, WV and select franchised locations. Such locations may offer 4-6 Months Same as Cash; 4 Months Same As Cash valid only on new agreements, and period ends 123 days after start of the new agreement.

- Standard Early Purchase : After your same as cash period has ended, you still save on the remaining total if you choose to own your item early. Most Rent-A-Center locations offer a 50 percent savings on the remaining total. This can vary by state, so be sure to review your agreement.

Figure 7 Buyout Price Descriptions: Top-left, Aaron's; Top-right, Progressive Leasing; Bottom, Rent-A-Center

For all three firms in Figure 7, in the initial interval of rental, each rental fee payment of s reduces the buyout price by exactly s. After this interval, at Rent-A-Center, the price is equal to half of the remaining rental fees required for completing the term, i.e., the buyout price in a particular period is $(sT - \text{total rental payments by this period})/2$. As the renter pays s in each period, the price drops by $s/2$ from one period to the next, which is more gradual than the earlier price drop s during the same-as-cash interval. Similarly, the buyout price at Progressive Leasing after the initial 90-day interval is "a percentage of the unpaid 12-month lease-to-own total," and Aaron's states that "after the same as cash option expires, you can purchase the merchandise for more than the cash price but less than the total of remaining lease payments."

A.2. Our Algorithm and Interface

We use the Dynamic Program for Pricing (DPP) algorithm below to solve the DP in (18) for a given pair of agreement initiation probability and utilization. For each $k \in [2:T-1]$, we first form a price-state grid using prices in $[s, ks]$ with increments ϵ_p and DP states being one of the n_a equidistant points belonging to the interval obtained by combining forward and backward constructions in Lemma 6. We then evaluate the value function v_k on the price-state grid to find the best price.

```
Function SolveDP(α,γ)
  Data: T, s, c, F_k for k ≥ 2, ρ, price increment ε_p > 0 that divides s, number of states n_a ≥ 2
  /* Construct sets of states and prices for each rental state k                      */
  for k ∈ [2:T−1] do
    a_k(lower) ← max{a^l_k, ã^l_k}; ā_k ← min{a^u_k, ã^u_k} // Upper and lower bounds on DP states
    ε_{a_k} ← (ā_k − a_k(lower))/n_a // Determine state increment
    𝐚_k ← {a_k(lower), a_k(lower) + ε_{a_k}, ..., ā_k − ε_{a_k}, ā_k} // Construct set of DP states for rental state k
    𝐩_k ← {s, s + ε_p, ..., ks − ε_p, ks} // Construct set of prices for rental state k
  end for
  /* Iteratively compute optimal policy                                                */
  v_1 ← ρ(s − c); a_1 ← 0 // Terminal values
  for k ∈ [2:T−1] do
    for a_k ∈ 𝐚_k do
      for p_k ∈ 𝐩_k do
        if (a_k/F_k(p_k) − 1)/ρ ∈ [a_k(lower), ā_k] then
          â_{k−1}(a_k, p_k) ← argmin_{a_{k−1} ∈ 𝐚_{k−1}} |a_{k−1} − (a_k/F_k(p_k) − 1)/ρ| // Find closest
             next state
          v̂_k(a_k, p_k) ← ρF̄_k(p_k)(p_k − c) + ρF_k(p_k)(s + v̂_{k−1}(â_{k−1}(a_k, p_k))) // Value for
             price-state pair
        else
          v̂_k(a_k, p_k) ← −∞
        end if
      end for
      v_k(a_k) ← max_{p_k ∈ 𝐩_k} v̂_k(a_k, p_k) // Optimal value function at this DP state
      p̂_k(a_k) ← argmax_{p_k ∈ 𝐩_k} v̂_k(a_k, p_k) // Optimal price at this DP state
    end for
  end for
  /* Retrieve optimal prices                                                           */
  a^{α,γ}_{T−1} ← (1 − γ − αγ)/(αγρ) // Known starting state
  p^{α,γ}_{T−1} ← p̂_{T−1}(a^{α,γ}_{T−1}) // Optimal price for T − 1 based on known starting state
  for k ∈ [(T−1) : −1 : 2] do
    a^{α,γ}_{k−1} ← â_{k−1}(a^{α,γ}_k, p^{α,γ}_k) // Determine incoming state for optimal policy
    p^{α,γ}_{k−1} ← p̂_{k−1}(a^{α,γ}_{k−1}) // Determine optimal price for rental state k
  end for
  Result: Optimal prices 𝐩^{α,γ} = (p^{α,γ}_1, ..., p^{α,γ}_{T−1})
```

DPP algorithm: Solve DP in (18) for fixed agreement initiation probability α and utilization $1-\gamma$

Recall that for $k=1$, $a_1=0$, $p_1=s$ and $v_1=\rho(s-c)$. Hence, no calculation is required for $k=1$. For $k\geq 2$, the number of price points increases linearly in $k-1$, i.e., if the number of price points for $k=2$ is n, then for $k=3,4,...T-1$, the number of price points respectively are $2n, 3n, ..., (T-2)n$. The number of DP state points, however, is fixed at n_a. So, the price-state grid consists of $n_a(k-1)n$ points. For each price-state combination, we first perform a fixed number of calculations to evaluate v_k each of which takes, say t_c, to be completed. Then, finding the price with the maximum value function takes $n_a(k-1)n-1$ comparison steps each of which takes t_m. After this procedure is followed for each $k \in [2:T-1]$, we need to retrieve optimal prices for each rental state $k \in [2:T-1]$. With t_r denoting the elapsed time for each price retrieval, the total retrieval time is $(T-2)t_r$. In conclusion, the total computation time for the DPP algorithm is

$$(T-2)t_r + \sum_{k=2}^{T-1} \big(n_a(k-1)n\big)t_c + \big(n_a(k-1)n-1\big)t_m = (T-2)(t_r - t_m) + \sum_{k=2}^{T-1} \big(n_a(k-1)n\big)(t_c + t_m).$$

For a given ϵ_p and n_a, the computation time above grows proportional to T^2.

To solve the RTO buyout pricing problem in (6) (i.e., to jointly optimize prices and utilization), we use the Single-Slot Search Procedure below. This procedure involves repetitions of the DPP algorithm over a set of achievable utilizations specified by Lemma 3. As such, its computation time grows proportional to T^2 and the number of utilizations considered.

Data: T, s, c, F_k for $k \geq 2$, ρ, α, set $\boldsymbol{\gamma}$ of γ values
for $\gamma \in \boldsymbol{\gamma}$ **do**
  $\boldsymbol{p}^\gamma \leftarrow$ `SolveDP`(α, γ) // Obtain optimal price path for a given γ
  $V^\gamma \leftarrow R_T(\boldsymbol{p}^\gamma; \alpha)$ // Compute associated profit rate
end for
$V^* \leftarrow \max_{\gamma \in \boldsymbol{\gamma}} V^\gamma$ // Obtain globally optimal profit rate
$\boldsymbol{p}^* \leftarrow \arg\max_{\gamma \in \boldsymbol{\gamma}} V^\gamma$ // Obtain globally optimal price paths
Result: $\boldsymbol{p}^*, V^*$

Single-Slot Search Procedure: Buyout price and utilization optimization

Finally, we use the Multi-Slot Search Procedure below to solve (22) and to jointly optimize prices and inventory. For each given pair of the inventory level and utilization, this procedure first checks if the given utilization yields a valid solution to (21) in α for the given inventory level. If so, it then invokes the DPP algorithm for that utilization and α pair. Therefore, its computation time grows proportional to T^2 as well as the number of inventory level and utilization pairs considered.

Data: T, s, c, F_k for $k \geq 2$, ρ, holding cost w, set $\boldsymbol{\gamma}$ of γ values, set $\boldsymbol{I}$ of numbers of inventory slots
for $(I, \gamma) \in \boldsymbol{I} \times \boldsymbol{\gamma}$ **do**
  /* Check whether γ is achievable for $\alpha = \alpha^{\mathcal{I}}(I, \gamma)$ calculated from (21) */;
  if $\gamma \in [\gamma_T^l(\alpha^{\mathcal{I}}(I,\gamma)), \gamma_T^u(\alpha^{\mathcal{I}}(I,\gamma))]$ **then**
    $\boldsymbol{p}^{I,\gamma} \leftarrow$ `SolveDP`$(\alpha^{\mathcal{I}}(I,\gamma), \gamma)$ // Obtain optimal prices via DPP algorithm
    $V^{I,\gamma} \leftarrow I\big(R_T(\boldsymbol{p}^{I,\gamma}; \alpha^{\mathcal{I}}(I,\gamma)) - w\big)$ // Compute the corresponding profit rate net of holding cost
  else
    $V^{I,\gamma} \leftarrow -\infty$ // Rule out (I,γ) pair if γ not achievable with $\alpha = \alpha^{\mathcal{I}}(I,\gamma)$
  end if
end for
$I^*, \gamma^* \leftarrow \arg\max_{I \in \boldsymbol{I}, \gamma \in \boldsymbol{\gamma}} V^{I,\gamma}$ // Retrieve optimal I and γ
$\boldsymbol{p}^* \leftarrow \boldsymbol{p}^{I^*,\gamma^*}$; $V^* \leftarrow V^{I^*,\gamma^*}$ // Retrieve optimal prices and profit rate net of holding cost
Result: $I^*, \gamma^*, \boldsymbol{p}^*, V^*$

Multi-Slot Search Procedure: Joint buyout pricing and inventory optimization

We embed our algorithm and procedures in a simple graphical user interface (GUI) for ease of application. The user only needs to input a few parameters and select the tab corresponding to the desired problem to solve (Inventory and Pricing, Inventory Only, or Pricing Only). For pricing-only optimization of a 20-period agreement with a rental fee of \$60 per period, price increments of \$1, and 100 DP grid points for each period's DP state, the tool returns a solution in roughly 5 minutes on a multi-core desktop computer using 20 processing threads. A screenshot of the GUI with this solution appears in Figure 8.

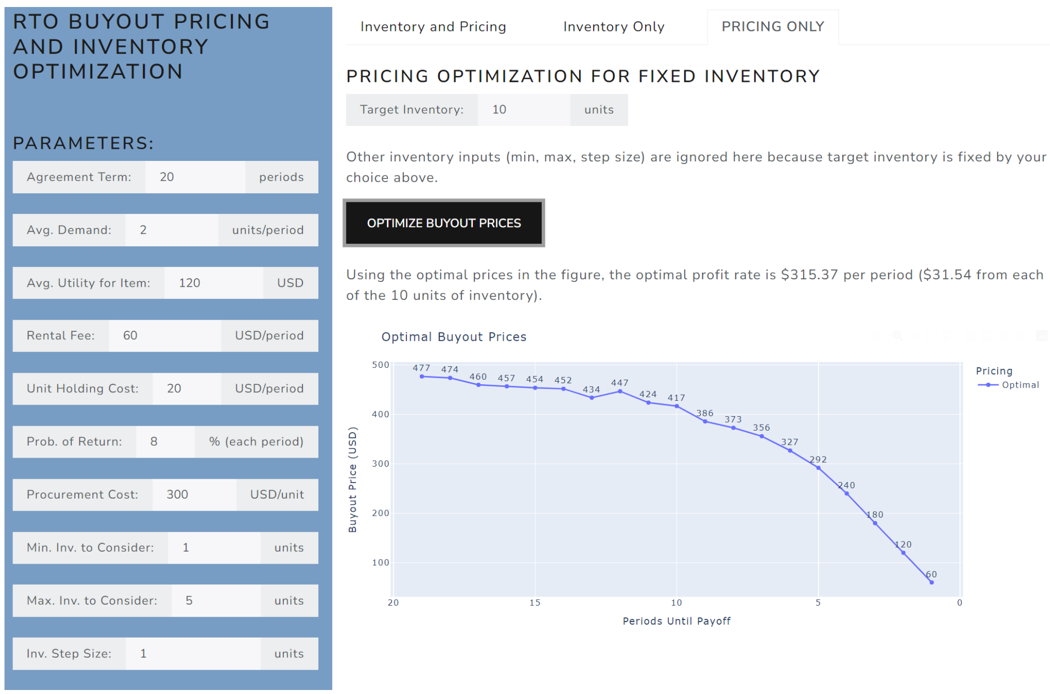


Figure 8 GUI to our algorithm and procedures with the Pricing Only tab active.

B. Proofs

Proof of Lemma 1: The stationary probabilities of the RTO MC in Fig 1 are the solution to its balance equations:

$$\begin{cases} \pi_T(\boldsymbol{p};\alpha) = (1-\alpha)\pi_T(\boldsymbol{p};\alpha) + \sum_{k=2}^{T-1} \pi_k(\boldsymbol{p};\alpha)\left(1-\rho+\rho\bar{F}_k(p_k)\right) + \pi_1(\boldsymbol{p};\alpha) \\ \pi_{T-1}(\boldsymbol{p};\alpha) = \alpha\pi_T(\boldsymbol{p};\alpha) \\ \pi_k(\boldsymbol{p};\alpha) = \rho F_{k+1}(p_{k+1})\pi_{k+1}(\boldsymbol{p};\alpha) \quad \text{for } k\in[1:T-2] \\ \sum_{k=1}^{T} \pi_k(\boldsymbol{p};\alpha) = 1. \end{cases}$$

Note that we do not include $\pi_k(\boldsymbol{p};\alpha) > 0$ for $k \in [1:T]$ above as they are implied by the last three equations. The second and third equations imply $\pi_k(\boldsymbol{p};\alpha) = \alpha\pi_T(\boldsymbol{p};\alpha)\rho^{T-k-1}\prod_{j=k+1}^{T-1} F_j(p_j)$ for $k\in[1:T-2]$, inserting which in the last equation yields

$$\pi_T(\boldsymbol{p};\alpha) = \left[1+\alpha+\sum_{i=2}^{T-1}\alpha\rho^{i-1}\prod_{j=T-i+1}^{T-1}F_j(p_j)\right]^{-1}. \tag{24}$$

Next, we show that the the first equation is redundant as it is implied by (24), the second equation and the third equation. Substituting π_T from (24), $\pi_{T-1}(\boldsymbol{p};\alpha)$ from the second equation and $\pi_k(\boldsymbol{p};\alpha)$'s from the third equation

$$\begin{aligned}
\pi_T(\boldsymbol{p};\alpha) &= (1-\alpha)\pi_T(\boldsymbol{p};\alpha) + \sum_{k=1}^{T-1} \pi_k(\boldsymbol{p};\alpha)\left(1-\rho+\rho\bar{F}_k(p_k)\right) \\
&= (1-\alpha)\pi_T(\boldsymbol{p};\alpha) + \sum_{k=2}^{T-1}\left(\alpha\pi_T(\boldsymbol{p};\alpha)\rho^{T-k-1}\prod_{j=k+1}^{T-1} F_j(p_j)\,(1-\rho F_k(p_k))\right) + \alpha\pi_T(\boldsymbol{p};\alpha)\rho^{T-2}\prod_{j=2}^{T} F_j(p_j) \\
&= (1-\alpha)\pi_T(\boldsymbol{p};\alpha) + \sum_{k=3}^{T-1}\left(\alpha\pi_T(\boldsymbol{p};\alpha)\rho^{T-k-1}\prod_{j=k+1}^{T-1} F_j(p_j)\,(1-\rho F_k(p_k))\right) \\
&\quad +\alpha\pi_T(\boldsymbol{p};\alpha)\rho^{T-3}\prod_{j=3}^{T-1} F_j(p_j)\,(1-\rho F_2(p_2)) + \alpha\pi_T(\boldsymbol{p};\alpha)\rho^{T-2}\prod_{j=2}^{T-1} F_j(p_j) \\
&= (1-\alpha)\pi_T(\boldsymbol{p};\alpha) + \sum_{k=3}^{T-1}\left(\alpha\pi_T(\boldsymbol{p};\alpha)\rho^{T-k-1}\prod_{j=k+1}^{T-1} F_j(p_j)\,(1-\rho F_k(p_k))\right) \\
&\quad +\alpha\pi_T(\boldsymbol{p};\alpha)\rho^{T-3}\prod_{j=3}^{T-1} F_j(p_j) - \alpha\pi_T(\boldsymbol{p};\alpha)\rho^{T-2}\prod_{j=2}^{T-1} F_j(p_j) + \alpha\pi_T(\boldsymbol{p};\alpha)\rho^{T-2}\prod_{j=2}^{T-1} F_j(p_j) \\
&= (1-\alpha)\pi_T(\boldsymbol{p};\alpha) + \sum_{k=3}^{T-1}\left(\alpha\pi_T(\boldsymbol{p};\alpha)\rho^{T-k-1}\prod_{j=k+1}^{T-1} F_j(p_j)\,(1-\rho F_k(p_k))\right) + \alpha\pi_T(\boldsymbol{p};\alpha)\rho^{T-3}\prod_{j=3}^{T-1} F_j(p_j). \quad (25)
\end{aligned}$$

By similar manipulations, we can increase the lower index of the summation above to $T-1$. That is, (25) is equivalent to $\pi_T(\boldsymbol{p};\alpha) = (1-\alpha)\pi_T(\boldsymbol{p};\alpha) + \alpha\pi_T(\boldsymbol{p};\alpha)\,(1-\rho F_{T-1}(p_{T-1})) + \alpha\rho\pi_T(\boldsymbol{p};\alpha)F_{T-1}(p_{T-1}) = \pi_T(\boldsymbol{p};\alpha)$. Therefore, the stationary probabilities specified by the last three equations of the system and (24) satisfy also the first equation of the system. □

Proof of Remark 1: For $T=4$, from Lemma 1 we have

$$\begin{aligned}
\pi_4(\boldsymbol{p};\alpha) &= \frac{1}{1+\alpha+\alpha\sum_{i=1}^{2}\rho^{3-i}\prod_{j=i+1}^{3}F_j(p_j)}; \quad \pi_3(\boldsymbol{p};\alpha) = \alpha\pi_4(\boldsymbol{p};\alpha); \\
\pi_2(\boldsymbol{p};\alpha) &= \alpha\pi_4(\boldsymbol{p};\alpha)\rho F_3(p_3); \quad \pi_1(\boldsymbol{p};\alpha) = \alpha\pi_4(\boldsymbol{p};\alpha)\rho^2\prod_{j=2}^{3}F_j(p_j).
\end{aligned}$$

To get the profit rate function R_4, we insert stationary probabilities above in (2) for $T=4$ and specialize F_k for $k\in[1:3]$ therein to exponential distribution of mean $\nu = 1/\mu$:

$$\begin{aligned}
R_4(\boldsymbol{p};\alpha) &= \Big(\alpha s + \alpha\rho\big((1-e^{-\mu p_3})s + e^{-\mu p_3}(p_3-c)\big) + \alpha\rho^2(1-e^{-\mu p_3})\big((1-e^{-\mu p_2})s + e^{-\mu p_2}(p_2-c)\big) \\
&\quad +\alpha\rho^3(1-e^{-\mu p_3})(1-e^{-\mu p_2})(s-c)\Big)\Big(1+\alpha+\alpha\rho(\rho(1-e^{-\mu p_3})(1-e^{-\mu p_2}) + 1 - e^{-\mu p_3})\Big)^{-1}.
\end{aligned}$$

Let $R_4^{ij}(\boldsymbol{p}) := \frac{\partial}{\partial p_i \partial p_j}R_4(\boldsymbol{p})$. Using Wolfram Mathematica application, we evaluate the cross derivative function $R_4^{32}(p_3,p_2)$ wrt p_3 and p_2, while the parameters are fixed; $s=300$, $c=300$, $\mu=1/120$, $\alpha=0.8$ and $\rho=0.78$. As an example, $R_4^{32}(500,400) = 3.57\times 10^{-7} > 0$, illustrating nonconcavity of R_4 in buyout prices. Recall that $p_2\in[300,600]$ and $p_3\in[100,900]$ constraints in (3). Over this range, $\min R_4^{22} \approxeq -10^{-4}$, $\min_{\boldsymbol{p}} R_4^{22} \approxeq 10^{-4}$, $\max_{\boldsymbol{p}} R_4^{22} \approxeq -10^{-6}$, $\min_{\boldsymbol{p}} R_4^{33} \approxeq -10^{-4}$, $\max_{\boldsymbol{p}} R_4^{33} \approxeq 10^{-6}$, $\min_{\boldsymbol{p}} R_4^{32} \approxeq -10^{-7}$ and $\max_{\boldsymbol{p}} R_4^{32} \approxeq 10^{-6}$. As such,

$|R_4^{32}(500,400)| \approxeq 10^{-1}\max_{\boldsymbol{p}}|R^{32}|$ and hence significant. Moreover, we can increase the magnitude of these derivatives by decreasing monetary values s, p_2, p_3, $1/\mu$ and c by the same scaling factor; for instance, we can make these magnitudes 1000 times larger if we use thousands of dollars as the monetary unit instead of dollars. The Wolfram Mathematica script we used to set up $R_4(\boldsymbol{p};\alpha)$ and evaluate $R_4^{32}(\boldsymbol{p};\alpha)$ wrt p_2 and p_3 appears in Appendix F. □

Proof of Lemma 2: For $T=3$, the firm only needs to decide on the buyout price p_2. For ease of reading, let p denote this price, and let $F_2=F$. Recall also that $c>s$. When $\rho=0$, the rental ends in period T without an opportunity to offer the item for sale and price it. Then buyout prices become irrelevant and the lemma holds trivially, so we consider $\rho>0$ in the rest of the proof. We first carry on our analysis using a generic valuation distribution F and later on specialize it to exponential.

From Lemma 1, the stationary probabilities for the RTO MC are

$$\pi_3(p;\alpha)=\frac{1}{1+\alpha+\alpha\rho F(p)},\pi_2(p;\alpha)=\frac{\alpha}{1+\alpha+\alpha\rho F(p)},\pi_1(p;\alpha)=\frac{\alpha\rho F(p)}{1+\alpha+\alpha\rho F(p)}.$$

The profit rate function in (2) for $T=3$ therefore is

$$R_3(p;\alpha)=\frac{\alpha s+\alpha\rho\Big(F(p)s+\bar{F}(p)(p-c)\Big)+\alpha\rho^2F(p)(s-c)}{1+\alpha+\alpha\rho F(p)}=\frac{s+\rho\Big(F(p)s+\bar{F}(p)(p-c)\Big)+\rho^2F(p)(s-c)}{1/\alpha+1+\rho F(p)}.$$

If the optimal price belongs to the interior $(s,2s)$, then it must solve the FOC equation

$$\rho\frac{\Big(f(p)s-f(p)(p-c)+\bar{F}(p)+\rho f(p)(s-c)\Big)\Big(1/\alpha+1+\rho F(p)\Big)-\rho f(p)\Big(s+F(p)s+\bar{F}(p)(p-c)+\rho F(p)(s-c)\Big)}{\Big(1/\alpha+1+\rho F(p)\Big)^2}=0.$$

Dividing the above equation by $\rho f(p)(1/\alpha+1+\rho F(p))^{-2}$ for a positive density over $(s,2s)$, we get

$$\begin{aligned}
&\Big(s-(p-c)+\bar{F}(p)/f(p)+\rho(s-c)\Big)\Big(1/\alpha+1+\rho F(p)\Big)-\rho\Big(s+F(p)s+\bar{F}(p)(p-c)+\rho F(p)(s-c)\Big)=0\\
&\Leftrightarrow\rho\Big(s+F(p)s+\bar{F}(p)(p-c)+\rho F(p)(s-c)\Big)=\Big(s-(p-c)+\bar{F}(p)/f(p)+\rho(s-c)\Big)\Big(1/\alpha+1+\rho F(p)\Big)\\
&\Leftrightarrow\rho\Big(s+p-c\Big)=\Big(s-(p-c)+\bar{F}(p)/f(p)+\rho(s-c)\Big)(1/\alpha+1)+\rho F(p)\bar{F}(p)/f(p)\\
&\Leftrightarrow(1+\rho)p-c-s-(1+\rho F(p))\bar{F}(p)/f(p)=\frac{s-(p-c)+\bar{F}(p)/f(p)+\rho(s-c)}{\alpha}.
\end{aligned}\tag{26}$$

The RHS of (26) is positive for $p\in(s,2s)$, because it strictly decreases in p and yields, at $p=2s$,

$$-2s+(1+\rho)s+(1-\rho)c+\frac{\bar{F}(2s)}{f(2s)}\geq-2s+(1+\rho)s+(1-\rho)s+\frac{\bar{F}(2s)}{f(2s)}=\frac{\bar{F}(2s)}{f(2s)}\geq0.$$

A solution to (26) cannot make its LHS zero or negative. So, any interior optimal price must solve

$$\frac{-p+(1+\rho)s+(1-\rho)c+\bar{F}(p)/f(p)}{(1+\rho)p-c-s-(1+\rho F(p))\bar{F}(p)/f(p)}=\alpha.\tag{27}$$

For exponential distribution with mean ν, $f(p)=e^{-p/\nu}/\nu$, $F(p)=1-e^{-p/\nu}$ (in particular, ξ has the cdf $1-e^{-(1+x/\nu)}$ over $[-\nu,\infty]$).) and (27) becomes

$$g_\nu(p):=\frac{-p+(1+\rho)s+(1-\rho)c+\nu}{(1+\rho)p-c-s-\nu-\rho\nu(1-e^{-p/\nu})}=\alpha,\tag{28}$$

where g_ν has domain $[0,\infty)$ and range $(-\infty,\infty)$. As mentioned above, the numerator of $g_\nu(p)$ decreases in p and always positive for an interior p. Moreover, its denominator is increasing in p since its derivative is $1+\rho(1-\rho e^{-p/\nu})>0$. The denominator achieves zero at its unique root and makes g_ν discontinuous at the root. When $g_\nu(p)$ is positive, it decreases in p. Since $g_\nu(p)$ is continuous and $s<c$,

$$g_\nu(s)=\frac{-s+(1+\rho)s+(1-\rho)c+\nu}{(1+\rho)s-c-s-\nu-\rho\nu(1-e^{-s/\nu})}=\frac{\rho s+(1-\rho)c+\nu}{\rho s-c-\nu-\rho\nu(1-e^{-s/\nu})}<0.$$

Also,

$$g_\nu(2s)=\frac{-(1-\rho)s+(1-\rho)c+\nu}{(1+2\rho)s-c-\nu-\rho\nu(1-e^{-2s/\nu})}.$$

As $\alpha\in(0,1)$, we are interested in p values with $g_\nu(p)\in(0,1)$. The root of the denominator of $g_\nu(p)$ always exceeds s and can possibly exceed $2s$. As the numerator of $g_\nu(p)$ is always positive, $g_\nu(p)$ becomes positive after the root (if it is below $2s$) and is less than 1 when its denominator exceeds its numerator.

Claim 1. *There exists a unique solution $\tilde p$ to $g_\nu(p)=1$.*

Proof of Claim 1: The numerator of $g_\nu(p)$ is decreasing in p, positive at $p=0$ and negative when p grows sufficiently large, while the denominator of $g_\nu(p)$ is increasing in p, negative at $p=0$ and positive as p grows sufficiently large. So, the numerator and the denominator cross each other at a single unique point since both are continuous. Moreover, the crossing point cannot be a point where both the numerator and the denominator are zero because the root of the numerator is $p=(1+\rho)s+(1-\rho)c+\nu$ at which the denominator is positive. ◇

As mentioned earlier, $g_\nu(p)$ is decreasing when it is positive. As $g_\nu(\tilde p)=1$, the solution to (28) is also unique and greater than $\tilde p$.

Claim 2. *For exponential valuations, the profit rate $R_3(p;\alpha)$ is either increasing or unimodal in $p\in[s,2s]$.*

Proof of Claim 2: For ease of writing, we define $\lambda=1/\nu$. The profit rate function specialized to exponential valuations is

$$\begin{aligned}R_3(p;\alpha) &= \frac{s/\rho+(1-e^{-\lambda p})s+e^{-\lambda p}(p-c)+\rho(1-e^{-\lambda p})(s-c)}{1/(\alpha\rho)+1/\rho+(1-e^{-\lambda p})}\\ &= \frac{s/\rho+s-se^{-\lambda p}+pe^{-\lambda p}-ce^{-\lambda p}+\rho s-\rho se^{-\lambda p}-\rho c+\rho ce^{-\lambda p}}{\kappa-e^{-\lambda p}},\end{aligned}$$

where $\kappa=1/(\alpha\rho)+1/\rho+1$. The derivative of $R_3(p;\alpha)$ wrt p is

$$\begin{aligned}R_3'(p;\alpha) &= (\kappa-e^{-\lambda p})\frac{\lambda se^{-\lambda p}-\lambda pe^{-\lambda p}+e^{-\lambda p}+\lambda ce^{-\lambda p}+\rho\lambda se^{-\lambda p}-\rho\lambda ce^{-\lambda p}}{(\kappa-e^{-\lambda p})^2}\\ &\quad-\lambda e^{-\lambda p}\frac{s/\rho+s-se^{-\lambda p}+pe^{-\lambda p}-ce^{-\lambda p}+\rho s-\rho se^{-\lambda p}-\rho c+\rho ce^{-\lambda p}}{(\kappa-e^{-\lambda p})^2}.\end{aligned}$$

As we are interested in the sign of $R_3'(p,\alpha)$, for ease of exposition, we consider $R_3'(p,\alpha)(\kappa-e^{-\lambda p})^2/(\lambda e^{-\lambda p})$ which has the same sign as $R_3'(p,\alpha)$. We have

$$(\kappa-e^{-\lambda p})^2R_3'(p;\alpha)/(\lambda e^{-\lambda p})$$

$$= (\kappa - e^{-\lambda p})\Big(- p + \frac{1}{\lambda} + (1+\rho)s + (1-\rho)c\Big) - \Big(\frac{s}{\rho} + s - se^{-\lambda p} + pe^{-\lambda p} - ce^{-\lambda p} + \rho s - \rho s e^{-\lambda p} - \rho c + \rho c e^{-\lambda p}\Big)$$
$$= \Big((1+\rho)(\kappa - e^{-\lambda p}) - \frac{1}{\rho} - 1 + e^{-\lambda p} - \rho + \rho e^{-\lambda p}\Big)s + \Big((1-\rho)(\kappa - e^{-\lambda p}) + e^{-\lambda p} + \rho - \rho e^{-\lambda p}\Big)c - \kappa p + \frac{\kappa - e^{-\lambda p}}{\lambda}$$
$$\geq \Big((1+\rho)(\kappa - e^{-\lambda p}) + (1+\rho)e^{-\lambda p} - (1+\rho) - 1/\rho\Big)s + \Big((1-\rho)(\kappa - e^{-\lambda p}) + (1-\rho)e^{-\lambda p} + \rho\Big)c - \kappa p$$
$$= \Big((1+\rho)(\kappa - 1) - 1/\rho\Big)s + \Big((1-\rho)\kappa + \rho\Big)c - \kappa p.$$

The above expression evaluated at $p = s$ is equal to

$$\Big((1+\rho)(\kappa-1) - 1/\rho\Big)s + \Big((1-\rho)\kappa + \rho\Big)c - \kappa s = \Big(\rho\kappa - (1+\rho) - 1/\rho\Big)s + \Big((1-\rho)\kappa + \rho\Big)c,$$

which by inserting $\kappa = 1/(\alpha\gamma) + 1/\rho + 1$ becomes positive:

$$\Big(\rho(1/(\alpha\rho) + 1/\rho + 1) - (1+\rho) - 1/\rho\Big)s + \Big((1-\rho)(1/(\alpha\rho) + 1/\rho + 1) + \rho\Big)c$$
$$= (1/\alpha + 1 + \rho - 1 - \rho - 1/\rho)s + (1/(\alpha\rho) + 1/\rho + 1 - 1/\alpha - 1 - \rho + \rho)c$$
$$= (1/\alpha - 1/\rho)s + (1/(\alpha\rho) + 1/\rho - 1/\alpha)c = (c-s)/\rho + (c/\rho - c + s)/\alpha$$
$$> 0.$$

So, the profit rate is increasing at $p = s$. Recall that the FOC equation (28) has a unique solution. If the solution is smaller than s or larger than $2s$, then $R_3(p;\alpha)$ is increasing in $[s, 2s]$; otherwise, $R_3(p;\alpha)$ is unimodal in $[s, 2s]$. ⋄

Based on Claim 2, if (28) does not admit an interior solution, then the optimal buyout price is $p = 2s$, and if it does, then the solution is the optimal price.

CLAIM 3. *The solution to* (28) *is interior if and only if (i)* $\tilde{p} < 2s$ *and (ii)* $g(2s) < \alpha$.

Proof of Claim 3: Suppose the solution to (28) is in the interior. Because $g_\nu(p)$ is either negative or positive decreasing in p for $p \leq \tilde{p}$, we cannot have $g_\nu(p) = \alpha$ for $p \leq \tilde{p}$. Moreover, since $g_\nu(p)$ is decreasing in p for $p > \tilde{p}$, and $g_\nu(\tilde{p}) = 1$, the solution to (28) is greater than $\tilde{p}$. This implies $\tilde{p} < 2s$. Also $g_\nu(2s) < \alpha$, otherwise the solution cannot be in the interior.

Now suppose (i) and (ii) hold, so $g_\nu(\tilde{p}) = 1$ and $g_\nu(2s) < \alpha$ for $\tilde{p} < 2s$. Since $g_\nu(p)$ is continuous in p, by intermediate value theorem, there exists an interior price that solves (28). ⋄

In Claim 3, (ii) does not imply (i); for instance, if the solution to $(1+\rho)p - c - s - \nu - \rho\nu(1 - e^{-p/\nu}) = 0$ exceeds $2s$ (for a large enough ν), then $g_\nu(p) < 0$ for $p \leq 2s$. However, $\tilde{p} > 2s$.

CLAIM 4. $\tilde{p} < 2s$ *if and only if* $(2+\rho)s - (2-\rho)c > \nu(2 + \rho - \rho e^{-2s/\nu})$.

Proof of Claim 4: The inequality $(2+\rho)s - (2-\rho)c > \nu(2 + \rho - \rho e^{-2s/\nu})$ is equivalent to $g_\nu(2s) < 1$. Suppose $\tilde{p} < 2s$. By definition, $g_\nu(\tilde{p}) = 1$. Because $g_\nu(p)$ is decreasing in p for $p \geq \tilde{p}$, we have $g_\nu(2s) < 1$ and in turn $(2+\rho)s - (2-\rho)c > \nu(2 + \rho - \rho e^{-2s/\nu})$.

Now suppose $g_\nu(2s) < 1$. We prove $\tilde{p} < 2s$ by contradiction. If $\tilde{p} \geq 2s$, the solution to the equation $-p+(1+\rho)s+(1-\rho)c+\nu = (1+\rho)p - c - s - \nu - \rho\nu(1-e^{-p/\nu})$ is no less than $2s$, which implies

$$(2+\rho)p = (2+\rho)s + (2-\rho)c + \nu(2+\rho-\rho e^{-p/\nu}). \tag{29}$$

At $p=s$, the LHS of (29) is $(2+\rho)s$ and its RHS is $(2+\rho)s+(2-\rho)c+(2+\rho-\rho e^{-s/\nu})\nu > (2+\rho)s$. Moreover, the derivative of the LHS is $(2+\rho)$ more than the derivative $\rho e^{-p/\nu}$ of the the RHS. Therefore, since $\tilde{p} \geq 2s$, the LHS is no greater than the RHS at $p=2s$. That is, $(2+\rho)2s \leq (2+\rho)s+(2-\rho)c+\nu(2+\rho-\rho e^{-2s/\nu})$, which is equivalent to $(2+\rho)s-(2-\rho)c \leq \nu(2+\rho-\rho e^{-2s/\nu})$, a contradiction. As such, $\tilde{p} < 2s$. $\diamond$

Based on Claims 3 and 4, (28) has a unique interior solution if and only if

$$(2+\rho)s-(2-\rho)c > \nu(2+\rho-\rho e^{-2s/\nu}) \quad \text{and} \quad \frac{-(1-\rho)s+(1-\rho)c+\nu}{(1+2\rho)s-c-\nu-\rho\nu(1-e^{-2s/\nu})} = g_\nu(2s) < \alpha.$$

The first condition does not depend on α. When it does not hold, (28) does not admit an interior solution by Claim 3, and the optimal price is $2s$ by Claim 2. This optimal solution remains the same even when α changes. We now consider cases where the first condition holds. For small values of α, the condition $g_\nu(2s) < \alpha$ may not hold, in which case the optimal price is $2s$ by Claims 2 and 3. As α increases, it might satisfy $g_\nu(2s) < \alpha$ in which case the optimal price is in the interior. As α further increases, the optimal price decreases because $g_\nu(p)$ is a decreasing function in p. $\square$

Proof of Proposition 1: We rewrite (5) as $R_T^a = R_T/(\alpha\pi_T)$ by dropping the fixed price $\boldsymbol{p}$ from the notation for brevity. The RHS of this equation has $\pi_i/(\alpha\pi_T)$ terms, which reduce to $\rho^{T-i-1}\prod_{j=i+1}^{T-1} F_j(p_j)$ for $i \in [1:T-1]$ by Lemma 1. As such, limiting probabilities are eliminated in the RHS, which facilitates its comparison with R_T^a in the LHS. Specifically from (2) and by inserting $p_1 = s$, we have

$$\frac{R_T}{\alpha\pi_T} = s + \sum_{i=2}^{T-1} \rho^{T-i} \prod_{j=i+1}^{T-1} F_j(p_j)\Big(F_i(p_i)s + \bar{F}_i(p_i)(p_i - c)\Big) + \rho^{T-1}\prod_{j=2}^{T-1} F_j(p_j)\Big(s-c\Big).$$

Both R_T^a and $R_T/(\alpha\pi_T)$ are linear in c and s. They can be thought of as sums of three components: those including s, those including c and the rest. We show that each of these components are identical in R_T^a and $R_T/(\alpha\pi_T)$.

The coefficient of s in R_T^a with $p_1 = s$ from (4) is

$$\begin{aligned}
&\sum_{i=2}^{T-1}(T-i)\Big(\rho^{T-i-1}\prod_{j=i+1}^{T-1}F_j(p_j)\Big)\Big(\rho\bar{F}_i(p_i)+(1-\rho)\Big) + \Big(\rho^{T-2}\prod_{j=2}^{T-1}F_j(p_j)\Big)\Big(\rho T + (1-\rho)(T-1)\Big) \\
&= \Big(\rho\bar{F}_{T-1}(p_{T-1})+1-\rho\Big) + 2\rho F_{T-1}(p_{T-1})\Big(\rho\bar{F}_{T-2}(p_{T-2})+1-\rho\Big) + 3\rho^2 F_{T-1}(p_{T-1})F_{T-2}(p_{T-2})\Big(\rho\bar{F}_{T-3}(p_{T-3})+1-\rho\Big) \\
&\quad + \cdots + (T-2)\Big(\rho^{T-3}\prod_{j=3}^{T-1}F_j(p_j)\Big)\Big(\rho\bar{F}_2(p_2)+(1-\rho)\Big) + \rho^{T-2}(T-1)\prod_{j=2}^{T-1}F_j(p_j) + \rho^{T-1}\prod_{j=2}^{T-1}F_j(p_j) \\
&= \Big(1-\rho F_{T-1}(p_{T-1})\Big) + 2\rho F_{T-1}(p_{T-1})\Big(1-\rho F_{T-2}(p_{T-2})\Big) + 3\rho^2 F_{T-1}(p_{T-1})F_{T-2}(p_{T-2})\Big(1-F_{T-3}(p_{T-3})\Big) \\
&\quad + \cdots + (T-2)\Big(\rho^{T-3}\prod_{j=3}^{T-1}F_j(p_j)\Big)\Big(1-\rho F_2(p_2)\Big) + \rho^{T-2}(T-1)\prod_{j=2}^{T-2}F_j(p_j) + \rho^{T-1}\prod_{j=2}^{T-1}F_j(p_j)
\end{aligned}$$

$$= 1 + \rho F_{T-1}(p_{T-1}) + \rho^2 F_{T-1}(p_{T-1}) F_{T-2}(p_{T-2}) + \rho^3 F_{T-1}(p_{T-1}) F_{T-2}(p_{T-2}) F_{T-3}(p_{T-3}) + \cdots + \rho^{T-2} \prod_{j=2}^{T-1} F_j(p_j) + \rho^{T-1} \prod_{j=2}^{T-1} F_j(p_j).$$

The last expression is the coefficient of s in $R_T/(\alpha\pi_T)$. So, the coefficients of s in R_T^a and $R_T/(\alpha\pi_T)$ are the same. Similarly, the coefficient of c in R_T^a is $-\sum_{i=2}^{T-1}\left(\rho^{T-i}\prod_{j=i+1}^{T-1} F_j(p_j)\right)\bar{F}_i(p_i) - \rho^{T-1}\prod_{j=2}^{T-1} F_j(p_j)$, which is the same as its counterpart in $R_T/(\alpha\pi_T)$. The remaining terms are $\sum_{i=2}^{T-1}\left(\rho^{T-i}\prod_{j=i+1}^{T-1} F_j(p_j)\right)\bar{F}_i(p_i)p_i$ in R_T^a, which are the same as those in $R_T/(\alpha\pi_T)$. □

Proof of Lemma 3: From Lemma 1 and for a given α value, the idleness $\pi_T(\boldsymbol{p};\alpha)$ is decreasing in $\boldsymbol{p}$. Therefore, it achieves its maximum when the buyout prices in different states are equal to their lowest possible value s, and conversely achieves its minimum when $\boldsymbol{p} = (s(T-1), s(T-2), \ldots, s)$. Therefore, the idleness for a buyout price vector $\boldsymbol{p}$ with $p_t \in [s, st]$ for $t \in [1 : T-1]$ satisfies

$$\gamma_T^l(\alpha) = \left[1 + \alpha + \alpha \sum_{i=1}^{T-2} \rho^{T-i-1} \Phi_{i+1}^{T-1}\right]^{-1} \leq \pi_T(\boldsymbol{p};\alpha) \leq \left[1 + \alpha + \alpha \sum_{i=1}^{T-2} \rho^{T-i-1} \phi_{i+1}^{T-1}\right]^{-1} = \gamma_T^u(\alpha).$$

An achievable utilization satisfies $\pi(\boldsymbol{p};\alpha) = \gamma \in [\gamma_T^l(\alpha), \gamma_T^u(\alpha)]$. As, $\pi_T(\boldsymbol{p};\alpha)$ is continuous in p_t, by the intermediate value theorem (Sundaram 1996, Theorem 1.73), for any $\gamma \in [\gamma_T^l(\alpha), \gamma_T^u(\alpha)]$, there exists a price vector $\boldsymbol{p}$ with $p_t \in [s, st]$ for $t \in [1 : T-1]$ such that $\pi_T(\boldsymbol{p};\alpha) = \gamma$. □

Proof of Lemma 4: A feasible solution to the inner maximization problem of (6) solves the balance equations of the RTO MC (see proof of Lemma 1) in addition to $\pi_T(\boldsymbol{p};\alpha) = \gamma$. These equations are implicit in the statement of the inner problem of (6) since they are needed to find $\pi_T(\boldsymbol{p};\alpha)$. To get the full statement of the inner problem, we include these equations as constraints:

$$\max_{\boldsymbol{p}} \Big\{ R_T(\boldsymbol{p};\alpha) : \text{(i)} s \leq p_k \leq ks \text{ for } k \in [1 : T-1], \text{(ii)} \pi_T(\boldsymbol{p};\alpha) = \gamma, \text{(iii)} \pi_{T-1}(\boldsymbol{p};\alpha) = \alpha\pi_T(\boldsymbol{p};\alpha), \text{(iv)} \pi_k(\boldsymbol{p};\alpha) = \rho F_{k+1}(p_{k+1})\pi_{k+1}(\boldsymbol{p}) \text{ for } k \in [1 : T-2], \text{(v)} \sum_{k=1}^{T} \pi_k(\boldsymbol{p};\alpha) = 1 \Big\}. \quad (30)$$

The last three equations above are inherited from the balance equations (see Lemma 1) and are used to find $\pi_k(\boldsymbol{p};\alpha)$'s used to obtain $R_T(\boldsymbol{p};\alpha)$. In presence of (ii) above, (iii) is equivalent to $\pi_{T-1}(\boldsymbol{p};\alpha) = \alpha\gamma$ in whose presence (iv) is equivalent to $\pi_k(\boldsymbol{p};\alpha) = \alpha\gamma\rho^{T-k-1}\prod_{j=k+1}^{T-1} F_j(p_j)$ for $k \in [1 : T-2]$. Therefore, (30) is equivalent to

$$\max_{\boldsymbol{p}} \Big\{ R_T(\boldsymbol{p};\alpha) : \text{(i)} s \leq p_k \leq ks \text{ for } k \in [1 : T-1], \text{(ii)} \pi_T(\boldsymbol{p};\alpha) = \gamma, \text{(iii-a)} \pi_{T-1}(\boldsymbol{p};\alpha) = \alpha\gamma, \text{(iv-a)} \pi_k(\boldsymbol{p};\alpha) = \alpha\gamma\rho^{T-k-1}\prod_{j=k+1}^{T-1} F_j(p_j) \text{ for } k \in [1 : T-2], \text{(v)} \sum_{k=1}^{T} \pi_k(\boldsymbol{p};\alpha) = 1 \Big\}.$$

Above, (iii-a) and (iv-a) can be combined into a single expression by extending the index in (iv-a) to include $k = T-1$. Also, in presence of (iii-a), (iv-a) and (ii), (v) is equivalent to $\gamma + \alpha\gamma + \alpha\gamma\sum_{k=1}^{T-2}\rho^{T-k-1}\prod_{j=k+1}^{T-1} F_j(p_j) = 1$, rearrangement of which gives $\sum_{k=1}^{T-2}\rho^{T-k-1}\prod_{j=k+1}^{T-1} F_j(p_j) = \frac{1}{\gamma\alpha} - \frac{1}{\alpha} - 1$. In presence of (ii) and in light of Proposition 1, we have $R_T(\boldsymbol{p};\alpha) = \alpha\gamma R_T^a(\boldsymbol{p})$. Therefore, (31) is equivalent to

$$\max_{\boldsymbol{p}} \Big\{ \alpha\gamma R_T^a(\boldsymbol{p}) : \text{(i)} s \leq p_k \leq ks \text{ for } k \in [1 : T-1], \text{(ii)} \pi_T(\boldsymbol{p};\alpha) = \gamma,$$

$$\text{(iv-b)}\pi_k(\boldsymbol{p};\alpha) = \alpha\gamma\rho^{T-k-1}\prod_{j=k+1}^{T-1} F_j(p_j) \text{ for } k \in [1:T-1], \text{(v-a)}\sum_{k=1}^{T-2}\rho^{T-k-1}\prod_{j=k+1}^{T-1} F_j(p_j) = \frac{1-\gamma-\alpha\gamma}{\alpha\gamma}\Big\}.$$

Equations (ii) and (iv-b) are used to determine $\pi_k(\boldsymbol{p};\alpha)$'s which are needed to obtain $R_T(\boldsymbol{p};\alpha)$, but are not needed to calculate $R_T^a(\boldsymbol{p})$. Moreover, they do not affect the feasible region for the problem above. Hence, they can be dropped without changing the solution. That is, the above problem is equivalent to

$$\alpha\gamma \max_{\boldsymbol{p}} \Big\{R_T^a(\boldsymbol{p}) : \text{(i)} s \le p_k \le ks \text{ for } k \in [1:T-1], \text{(v-a)}\sum_{k=1}^{T-2}\rho^{T-k-2}\prod_{j=k+1}^{T-1} F_j(p_j) = \frac{1-\gamma-\alpha\gamma}{\alpha\gamma\rho}\Big\}.$$

This completes the proof of the lemma. □

Proof of Proposition 2: From (8), we obtain $\mathrm{P}(\tau(\boldsymbol{p}) \ge t) = \mathrm{P}(\tau(\boldsymbol{p}) \ge t-1)\rho F_{T-t+1}(p_{T-t+1})$ for $t \in [2:T-1]$, so

$$\mathrm{P}(\tau(\boldsymbol{p}) = t) \;=\; \mathrm{P}(\tau(\boldsymbol{p}) \ge t)(\rho\bar{F}_{T-t}(p_{T-t}) + 1-\rho) = \rho^{t-1}(\rho\bar{F}_{T-t}(p_{T-t}) + 1-\rho)\prod_{j=T-t+1}^{T-1} F_j(p_j) \quad \text{for } t \in [1, T-2].$$

Then inserting $p_1 = s$, the objective $R_T^a(\boldsymbol{p})$ of (4) in terms of $\tau(\boldsymbol{p})$ is

$$\begin{aligned} R_T^a(\boldsymbol{p}) \;=\; & \sum_{t=1}^{T-2}\mathrm{P}(\tau(\boldsymbol{p}) \ge t)\Big(\rho\bar{F}_{T-t}(p_{T-t})(ts + p_{T-t}) + (1-\rho)st\Big) \\ & + \mathrm{P}(\tau(\boldsymbol{p}) \ge T-1)\Big(\rho((T-1)s + p_1) + (1-\rho)(T-1)s\Big) \\ & - c\sum_{i=1}^{T-2}\mathrm{P}(\tau(\boldsymbol{p}) \ge t)\rho\bar{F}_{T-t}(p_{T-t}) - c\mathrm{P}(\tau(\boldsymbol{p}) \ge T-1)\rho. \end{aligned} \tag{31}$$

The terms that have s as the monetary coefficient in (31) boil down to the expected rental duration:

$$\begin{aligned} & \mathrm{P}(\tau(\boldsymbol{p}) \ge T-1)\Big(\rho(T-1) + (1-\rho)(T-1)\Big) + \sum_{t=1}^{T-2}\mathrm{P}(\tau(\boldsymbol{p}) \ge t)\Big(\rho\bar{F}_{T-t}(p_{T-t}) + 1-\rho\Big)t \\ =\; & \mathrm{P}(\tau(\boldsymbol{p}) = T-1)(T-1) + \sum_{t=1}^{T-2}\mathrm{P}(\tau(\boldsymbol{p}) = t)t \;=\; \mathbb{E}(\tau(\boldsymbol{p})). \end{aligned}$$

In presence of the utilization constraint of (7), the expected rental duration is independent of the price path, as it depends only on α and γ:

$$\begin{aligned} \mathbb{E}(\tau(\boldsymbol{p})) \;=\; & \sum_{i=1}^{T-1}\mathrm{P}(\tau(\boldsymbol{p}) \ge i) = \sum_{i=1}^{T-1}\rho^{i-1}\prod_{j=T-i+1}^{T-1} F_j(p_j) = \sum_{k=1}^{T-1}\rho^{T-k-1}\prod_{j=k+1}^{T-1} F_j(p_j) \\ =\; & 1 + \sum_{k=1}^{T-2}\rho^{T-k-1}\prod_{j=k+1}^{T-1} F_j(p_j) = 1 + \rho\left(\frac{1-\gamma}{\alpha\gamma\rho} - \frac{1}{\rho}\right) = \frac{1-\gamma}{\alpha\gamma}, \end{aligned} \tag{32}$$

where the next-to-last equality is the utilization constraint. That is, the utilization constraint becomes $\mathbb{E}(\tau(\boldsymbol{p})) = (1-\gamma)/(\alpha\gamma)$ in (9). As such, this constraint sets the expected rental revenue at $(1-\gamma)/(\alpha\gamma)s$.

If the firm sets each price sufficiently high to render a buyout unjustifiable during the agreement, the rental duration can be represented with the random variable $\tau(\infty)$, which has the probability mass function of $\mathrm{P}(\tau(\infty) = t) = \rho^{t-1}(1-\rho)$ for $t \in [1:T-2]$ and $\mathrm{P}(\tau(\infty) = T-1) = \rho^{T-1}$. It has the same distribution as $\min\{Geo(1-\rho), T-1\}$, where the geometric random variable $Geo(1-\rho)$ is the number of Bernoulli trials until the first success whose probability is $1-\rho$. By construction $\tau(\boldsymbol{p})$ is stochastically smaller than $\min\{Geo(1-\rho), T-1\}$, which in turn is stochastically smaller than $Geo(1-\rho)$. Hence, $\mathbb{E}(\tau(\boldsymbol{p})) \le 1/(1-\rho)$.

The terms that have $-c$ as the monetary coefficient in (31) sum to the probability of a sale.

$$\begin{aligned}
\mathrm{P}(S(\boldsymbol{p})=1) &= \sum_{t=1}^{T-2} \mathrm{P}(\tau(\boldsymbol{p}) \ge t)\rho \bar{F}_{T-t}(p_{T-t}) + \mathrm{P}(\tau(\boldsymbol{p}) \ge T-1)\rho \\
&= \rho \sum_{t=1}^{T-2} \mathrm{P}(\tau(\boldsymbol{p}) \ge t) - \sum_{t=1}^{T-2} \mathrm{P}(\tau(\boldsymbol{p}) \ge t)\rho F_{T-t}(p_{T-t}) + \rho \mathrm{P}(\tau(\boldsymbol{p}) \ge T-1) \\
&= \rho \sum_{t=1}^{T-1} \mathrm{P}(\tau(\boldsymbol{p}) \ge t) - \sum_{t=1}^{T-2} \mathrm{P}(\tau(\boldsymbol{p}) \ge t+1) \;=\; 1-(1-\rho)\mathbb{E}(\tau(\boldsymbol{p})).
\end{aligned}$$

In presence of the utilization constraint, the probability of a sale is independent of the price path. This probability is nonnegative as $\mathbb{E}(\tau(\boldsymbol{p})) \le 1/(1-\rho)$. The utilization constraint sets the expected replacement cost per agreement at $(1-(1-\rho)(1-\gamma)/\alpha\gamma)c$.

Given the utilization constraint, we can exclude the expected rental revenue and replacement cost from $R_T^a(\boldsymbol{p})$ when solving (7), and we instead consider $R_T^{a\setminus}(\boldsymbol{p})$. $R_T^{a\setminus}(\boldsymbol{p})$ is the expected sales revenue in an agreement when the appropriate random variables are incorporated into the revenue.

$$\begin{aligned}
R_T^{a\setminus}(\boldsymbol{p}) &= \mathrm{P}(S(\boldsymbol{p})=1)\left(\sum_{t=1}^{T-2} \frac{\rho \bar{F}_{T-t}(p_{T-t})\mathrm{P}(\tau(\boldsymbol{p}) \ge t)}{\mathrm{P}(S(\boldsymbol{p})=1)} p_{T-t} + \frac{\mathrm{P}(\tau(\boldsymbol{p}) \ge T-1)\rho}{\mathrm{P}(S(\boldsymbol{p})=1)} p_1\right) \\
&= \mathrm{P}(S(\boldsymbol{p})=1) \sum_{t=1}^{T-1} \mathrm{P}(\tau(\boldsymbol{p})=t | S(\boldsymbol{p})=1) p_{T-t} \;=\; \mathrm{P}(S(\boldsymbol{p})=1)\mathbb{E}\big(p_{T-\tau(\boldsymbol{p})} | S(\boldsymbol{p})=1\big).
\end{aligned}$$

The operations above induce a conditional random variable $[\tau(\boldsymbol{p})|S(\boldsymbol{p})=1]$ whose range is $[1:T-1]$. Specifically, $\mathrm{P}(\tau(\boldsymbol{p})=t|S(\boldsymbol{p})=1) = \rho\bar{F}_{T-t}(p_{T-t})\mathrm{P}(\tau(\boldsymbol{p}) \ge t)/\mathrm{P}(S(\boldsymbol{p})=1)$ for $t \in [1:T-2]$ and $\mathrm{P}(\tau(\boldsymbol{p}) = T-1|S(\boldsymbol{p})=1) = \rho\mathrm{P}(\tau(\boldsymbol{p}) \ge T-1)/\mathrm{P}(S(\boldsymbol{p})=1)$. This conditional random variable is deployed in $\mathbb{E}\big(p_{T-\tau(\boldsymbol{p})}|S(\boldsymbol{p})=1\big)$. Since $\mathrm{P}(S(\boldsymbol{p})=1)$ is independent of price path in the presence of the utilization constraint, we can maximize $\mathbb{E}\big(p_{T-\tau(\boldsymbol{p})}|S(\boldsymbol{p})=1\big)$ as in (9). □

Proof of Lemma 5: Since $\bar{G}^r \in \mathcal{G}^r$, we have $\int_0^T \bar{G}^r(t)dt \le T$, which implies $\delta^r \le T$. Using $\delta^r > 0$, we can express $\mathcal{G}$ parameterically via

$$\mathcal{G}(\delta^r) = \left\{ \bar{G} \;:\; \bar{G}(0) = \frac{1}{\delta}\int_0^T \bar{G}(t)dt,\ \bar{G}(t) \ge 0,\ -\dot{\bar{G}}(t)/\bar{G}(t) \in [0,1],\ \int_0^T \bar{G}(t)dt = \delta,\ \int_0^T \bar{G}(t)dt \le \delta\frac{T}{\delta^r} \right\}.$$

In $\mathcal{G}(\delta^r)$, we still have $\bar{G}(0)=1$. Moreover, the last inequality $\int_0^T \bar{G}(t)dt \le \delta T/\delta^r$ is never binding, because $\delta^r \le T$. Hence, $\mathcal{G} = \mathcal{G}(\delta^r)$. If $\delta^r \ge \delta$, $\mathcal{G}^r$ includes $\mathcal{G}(\delta^r)$; otherwise, no subset relation holds.

Since $\mathcal{G} = \mathcal{G}(\delta^r)$, we connect $\max\{R_T^{a\setminus}(\bar{G}) : \bar{G} \in \mathcal{G}^r\}$ to $\max\{R_T^{a\setminus}(\bar{G}) : \bar{G} \in \mathcal{G}(\delta^r)\}$ in the rest of the proof. The first three constraints in $\mathcal{G}(\delta^r)$ and $\mathcal{G}^r$ are identical:

$$\bar{G}(0) = \frac{1}{\delta}\int_0^T \bar{G}(t)dt, \quad \bar{G}(t) \ge 0, \quad -\dot{\bar{G}}(t)/\bar{G}(t) \in [0,1],$$

whose validity is unaffected from scaling of $\bar{G}(t)$ by a positive multiplier. So we focus on the other constraints in $\mathcal{G}(\delta^r)$ and $\mathcal{G}^r$ when arguing for feasibility below.

We first argue that $(\delta/\delta^r)\bar{G}^r \in \mathcal{G}(\delta^r) = \mathcal{G}$. We focus on the constraints that are affected by positive scaling of $\bar{G}^r$ by δ/δ^r. Since $\int_0^T (\delta/\delta^r)\bar{G}^r(t)dt = (\delta/\delta^r)\delta^r = \delta \le \delta T/\delta^r$, $(\delta/\delta^r)\bar{G}^r$ satisfies the equality $\int_0^T \bar{G}(t)dt = \delta$ and the inequality $\int_0^T \bar{G}(t)dt \le \delta T/\delta^r$ in $\mathcal{G}(\delta^r)$, i.e., $(\delta/\delta^r)\bar{G}^r \in \mathcal{G}(\delta^r)$.

It suffices to argue that $(\delta/\delta^r)\bar{G}^r$ solves $\max\{R_T^{a\searrow}(\bar{G}) : \bar{G} \in \mathcal{G}\}$. Proof is by contradiction. Suppose $(\delta/\delta^r)\bar{G}^r$ is not the maximizer, then there exists $\bar{G}^b \in \mathcal{G}(\delta^r)$ such that $R_T^{a\searrow}(\bar{G}^b) > R_T^{a\searrow}((\delta/\delta^r)\bar{G}^r)$. Let $\bar{G}^m = (\delta^r/\delta)\bar{G}^b$, we show $\bar{G}^m \in \mathcal{G}^r$. $\bar{G}^m$ is obtained from $\bar{G}^b \in \mathcal{G}(\delta^r)$ via scaling by a positive number, so we need to check the only constraint $\int_0^T \bar{G}(t)dt \le T$ that is affected by scaling. $\bar{G}^m$ satisfies this constraint, as $\int_0^T \bar{G}^m(t)dt = (\delta^r/\delta)\int_0^T \bar{G}^b(t)dt = \delta^r$ and $\delta^r \le T$, so $\bar{G}^m \in \mathcal{G}^r$. On the other hand, $R_T^{a\searrow}((\delta/\delta^r)\bar{G}^r) < R_T^{a\searrow}(\bar{G}^b) = R_T^{a\searrow}((\delta/\delta^r)\bar{G}^m)$, which by the sales revenue linearity implies $R_T^{a\searrow}(\bar{G}^r) < R_T^{a\searrow}(\bar{G}^m)$ for $\bar{G}^m \in \mathcal{G}^r$. The last inequality violates the optimality of $\bar{G}^r$ and establishes the contradiction. □

Proof of Proposition 3: We follow the convention of van Brunt (2003) and denote our variable $\bar{G}$ by y and its derivative by $\dot{y}$ and deploy methods of Calculus of Variations. The expected sales revenue is

$$\max_{\bar{G}} R_T^{a\searrow s}(\bar{G}) = \max_y \left\{ -\int_0^T \dot{y}(t)\left(1 + s + \dot{y}(t)/y(t)\right) dt \right\} = \max_y \int_0^T H(t; y(t), \dot{y}(t))dt,$$

where $H(\cdot; y, \dot{y}) = -\dot{y}(\cdot)\left(1 + s + \dot{y}(\cdot)/y(\cdot)\right)$.

Claim 5. *The expected sales revenue is maximized by* $y(t) = \exp(c_y)\exp\left(-\int_0^t \lambda(x)dx\right)$, *where*

$$\lambda(t) = \frac{2}{c_\lambda - t} \quad \textit{for } t \in [0, T]. \tag{33}$$

Above c_y and c_λ are constants to be determined.

Proof of Claim 5: The necessary condition for an optimal solution is the Euler-Lagrange differential equation (Theorem 2.2.3 in van Brunt 2003):

$$\frac{d}{dt}\frac{\partial}{\partial \dot{y}} H(t; y(t), \dot{y}(t)) - \frac{\partial}{\partial y} H(t; y(t), \dot{y}(t)) = 0 \quad \text{for } t \in [0, T].$$

We next derive the terms involved in the equation above:

$$\begin{aligned}
\frac{\partial}{\partial y} H(t; y, \dot{y}) &= -\dot{y}(t)(-\dot{y}(t)/(y(t))^2) = (\dot{y}(t))^2/(y(t))^2 \ge 0, \\
\frac{\partial}{\partial \dot{y}} H(t; y, \dot{y}) &= -1 - s - 2\dot{y}(t)/y(t), \\
\frac{d}{dt}\frac{\partial}{\partial \dot{y}} H(t; y, \dot{y}) &= -2\left(\ddot{y}(t)/y(t) - (\dot{y}(t))^2(t)/(y(t))^2\right).
\end{aligned}$$

Inserting the above equalities in the Euler-Lagrange differential equation, we get

$$2\left(\ddot{y}(t)/y(t) - (\dot{y}(t))^2/(y(t))^2\right) = -(\dot{y}(t))^2/(y(t))^2 \quad \Leftrightarrow \quad 2\ddot{y}(t)y(t) = (\dot{y}(t))^2. \tag{34}$$

The Legendre condition for a local extremum (Theorem 10.3.1 in van Brunt 2003) is

$$\frac{\partial}{\partial \dot{y}}\frac{\partial}{\partial \dot{y}} H(t; y, \dot{y}) = \frac{\partial}{\partial \dot{y}}\left[-s - 1 - 2\dot{y}(t)/y(t)\right] = -2/y(t) \le 0.$$

We are interested in $y(t)$ for $t \in [0, T]$. Any such solution to (34) is a local maximizer from the Legendre condition above.

Next, we transform (34) by writing it in terms $\lambda(t) = -\dot{y}(t)/y(t)$. Solving the differential equation $\lambda(t) = -\dot{y}(t)/y(t)$ gives $y(t) = \exp(c_y)\exp\left(-\int_0^t \lambda(x)dx\right)$ for a constant $\exp(c_y)$. The Euler equation (34) after dropping $\exp(2c_y)$ terms from both sides can be written in terms of $\lambda(t)$:

$$\exp\left(-\int_0^t \lambda(x)dx\right)\left((\lambda(t))^2 - \dot{\lambda}(t)\right)\exp\left(-\int_0^t \lambda(x)dx\right) = (\lambda(t))^2 \exp\left(-2\int_0^t \lambda(x)dx\right)$$
$$\iff \quad 2((\lambda(t))^2 - \dot{\lambda}(t)) = (\lambda(t))^2 \quad \iff \quad (\lambda(t))^2 = 2\dot{\lambda}(t).$$

The solution to the above differential equation is given by (33) in the statement of the claim. Moreover, this solution is unique upto the constant c_λ. ◇

The maximizer of expected sales revenue is not necessarily the tail probability of rental duration, as it is likely to be scaled by a positive constant. Before such scaling, we can remark about the optimal tail probability and optimal price behavior in time. When $y(t)$ is positive, $\ddot{y}(t)$ is nonnegative in (34). So, any positive maximizer y of expected sales revenue is convex in time. This convexity is preserved when y is scaled by a positive constant to obtain the optimal tail probability of rental duration. Scaling of the maximizer y does not affect $\lambda(t) = -\dot{y}(t)/y(t)$, except that $\lambda(t)$ becomes the failure rate of the duration when y is the tail probability. Moreover, the optimal price is $p(t) = \bar{F}^{-1}(1-\lambda(t)) = 1 + s - \lambda(t)$ and

$$\dot{\lambda}(t) = \frac{d}{dt}\lambda(t) = -\frac{\ddot{y}(t)y(t) - (\dot{y}(t))^2}{(y(t))^2} \geq 0 \iff (\dot{y}(t))^2 - \ddot{y}(t)y(t) \geq 0.$$

Deducting $\ddot{y}(t)y(t) \geq 0$ from both sides of (34), $(\dot{y}(t))^2 - \ddot{y}(t)y(t) = \ddot{y}(t)y(t) \geq 0$, so $\dot{\lambda}(t) \geq 0$. This implies that the optimal price path is decreasing in time.

Claim 6. *The constant*

$$c_\lambda(T,\delta) = \frac{3 + \sqrt{9 - 12(1-\delta/T)}}{6(1-\delta/T)}T$$

deployed in (33) *yields* $\lambda(t)$ *and then* $y(t)$ *such that* y *belongs to* $\mathcal{G}^r$ *for* $\delta/T \in [(T^2+6T+12)(3(T+2)^2), 1]$. *In particular, this* y *is* $y(t) = \exp(c_y)(1 - t/c_\lambda(T,\delta))^2$ *for* $t \in [0,T]$ *and* $c_y < 0$.

Proof of Claim 6: To determine constant $c_\lambda(T,\delta)$, we start with y characterized by Claim 5 and check the constraints in $\mathcal{G}^r$: $y(t) \geq 0$, $-\dot{y}(t)/y(t) \in [0,1]$, $\delta y(0) = \int_0^T y(t)dt$ and $\int_0^T y(t)dt \leq T$.

We move on to translate the optimal failure rate to the tail probability $y(t)$:

$$y(t) = \exp(c_y)\exp\left(\int_0^t \frac{-2}{c_\lambda - x}dx\right) = \exp(c_y)\exp\left(2\ln(c_\lambda - x)|_0^t\right) = \exp(c_y)\left(1 - \frac{t}{c_\lambda}\right)^2 \quad \text{for } t \in [0,T].$$

Note that $y(t) \geq 0$, $\dot{y}(t) \leq 0$ for $t \in [0,T]$, so $-\dot{y}(t)/y(t) \geq 0$ for $t \in [0,T]$. Since $y(t)$ is decreasing, it achieves the highest value at $y(0) = \exp(c_y)$. Then with $c_y \leq 0$

$$\int_0^T y(t)dt \leq \exp(c_y)T \leq T.$$

The exact value of c_y is not important as long as $c_y \leq 0$. This is because $\exp(c_y)$ is removed by scaling of $y(t)$ with $\int_0^T y(t)dt$, as we show below.

The only remaining conditions to check to ensure $y \in \mathcal{G}^r$ are $\delta y(0) = \int_0^T y(t)dt$ and $-\dot{y}(t)/y(t) \leq 1$. The first condition will yield an expression for the constant c_λ. We have $\delta y(0) = \delta \exp(c_y)$ and

$$\int_0^T y(t)dt = \exp(c_y)\int_0^T (1-t/c_\lambda)^2 dt = \exp(c_y)c_\lambda \int_1^{1-T/c_\lambda} u^2(-du) = \exp(c_y)(c_\lambda/3)(1-(1-T/c_\lambda)^3).$$

Setting $\delta y(0) = \int_0^T y(t)dt$ first leads to $3\delta/c_\lambda = 1-(1-T/c_\lambda)^3 = 3T/c_\lambda - 3T^2/c_\lambda^2 + T^3/c_\lambda^3$ then to the following quadratic equation in c_λ where T and δ are parameters:

$$3(T-\delta)c_\lambda^2 - 3T^2 c_\lambda + T^3 = 0. \tag{35}$$

The larger root of (35) is real for $\delta \geq T/4$ and it is given by $c_\lambda(T,\delta)$ in the statement of the claim. We must require $c_\lambda(T,\delta) \geq T+2$ so that $-\dot{y}(T)/y(T) \leq 1$:

$$c_\lambda(T,\delta)/T = \frac{3+\sqrt{9-12(1-\delta/T)}}{6(1-\delta/T)} \geq 1+2/T.$$

We look for the range of $\tilde{\delta} = 1-\delta/T$ such that the above inequality holds. This requires

$$3+\sqrt{9-12\tilde{\delta}} \geq (1+2/T)6\tilde{\delta} \quad \Leftrightarrow \quad \sqrt{9-12\tilde{\delta}} \geq (1+2/T)6\tilde{\delta} - 3.$$

Depending on the sign of $(1+2/T)6\tilde{\delta}-3$, we have two cases.

- If $(1+2/T)6\tilde{\delta}-3 \leq 0$, the inequality is trivially satisfied, i.e., when

$$\begin{aligned}
&(1+2/T)6\tilde{\delta} \leq 3 \;\Leftrightarrow\; 6\tilde{\delta} \leq 3/(1+2/T) \;\Leftrightarrow\; \tilde{\delta} \leq 1/(2+4/T) \;\Leftrightarrow\; 1-\delta/T \leq 1/(2+4/T)\\
&\Leftrightarrow\; \delta/T \geq 1-1/(2+4/T) \;\Leftrightarrow\; \delta \geq T - T/(2+4/T) \;\Leftrightarrow\; \delta \geq \frac{T+4}{2+4/T}.
\end{aligned}$$

Note for $T \geq 2$ that $(T+4)/(2+4/T) \geq T/4$. So, $\delta \in [(T+4)/(2+4/T), T]$ ensures real $c_\lambda(T,\delta)$ and $c_\lambda(T,\delta) \geq T+2$.

- If $(1+2/T)6\tilde{\delta}-3 \geq 0$, we require

$$\begin{aligned}
&9-12\tilde{\delta} \geq (6\tilde{\delta}-3+12\tilde{\delta}/T)^2 = 36\tilde{\delta}^2 - 36\tilde{\delta} + 9 + 2(6\tilde{\delta}-3)12\tilde{\delta}/T + 144\tilde{\delta}^2/T^2\\
\Leftrightarrow\;\; &0 \geq (36+144/T+144/T^2)\tilde{\delta}^2 - (24+72/T)\tilde{\delta} \;\Leftrightarrow\; 0 \geq \tilde{\delta}(\tilde{\delta}(36+144/T+144/T^2) - 24 - 72/T).
\end{aligned}$$

The final inequality above holds between the roots of $\tilde{\delta}$, i.e., when

$$\tilde{\delta} \in \Big[0, \frac{24+72/T}{36+144/T+144/T^2}\Big] = \Big[0, \frac{2T^2+6T}{3T^2+12T+12}\Big].$$

The above condition is equivalent to $\delta \leq T$ and

$$1-\delta/T \leq \frac{2T^2+6T}{3T^2+12T+12} \;\Leftrightarrow\; T - \frac{2T^3+6T^2}{3(T+2)^2} \leq \delta \;\Leftrightarrow\; \frac{T}{3}\frac{T^2+6T+12}{(T+2)^2} \leq \delta.$$

Let $l(T) = (T^2+6T+12)(3(T+2)^2)$. So, $\delta \in [T/4, (T+4)/(2+4/T)] \cap \big[l(T)T, T\big]$ ensures real $c_\lambda(T,\delta)$ and $c_\lambda(T,\delta) \geq T+2$.

Uniting the intervals obtained according to the sign of $(1+2/T)6\tilde{\delta}-3$,

$$\begin{aligned}
&[(T+4)/(2+4/T), T] \cup \Big([T/4, (T+4)/(2+4/T)] \cap \big[l(T)T, T\big]\Big)\\
&= \Big([(T+4)/(2+4/T), T] \cup [T/4, (T+4)/(2+4/T)]\Big) \cap \Big([(T+4)/(2+4/T), T] \cup \big[l(T)T, T\big]\Big)
\end{aligned}$$

$$= [T/4, T] \cap \big[l(T)T, T\big] \quad \Leftarrow \; l(T)T \leq (T+4)/(2+4/T)$$
$$= \big[l(T)T, T\big] \quad \Leftarrow \; l(T)T \geq T/4.$$

In summary, $c_\lambda(T,\delta)$ is real and at least $T+2$ if $\delta \in [l(T)T, T]$.

The analysis for the larger root of (35) can be repeated for the smaller root. But this analysis yields a more restrictive condition than $\delta \geq l(T)$, so we keep the condition $\delta \geq l(T)T$ unaltered. As T grows large, $l(T) \downarrow 1/3$, and the condition simplifies to $\delta \in [T/3, T]$. In conclusion, for given T and $\delta \in [0,T]$, if $\delta \in [l(T)T, T]$, then we have $c_\lambda(T,\delta) \geq T+2$ and y given in the statement of the claim belongs to $\mathcal{G}^r$. $\diamond$

The process used in claims above guarantees that the resulting $y \in \mathcal{G}^r$ maximizes $R_T^{a \searrow s}$. This y solves $\max\{R_T^{a\searrow s}(\bar{G}) : \bar{G} \in \mathcal{G}^r\}$. If there is another $y^b \in \mathcal{G}^r$ allegedly maximizing $R_T^{a\searrow s}$, it must satisfy the functional form in Claim 5 with the constant in Claim 6. Then $y^b = y$ because the construction in the claims leads to a unique solution.

With y in Claim 6, $\int_0^T y(t)dt = \exp(c_y)(c_\lambda(T,\delta)/3)(1-(1-T/c_\lambda(T,\delta))^3) = \delta^r$. By using Lemma 5, the solution to $\max\{R_T^{a\searrow s}(\bar{G}) : \bar{G} \in \mathcal{G}\}$ is

$$y(t) = \frac{\delta}{(c_\lambda(T,\delta)/3)(1-(1-T/c_\lambda(T,\delta))^3)}\left(1 - \frac{t}{c_\lambda(T,\delta)}\right)^2 \quad \text{for } t \in [0,T]. \tag{36}$$

The payoff probability with optimal prices is $y(T)$ and can be computed from (36). This probability is positive for a finite T, which makes δ finite, too.

Finally, by (33), the optimal price path is $p(t) = s + 1 - \lambda(t)$, which is given in the statement of the proposition. The optimal price path is decreasing and concave in time.

Claim 7. *The constant $c_\lambda(T,\delta)$ increases in δ for $\delta \geq l(T)$.*

Proof of Claim 7: The first term $1/(2-2\delta/T)$ in $c_\lambda(T,\delta)$ increases in δ. It suffices to check the derivative of the second term wrt δ.

$$\frac{\partial}{\partial\delta}\frac{\sqrt{1-(4/3)(1-\delta/T)}}{2(1-\delta/T)}T = \frac{T}{2}\frac{\partial(1-\delta/T)}{\partial\delta}\frac{\partial}{\partial\tilde{\delta}}\sqrt{\tilde{\delta}^{-2}-(4/3)\tilde{\delta}^{-1}} = -\frac{1}{4}\frac{-2\tilde{\delta}^{-3}+(4/3)\tilde{\delta}^{-2}}{\sqrt{\tilde{\delta}^{-2}-(4/3)\tilde{\delta}^{-1}}}$$
$$= \frac{1}{2\tilde{\delta}^2}\frac{1-(2/3)\tilde{\delta}}{\sqrt{1-(4/3)\tilde{\delta}}} \geq 0,$$

where $\tilde{\delta} = 1 - \delta/T \leq 1 - l(T) \leq 3/4$. $\diamond$

If the initiation probability α or the idleness γ increases, the expected duration δ decreases. Then, $c_\lambda(T,\delta)$ decreases by Claim 7 and the optimal price path shifts down. $\square$

Proof of Proposition 4: To express the objective function $R_T^a(\boldsymbol{p})$ of the maximization problem (7), we delineate the profit-to-go function $R_{\leq k}^a(\boldsymbol{p}_{\leq k})$ in a state $k \in [1:T-1]$. This function represents the expected profit to be gained in state k and afterwards with the buyout price vector $\boldsymbol{p}_{\leq k}$. For $i \in [2:k]$, the agreement terminates in state i either via a buyout wp $\rho^{k-i}\prod_{j=i+1}^{k} F_j(p_j)\rho\bar{F}_i(p_i)$ and the profit $(k-i)s + p_i - c$, or an item return wp $\rho^{k-i}\prod_{j=i+1}^{k}F_j(p_j)(1-\rho)$ and the profit $(k-i)s$. Alternatively, since $p_1=s$, the agreement terminates in state 1 by selling the item (either through a payoff or a buyout) wp $\rho^{k-1}\prod_{j=2}^{k}F_j(p_j)\rho$ and profit of $ks-c$, or by the renter's returning the item wp $\rho^{k-1}\prod_{j=2}^{k}F_j(p_j)(1-\rho)$ and $(k-1)s$ in profit. It easy to show that $R^a_{\le k}(\boldsymbol{p}_{\le k})$ has the following form:

$$\begin{aligned}R^*_T(\gamma;\alpha):=R^a_{\le k}(\boldsymbol{p}_{\le k}) &= \sum_{i=2}^{k}\Big(\rho^{k-i}\prod_{j=i+1}^{k}F_j(p_j)\Big)\Big(\rho\bar{F}_i(p_i)((k-i)s+p_i-c)+(1-\rho)(k-i)s\Big)\\ &\quad+\Big(\rho^{k-1}\prod_{j=2}^{k}F_j(p_j)\Big)\Big(\rho(ks-c)+(1-\rho)(k-1)s\Big).\end{aligned} \tag{37}$$

We can check that $\sum_{i=2}^{k}\Big(\rho^{k-i}\prod_{j=i+1}^{k}F_j(p_j)\Big)(\rho\bar{F}_i(p_i)+(1-\rho))+\rho^{k-1}\prod_{j=2}^{k}F_j(p_j)=1$. Note that $\boldsymbol{p}_{\le T-1}=\boldsymbol{p}$ and $R^a_{\le 1}(\boldsymbol{p}_{\le 1})=\rho(s-c)$ as $\boldsymbol{p}_{\le 1}=s$. The following claim connects $R^a_{\le T-1}$ to the expected profit R^a_T in an agreement of term T.

CLAIM 8. *The profit-to-go function in state $T-1$ and the expected profit in the agreement are related to each other through $R^a_{\le T-1}(\boldsymbol{p})=R^a_T(\boldsymbol{p})-s$.*

Proof of Claim 8: We can reorganize R^a_T from (4) and insert $p_1=s$ as follows:

$$\begin{aligned}R^a_T(\boldsymbol{p}) &= \sum_{i=2}^{T-1}\Big(\rho^{T-i-1}\prod_{j=i+1}^{T-1}F_j(p_j)\Big)\Big(\rho\bar{F}_i(p_i)((T-i)s+p_i-c)+(1-\rho)(T-i)s\Big)\\ &\quad+\Big(\rho^{T-2}\prod_{j=2}^{T-1}F_j(p_j)\Big)\Big(\rho(Ts-c)+(1-\rho)(T-1)s\Big)\\ &= \sum_{i=2}^{T-1}\Big(\rho^{T-i-1}\prod_{j=i+1}^{T-1}F_j(p_j)\Big)\Big(\rho\bar{F}_i(p_i)((T-i-1)s+p_i-c)+(1-\rho)(T-i-1)s+(1-\rho F_i(p_i))s\Big)\\ &\quad+\Big(\rho^{T-2}\prod_{j=2}^{T-1}F_j(p_j)\Big)\Big(\rho(((T-1)s-c)+(1-\rho)(T-2)s+s\Big)\\ &= \Big(\sum_{i=2}^{T-1}\big(\rho^{T-i-1}\prod_{j=i+1}^{T-1}F_j(p_j)\big)\big(1-\rho F_i(p_i)\big)+\rho^{T-2}\prod_{j=2}^{T-1}F_j(p_j)\Big)s+R^a_{\le T-1}(\boldsymbol{p})\\ &= \Big(\sum_{i=3}^{T-1}\big(\rho^{T-i-1}\prod_{j=i+1}^{T-1}F_j(p_j)\big)\big(1-\rho F_i(p_i)\big)+\rho^{T-3}\prod_{j=3}^{T-1}F_j(p_j)\Big)s+R^a_{\le T-1}(\boldsymbol{p}).\end{aligned}$$

The second equality above holds because $\rho\bar{F}_i(p_i)+(1-\rho)=1-\rho F_i(p_i)$. With similar manipulations, we can increase the lower indices of the summation and the product in the last equality to get $R^a_T=(1-\rho F_{T-1}(p_{T-1})+\rho F_{T-1}(p_{T-1}))s+R^a_{\le T-1}=s+R^a_{\le T-1}$, which completes the proof of Claim 8. $\diamond$

The difference between $R^a_{\le T-1}$ and R^a_T is the profit s gained in state T. This profit is absent in $R^a_{\le T-1}$, for which expected profit are calculated from state $T-1$ on.

CLAIM 9. *The profit-to-go function in state $k\in[2:T-1]$ is related to the profit-to-go function in state $k-1$ through the recursion*

$$R^a_{\le k}(\boldsymbol{p}_{\le k}) = \rho\bar{F}_k(p_k)(p_k-c)+\rho F_k(p_k)\Big(s+R^a_{\le k-1}(\boldsymbol{p}_{\le k-1})\Big).$$

Proof of Claim 9: From (37), we write $R^a_{\le k}(\boldsymbol{p}_{\le k})$ as

$$\begin{aligned}
&\sum_{i=2}^{k}\Big(\rho^{k-i}\prod_{j=i+1}^{k}F_j(p_j)\Big)\Big(\rho\bar{F}_i(p_i)((k-i)s+p_i-c)+(1-\rho)(k-i)s\Big)\\
&\qquad+\Big(\rho^{k-1}\prod_{j=2}^{k}F_j(p_j)\Big)\Big(\rho(ks-c)+(1-\rho)(k-1)s\Big)\\
&=\rho\bar{F}_k(p_k)(p_k-c)+\sum_{i=2}^{k-1}\Big(\rho^{k-i}\prod_{j=i+1}^{k}F_j(p_j)\Big)\Big(\rho\bar{F}_i(p_i)\big((k-1-i)s+p_i-c\big)+(1-\rho)(k-1-i)s+(1-\rho F_i(p_i))s\Big)\\
&\qquad+\Big(\rho^{k-1}\prod_{j=2}^{k}F_j(p_j)\Big)\Big(\rho((k-1)s-c)+(1-\rho)(k-2)s+s\Big)\\
&=\rho\bar{F}_k(p_k)(p_k-c)+\rho F_k(p_k)\sum_{i=2}^{k-1}\Big(\rho^{k-1-i}\prod_{j=i+1}^{k-1}F_j(p_j)\Big)\Big(\rho\bar{F}_i(p_i)\big((k-1-i)s+p_i-c\big)+(1-\rho)(k-1-i)s\Big)\\
&\qquad+\rho F_k(p_k)\Big(\rho^{k-2}\prod_{j=2}^{k-1}F_j(p_j)\Big)\Big(\rho((k-1)s-c)+(1-\rho)(k-2)s\Big)\\
&\qquad+\Big(\sum_{i=2}^{k-1}\big(\rho^{k-i}\prod_{j=i+1}^{k}F_j(p_j)\big)\big(1-\rho F_i(p_i)\big)+\rho^{k-1}\prod_{i=2}^{k}F_i(p_i)\Big)s\\
&=\rho\bar{F}_k(p_k)(p_k-c)+\rho F_k(p_k)R^a_{\le k-1}(\boldsymbol{p}_{\le k-1})+\Big(\sum_{i=2}^{k-1}\big(\rho^{k-i}\prod_{j=i+1}^{k}F_j(p_j)\big)\big(1-\rho F_i(p_i)\big)+\rho^{k-1}\prod_{i=2}^{k}F_i(p_i)\Big)s\\
&=\rho\bar{F}_k(p_k)(p_k-c)+\rho F_k(p_k)R^a_{\le k-1}(\boldsymbol{p}_{\le k-1})+\Big(\sum_{i=3}^{k-1}\big(\rho^{k-i}\prod_{j=i+1}^{k}F_j(p_j)\big)\big(1-\rho F_i(p_i)\big)+\rho^{k-2}\prod_{i=3}^{k}F_i(p_i)\Big)s.
\end{aligned}$$

With similar manipulations, we can increase the lower index of i in the sum and product up to $k-1$ to ultimately get

$$\begin{aligned}
R^a_{\le k}(\boldsymbol{p}_{\le k}) &= \rho\bar{F}_k(p_k)(p_k-c)+\rho F_k(p_k)R^a_{\le k-1}(\boldsymbol{p}_{\le k-1})+\Big(\rho F_k(p_k)\big)\big(1-\rho F_k(p_{k-1})\big)+\rho^2F_k(p_k)F_{k-1}(p_{k-1})\Big)s\\
&= \rho\bar{F}_k(p_k)(p_k-c)+\rho F_k(p_k)\big(s+R^a_{\le k-1}(\boldsymbol{p}_{\le k-1})\big).
\end{aligned}$$

This concludes the proof for Claim 9. ◇

For feasible prices to (7), $a_1(\boldsymbol{p}_{>1})=0$, which becomes the hypothesis of the next claim. The claim provides necessary condition for each price to be feasible. There, $\boldsymbol{p}^{\le k}_{>j}:=(p_k,\dots,p_{j+1})$ for $j\le k$.

Claim 10. *Consider a price vector $\boldsymbol{p}$ with $s\le p_j\le js$ for $j\in[1:k]$ and a family of functions $\{\tilde{a}_j(\boldsymbol{p}^{\le k}_{>j}):j\in[1:k]\}$ that satisfy recursion (13). If $\tilde{a}_1(\boldsymbol{p}^{\le k}_{>1})=0$, then*

$$\frac{\tilde{a}_j(\boldsymbol{p}^{\le k}_{>j})}{1+\sum_{i=1}^{j-2}\rho^i\Phi^{j-1}_{j-i}}\le F_j(p_j)\le\frac{\tilde{a}_j(\boldsymbol{p}^{\le k}_{>j})}{1+\sum_{i=1}^{j-2}\rho^i\phi^i}\qquad \textit{for } j\in[2:k]. \tag{38}$$

Proof of Claim 10: For ease of exposition, let $a_j=a_j(\boldsymbol{p}^{\le k}_{>j})$. So, $a_1=0$. Using recursion (13), $a_2=(1+\rho a_1)F_2(p_2)=F_2(p_2)$, i.e., (38) holds for $j=2$ if $a_1=0$. Starting with $j=2$, we provide a proof through an induction on j. As the induction hypothesis, we assume (38) holds for some $j=l\in[2:k-1]$. In the induction step, we show that the induction hypothesis holds for $l+1$.

To get an upper bound on $F_{l+1}(p_{l+1})$, we start with $F_l(p_l) \le a_l/(1+\sum_{i=1}^{l-2}\rho^i\phi_{l-i}^{l-1})$ and replace a_l by a_{l+1} by using the recursion (13) to obtain $1+\rho F_l(p_l)(1+\sum_{i=1}^{l-2}\rho^i\phi_{l-i}^{l-1}) \le a_{l+1}/F_{l+1}(p_{l+1})$. Since $s \le p_l$, this inequality implies

$$1+\rho F_l(s)(1+\sum_{i=1}^{l-2}\rho^i\phi_{l-i}^{l-1}) \le a_{l+1}/F_{l+1}(p_{l+1}) \Leftrightarrow 1+\sum_{i=1}^{l-1}\rho^i\phi_{l-i-1}^{l} \le a_{l+1}/F_{l+1}(p_{l+1}) \Leftrightarrow F_{l+1}(p_{l+1}) \le \frac{a_{l+1}}{1+\sum_{i=1}^{l-1}\rho^i\phi_{l-i-1}^{l}}.$$

To get a lower bound on $F_{l+1}(p_{l+1})$, we start with $a_l/(1+\sum_{i=1}^{l-2}\rho^i\Phi_{l-i}^{l-1}) \le F_l(p_l)$ and replace a_l by a_{l+1} by using the recursion (13) to obtain $a_{l+1}/F_{l+1}(p_{l+1}) \le 1+\rho F_l(p_l)(1+\sum_{i=1}^{l-2}\rho^i\Phi_{l-i}^{l-1})$. Since $p_l \le ls$, this inequality implies $a_{l+1}/F_{l+1}(p_{l+1}) \le 1+\rho F_l(ls)+\rho F_l(ls)\sum_{i=1}^{l-2}\rho^i\Phi_{l-i}^{l-1} = 1+\sum_{i=1}^{l-1}\rho^i\Phi_{l+1-i}^{l}$, which is equivalent to $a_{l+1}/(1+\sum_{i=1}^{l-1}\rho^i\Phi_{l+1-i}^{l}) \le F_{l+1}(p_{l+1})$. In conclusion, when (38) holds for $j=l$, it also holds for $j=l+1$. This completes the induction step and the proof of the claim. $\diamond$

Equipped with Claim 10, we present the following claim that facilitates the characterization of the feasible region of (7). In particular, with the definition of $\mathcal{A}_k^O$ below, the feasible region is

$$\{\boldsymbol{p} : \boldsymbol{p} \in \mathcal{A}_{T-1}^O(a_{T-1}),\ a_{T-1} = (1-\gamma-\alpha\gamma)/(\alpha\gamma\rho)\}.$$

CLAIM 11. *For a given $k \ge 2$ and a number a_k', the sets $\mathcal{A}_k^O(a_k')$ and $\mathcal{A}_k^D(a_k')$ defined below are the same.*

$$\mathcal{A}_k^O(a_k') = \Big\{\boldsymbol{p}_{\le k} : s \le p_j \le js \text{ for } j \in [1:k];\ \sum_{i=1}^{k-1}\rho^{k-1-i}\prod_{j=i+1}^{k}F_j(p_j) = a_k'\Big\}$$

$$\mathcal{A}_k^D(a_k') = \Big\{\boldsymbol{p}_{\le k} : s \le p_j \le js \text{ for } j \in [1:k];\ \frac{a_j'}{1+\sum_{i=1}^{j-2}\rho^i\Phi_{j-i}^{j-1}} \le F_j(p_j) \le \frac{a_j'}{1+\sum_{i=1}^{j-2}\rho^i\phi_{j-i}^{j-1}}$$
$$\text{where } a_{j-1}' = (a_j'/F_j(p_j)-1)/\rho \text{ for } j \in [2:k]\Big\}.$$

Proof of Claim 11: It suffices to prove $\mathcal{A}_k^D(a_k') \subseteq \mathcal{A}_k^O(a_k')$ and $\mathcal{A}_k^O(a_k') \subseteq \mathcal{A}_k^D(a_k')$.

To obtain $\mathcal{A}_k^D(a_k') \subseteq \mathcal{A}_k^O(a_k')$, consider an arbitrary price vector $\boldsymbol{p}_{\le k} \in \mathcal{A}_k^D(a_k')$. The inequalities on $F_j(p_j)$, when specialized to $j=2$, yield $a_2' = F_2(p_2)$. Using the recursive equation of a_j''s for $j=k$, we obtain $a_k' = F_k(p_k)+\rho F_k(p_k)a_{k-1}'$. Using the same equations for $j=k-1,\dots,2$,

$$\begin{aligned} a_k' &= F_k(p_k)+\rho F_k(p_k)\Big(F_{k-1}(p_{k-1})+\rho F_{k-1}(p_{k-1})a_{k-2}'\Big) = \sum_{i=k-2}^{k-1}\rho^{k-i-1}\prod_{j=i+1}^{k}F_j(p_j)+\rho^2\Big(\prod_{j=k-1}^{k}F_j(p_j)\Big)a_{k-2}' \\ &= \sum_{i=k-3}^{k-1}\rho^{k-i-1}\prod_{j=i+1}^{k}F_j(p_j)+\rho^3\Big(\prod_{j=k-2}^{k}F_j(p_j)\Big)a_{k-3}' \\ &= \sum_{i=2}^{k-1}\rho^{k-i-1}\prod_{j=i+1}^{k}F_j(p_j)+\rho^{k-2}\Big(\prod_{j=3}^{k}F_j(p_j)\Big)a_2' = \sum_{i=2}^{k-1}\rho^{k-i-1}\prod_{j=i+1}^{k}F_j(p_j)+\rho^{k-2}\Big(\prod_{j=3}^{k}F_j(p_j)\Big)F_2(p_2) \\ &= \sum_{i=1}^{k-1}\rho^{k-i-1}\prod_{j=i+1}^{k}F_j(p_j). \end{aligned}$$

This coupled with $p_j \in [s, js]$ for $j \in [1:k]$ yields $\boldsymbol{p}_{\le k} \in \mathcal{A}_k^O(a_k')$. Hence, $\mathcal{A}_k^D(a_k') \subseteq \mathcal{A}_k^O(a_k')$.

To obtain $\mathcal{A}_k^O(a_k') \subseteq \mathcal{A}_k^D(a_k')$, consider an arbitrary price vector $\boldsymbol{p}_{\le k} \in \mathcal{A}_k^O(a_k')$. We define the sequence $\{a_j' : j \in [1:k]\}$ as follows:

$$a_j' = \sum_{i=1}^{j-1}\rho^{j-1-i}\prod_{t=i+1}^{j}F_t(p_t). \tag{39}$$

So, $a'_1 = 0$. The number a'_k given in the claim statement satisfies (39) as $a'_k = \sum_{i=1}^{k-1} \rho^{k-1-i} \prod_{t=i+1}^{k} F_t(p_t)$ by $\boldsymbol{p}_{\le k} \in \mathcal{A}_k^O(a'_k)$. Also, a'_j's satisfy the recursion in $\mathcal{A}_k^D$ because

$$(a'_j/F_j(p_j) - 1)/\rho = (\sum_{i=1}^{j-1} \rho^{j-i-1} \prod_{t=i+1}^{j} F_t(p_t)/F_j(p_j) - 1)/\rho = (\sum_{i=1}^{j-2} \rho^{j-i-2} \prod_{t=i+1}^{j-1} F_t(p_t)) = a'_{j-1}.$$

What remains is to establish the inequalities on $F_j(p_j)$ in $\mathcal{A}_k^D(a'_k)$ with parameters a'_j's for $\boldsymbol{p}_{\le k} \in \mathcal{A}_k^O(a'_k)$. This is done by using Claim 10 with the new function $\tilde{a}_j(\boldsymbol{p}_{>j}^{\le k})$ defined for $\boldsymbol{p}_{>j}^{\le k} := (p_k, \ldots, p_{j+1})$:

$$\tilde{a}_j(\boldsymbol{p}_{>j}^{\le k}) := \frac{a'_k - \sum_{i=j}^{k-1} \rho^{k-i-1} \prod_{t=i+1}^{k} F_t(p_t)}{\rho^{k-j} \prod_{t=j+1}^{k} F_t(p_t)} \quad \text{for } j \in [1:k],$$

which is for a problem of k prices and a counterpart of (12). Also, $\tilde{a}_1(\boldsymbol{p}_{>1}^{\le k}) = 0$ as $a'_k = \sum_{i=1}^{k-1} \rho^{k-i-1} \prod_{t=i+1}^{k}$ $F_t(p_t)$ by $\boldsymbol{p}_{\le k} \in \mathcal{A}_k^O(a'_k)$. We now relate a'_j to $\tilde{a}_j(\boldsymbol{p}_{>j}^{\le k})$. Writing (39) with index $k = j$, we obtain a series of equivalent equalities

$$\begin{aligned}
\sum_{i=1}^{k-1} \rho^{k-i-1} \prod_{t=i+1}^{k} F_t(p_t) &= a'_k. \\
\sum_{i=1}^{j-1} \rho^{k-i-1} \prod_{t=i+1}^{k} F_t(p_t) &= a'_k - \sum_{i=j}^{k-1} \rho^{k-i-1} \prod_{t=i+1}^{k} F_t(p_t). \\
\Big(\prod_{t=j+1}^{k} F_t(p_t)\Big) \sum_{i=1}^{j-1} \rho^{j-1-i+k-j} \prod_{t=i+1}^{j} F_t(p_t) &= a'_k - \sum_{i=j}^{k-1} \rho^{k-i-1} \prod_{t=i+1}^{k} F_t(p_t). \\
\sum_{i=1}^{j-1} \rho^{j-1-i} \prod_{t=i+1}^{j} F_t(p_t) &= \frac{a'_k - \sum_{i=j}^{k-1} \rho^{k-i-1} \prod_{t=i+1}^{k} F_t(p_t)}{\rho^{k-j} \prod_{t=j+1}^{k} F_t(p_t)}.
\end{aligned}$$

The LHS above is a'_j given by (39) and the RHS is $\tilde{a}_j(\boldsymbol{p}_{>j}^{\le k})$. Therefore, $a'_j = \tilde{a}_j(\boldsymbol{p}_{>j}^{\le k})$ for $j \in [1:k]$. This together with $\tilde{a}_1(\boldsymbol{p}_{>1}^{\le k}) = 0$ imply that $\{\tilde{a}_j(\boldsymbol{p}_{>j}^{\le k}) : j \in [1:k]\}$ satisfy the recursion (13). With $\tilde{a}_1(\boldsymbol{p}_{>1}^{\le k}) = 0$, Claim 10 applies and yields

$$\frac{\tilde{a}_j(\boldsymbol{p}_{>j}^{\le k})}{1 + \sum_{i=1}^{k-2} \rho^i \Phi_{j-i}^{j-1}} \le F_j(p_j) \le \frac{\tilde{a}_j(\boldsymbol{p}_{>j}^{\le k})}{1 + \sum_{i=1}^{j-2} \rho^i \phi_{j-i}^{j-1}} \qquad \text{for } j \in [2:k].$$

Replacing $\tilde{a}_j(\boldsymbol{p}_{>j}^{\le k})$ by a'_j confirm that $\boldsymbol{p}_{\le k}$ satisfies the inequalities in $\mathcal{A}_k^D(a'_k)$. Therefore, we obtain $\boldsymbol{p}_{\le k} \in \mathcal{A}_k^D(a'_k)$ and $\mathcal{A}_k^O(a'_k) \subseteq \mathcal{A}_k^D(a'_k)$. This completes the proof. ◇

For the forthcoming derivations, we recursively specify $\mathcal{A}_k^D(a_k)$ for a given number a_k.

$$\begin{aligned}
\mathcal{A}_k^D(a_k) = \Big\{ \boldsymbol{p}_{\le k} = (p_k, \boldsymbol{p}_{\le k-1}) : s \le p_k \le ks; \frac{a_k}{1 + \sum_{i=1}^{k-2} \rho^i \Phi_{k-i}^{k-1}} \le F_k(p_k) \le \frac{a_k}{1 + \sum_{i=1}^{k-2} \rho^i \phi_{k-i}^{k-1}}; \\
\boldsymbol{p}_{\le k-1} \in \mathcal{A}_{k-1}^D(a_{k-1}) \text{ where } a_{k-1} = (a_k/F_k(p_k) - 1)/\rho \Big\}.
\end{aligned}$$

By Claim 11, the feasible region of the maximization problem (7) is $\{\boldsymbol{p} : \boldsymbol{p} \in \mathcal{A}_{T-1}^O(a_{T-1}),\ a_{T-1} = (1 - \gamma - \alpha\gamma)/(\alpha\gamma\rho)\}$ and its objective function is $R_T^a(\boldsymbol{p}) = R_{\le T-1}^a(\boldsymbol{p}) + s$. So, we can instead maximize $R_{\le T-1}^a(\boldsymbol{p})$. Hence,

$$\max_{\boldsymbol{p}} \Big\{ R_{\le T-1}^a(\boldsymbol{p}) : \boldsymbol{p} \in \mathcal{A}_{T-1}^O((1 - \gamma - \alpha\gamma)/(\alpha\gamma\rho)) \Big\}$$

has the same solution as the maximization problem (7). For $k \in [2:T-1]$ and the feasible set parameterized by a_k, we consider

$$\begin{aligned}
&\max_{\boldsymbol{p}_{\le k}} \Big\{ R^a_{\le k}(\boldsymbol{p}_{\le k}) : \boldsymbol{p}_{\le k} \in \mathcal{A}^O_k(a_k) \Big\} \\
&= \max_{p_k} \Big\{ \rho \bar{F}_k(p_k)(p_k - c) + \rho F_k(p_k) \Big(s + \max_{\boldsymbol{p}_{\le k-1}} \{ R^a_{\le k-1}(\boldsymbol{p}_{\le k-1}) \} \Big) : \boldsymbol{p}_{\le k} \in \mathcal{A}^D_k(a_k) \Big\} \\
&= \max_{p_k} \Big\{ \rho \bar{F}_k(p_k)(p_k - c) + \rho F_k(p_k) \Big(s + \max_{\boldsymbol{p}_{\le k-1}} \{ R^a_{\le k-1}(\boldsymbol{p}_{\le k-1}) \} \Big) : \\
&\qquad \boldsymbol{p}_{\le k-1} \in \mathcal{A}^D_{k-1}(a_{k-1}); s \le p_k \le ks; \\
&\qquad \frac{a_k}{1 + \sum_{i=1}^{k-2} \rho^i \Phi^{k-1}_{k-i}} \le F_k(p_k) \le \frac{a_k}{1 + \sum_{i=1}^{k-2} \rho^i \phi^{k-1}_{k-i}} \text{ where } a_{k-1} = (a_k / F_k(p_k) - 1)/\rho \Big\} \\
&= \max_{p_k} \Big\{ \rho \bar{F}_k(p_k)(p_k - c) + \rho F_k(p_k) \Big(s + \max_{\boldsymbol{p}_{\le k-1}} \{ R^a_{\le k-1}(\boldsymbol{p}_{\le k-1}) : \boldsymbol{p}_{\le k-1} \in \mathcal{A}^O_{k-1}(a_{k-1}) \} \Big) : s \le p_k \le ks; \\
&\qquad \frac{a_k}{1 + \sum_{i=1}^{k-2} \rho^i \Phi^{k-1}_{k-i}} \le F_k(p_k) \le \frac{a_k}{1 + \sum_{i=1}^{k-2} \rho^i \phi^{k-1}_{k-i}} \text{ where } a_{k-1} = (a_k / F_k(p_k) - 1)/\rho \Big\}.
\end{aligned}$$

Above, Claim 9 enables us to cast the initial problem as a nested maximization problem, and Claim 11 enables us to draw alternative representations of the feasible region of the nested problem. These manipulations, with $v_k(a_k) = \max \big\{ R^a_{\le k}(\boldsymbol{p}_{\le k}) : \boldsymbol{p}_{\le k} \in \mathcal{A}^O_k(a_k) \big\}$, yield

$$\begin{aligned}
v_k(a_k) = \max_{p_k} \Big\{ &\rho \bar{F}_k(p_k)(p_k - c) + \rho F_k(p_k) \Big(s + v_{k-1}((a_k / F_k(p_k) - 1)/\rho) \Big) : \\
&s \le p_k \le ks; \frac{a_k}{1 + \sum_{i=1}^{k-2} \rho^i \Phi^{k-1}_{k-i}} \le F_k(p_k) \le \frac{a_k}{1 + \sum_{i=1}^{k-2} \rho^i \phi^{k-1}_{k-i}}.
\end{aligned}$$

Recall that $R^a_{\le 1}(p_{\le 1}) = \rho(s - c)$, which coincides with v_1 in the statement of the lemma. The value functions $\{v_k : k \ge 1\}$ defined above coincide with those in the statement of the lemma. Moreover,

$$\begin{aligned}
&v_{T-1}((1 - \gamma - \alpha\gamma)/(\alpha\gamma\rho)) \\
&= \max_{\boldsymbol{p}_{\le T-1}} \Big\{ R^a_{\le T-1}(\boldsymbol{p}_{\le T-1}) : s \le p_j \le js,\ j \in [1:T-1];\ \sum_{i=1}^{T-2} \rho^{T-i-2} \prod_{j=i+1}^{T-1} F_j(p_j) = (1 - \gamma - \alpha\gamma)/(\alpha\gamma\rho) \Big\} \\
&= \max_{\boldsymbol{p}} \Big\{ R^a_T(\boldsymbol{p}) : s \le p_j \le js,\ j \in [1:T-1];\ \sum_{i=1}^{T-2} \rho^{T-i-2} \prod_{j=i+1}^{T-1} F_j(p_j) = (1 - \gamma - \alpha\gamma)/(\alpha\gamma\rho) \Big\} - s \\
&= R^*_T(\gamma; \alpha)/(\alpha\gamma) - s.
\end{aligned}$$

This completes the proof of the lemma. □

Proof of Lemma 6: We first consider forward construction of intervals. As an induction hypothesis in our proof by induction, we assume $a_k \in [a^l_k, a^u_k]$ for rental state k. By recursion (13), $a_k = (\rho a_{k-1} + 1) F_k(p_k)$, and therefore $a^l_k \le a_k$ is equivalent to $(a^l_k / F_k(p_k) - 1)/\rho \le a_{k-1}$. Since $p_k \le ks$, this inequality implies $(a^l_k / F_k(ks) - 1)/\rho \le a_{k-1}$. By (19), the LHS of this inequality can be simplified to

$$\begin{aligned}
(a^l_k / F_k(ks) - 1)/\rho &= \frac{\frac{(1-\gamma)/(\alpha\gamma)}{\rho^{T-k} \Phi^{T-1}_{k+1}} - \sum_{i=k+1}^{T} \frac{1}{\rho^{i-k} \Phi^{i-1}_{k+1}}}{\rho F_k(ks)} - \frac{1}{\rho} = \frac{(1-\gamma)/(\alpha\gamma)}{\rho^{T-k+1} \Phi^{T-1}_k} - \sum_{i=k+1}^{T} \frac{1}{\rho^{i-k+1} \Phi^{i-1}_k} - \frac{1}{\rho} \\
&= \frac{(1-\gamma)/(\alpha\gamma)}{\rho^{T-k+1} \Phi^{T-1}_k} - \sum_{i=k}^{T} \frac{1}{\rho^{i-k+1} \Phi^{i-1}_k} = a^l_{k-1}.
\end{aligned}$$

So, $a_k^l \le a_k$ implies that $a_{k-1}^l \le a_{k-1}$.

Next, by recursion (13), $a_k \le a_k^u$ is equivalent to $a_{k-1} \le (a_k^u/F_k(p_k) - 1)/\rho$. Since $s \le p_k$, this inequality implies $a_{k-1} \le (a_k^u/F_k(s) - 1)/\rho$. By (19), the RHS of this inequality can be simplified to

$$\begin{aligned}(a_k^u/F_k(s)-1)/\rho &= \frac{\frac{(1-\gamma)/(\alpha\gamma)}{\rho^{T-k}\phi^{T-k-1}} - \sum_{i=k+1}^{T}\frac{1}{\rho^{i-k}\phi_{k+1}^{i-1}}}{\rho F_k(s)} - \frac{1}{\rho} = \frac{(1-\gamma)/(\alpha\gamma)}{\rho^{T-k+1}\phi_k^{T-1}} - \sum_{i=k+1}^{T}\frac{1}{\rho^{i-k+1}\phi_k^{i-1}} - \frac{1}{\rho} \\ &= \frac{(1-\gamma)/(\alpha\gamma)}{\rho^{T-k+1}\phi_k^{T-1}} - \sum_{i=k}^{T}\frac{1}{\rho^{i-k+1}\phi_k^{i-1}} = a_{k-1}^u.\end{aligned}$$

Therefore, $a_k \le a_k^u$ implies that $a_{k-1} \le a_{k-1}^u$. Last, we have $a_{T-1} = (1-\gamma-\alpha\gamma)/(\alpha\gamma\rho) \in [a_{T-1}^l, a_{T-1}^u] = [(1-\gamma-\alpha\gamma)/(\alpha\gamma\rho), (1-\gamma-\alpha\gamma)/(\alpha\gamma\rho)]$, which initiates our induction and completes the proof.

We now consider backward construction of intervals. For the proof of the only-if part, suppose that the constraint in (18) holds. Then in particular for $k=2$, (20) gives $p_2 = F_2^{-1}(a_2)$. We can rewrite $p_2 \in [s, 2s]$ as $a_2 \in [F_2(s), F_2(2s)] = [\tilde{a}_2^l, \tilde{a}_2^u]$. To complete the proof of the only-if part with induction, we show that $a_{t-1} \in [\tilde{a}_{t-1}^l, \tilde{a}_{t-1}^u]$ implies $a_t \in [\tilde{a}_t^l, \tilde{a}_t^u]$. From (13), $a_{t-1} = (a_t/F_t(p_t) - 1)/\rho$ is decreasing in $F_t(p_t)$. Therefore for a given a_t, its minimum is $(a_t/F_t(ts) - 1)/\rho$ and its maximum is $(a_t/F_t(s) - 1)/\rho$. As $a_{t-1} \in [\tilde{a}_{t-1}^l, \tilde{a}_{t-1}^u]$, we have

$$\begin{aligned}\frac{1}{\rho}\Big(\frac{a_t}{F_t(s)} - 1\Big) \ge a_{t-1} \ge \sum_{i=1}^{t-2}\rho^{i-1}\phi_{t-i}^{t-1} \Leftrightarrow a_t \ge F_t(s) + \sum_{i=1}^{t-2}\rho^{i}\phi_{t-i}^{t-1} = \sum_{i=1}^{t-1}\rho^{i-1}\phi_{t-i+1}^{t} = \tilde{a}_t^l,\\ \frac{1}{\rho}\Big(\frac{a_t}{F_t(ts)} - 1\Big) \le a_{t-1} \le \sum_{i=1}^{t-2}\rho^{i-1}\Phi_{t-i}^{t-1} \Leftrightarrow a_t \le F_t(ts) + \sum_{i=1}^{t-2}\rho^{i}\Phi_{t-i}^{t-1} = \sum_{i=1}^{t-1}\rho^{i-1}\Phi_{t-i+1}^{t} = \tilde{a}_t^u.\end{aligned}$$

Therefore, $a_t \in [\tilde{a}_t^l, \tilde{a}_t^u]$ and the proof of the only-if is complete.

For the proof of the if part, suppose that $a_2 = F_2(p_2)$ and $a_k \in [\tilde{a}_k^l, \tilde{a}_k^u]$ holds for (20) and $k \ge 2$. As such, $a_1 = (a_2/F_2(p_2) - 1)/\rho = 0$ from recursion (13). From Claim 10 in the proof of Proposition 4, the constraint in (18) holds for all $k \ge 2$, and the if part is proved. □

C. Derivations and Supplementary Analyses

C.1. Equivalence of (3) and (6)

Since we consider fixed α, we drop it from the argument of the functions below. Let $\boldsymbol{p}^* = (p_{T-1}^*, \ldots, p_1^*)$ be the solution to (3) and γ^* be the solution to (6). We need to show that $R_T^*(\gamma^*) = R_T(\boldsymbol{p}^*)$. Let $\hat{\gamma} = \pi_T(\boldsymbol{p}^*)$, so $R_T^*(\hat{\gamma}) = R_T(\boldsymbol{p}^*)$. We have $R_T^*(\hat{\gamma}) \le R_T^*(\gamma^*)$ by the optimality of γ^* for (6). Moreover, we cannot have $R_T^*(\hat{\gamma}) < R_T^*(\gamma^*)$. If so, then the solution buyout price path to the inner problem of (6) for $\gamma = \gamma^*$ results in a profit rate higher than $R_T(\boldsymbol{p}^*)$ contradicting the optimality of $\boldsymbol{p}^*$ for (3). Therefore, we have $R_T(\boldsymbol{p}^*) = R_T^*(\hat{\gamma}) = R_T^*(\gamma^*)$ and solutions of (3) and (6) yield the same profit rate.

C.2. Profit Rate via Renewal Theory

An alternative route to reach the maximization problem in (7) is via the renewal reward theorem. Rental initiation for each slot also initiates a renewal cycle. Each cycle includes periods of rental followed by periods of idleness. The expected profit obtained during each cycle with buyout price path $\boldsymbol{p}$ is $R_T^a(\boldsymbol{p})$.

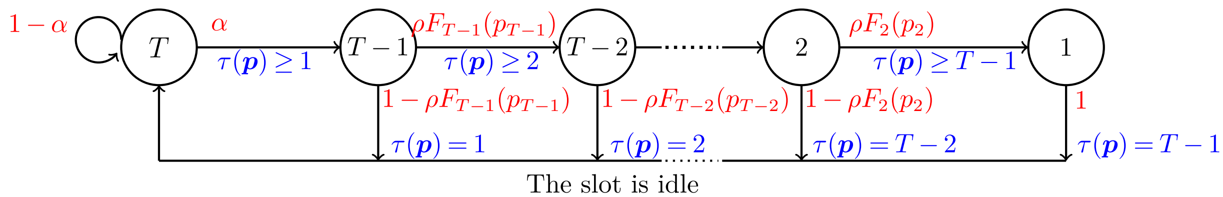


Figure 9 **The inventory slot transition probabilities and associated rental duration $\tau(\boldsymbol{p})$.**

The number of idle periods depends only on the initiation probability α and it is a geometric random variable with success probability α. Hence, the expected number of idle periods in a renewal cycle is $1/\alpha$. The number of rental periods with buyout prices $\boldsymbol{p}$ is also random and denoted by $\tau(\boldsymbol{p})$; see Figure 9. We have $\tau(\boldsymbol{p}) \in [1:T-1]$ and the tail probability of $\tau(\boldsymbol{p})$ is given by (8). That is, a rental agreement lasts at least i periods if the slot transitions from state j to state $j-1$ wp $\rho F_j(p_j)$ for $j \in [T-i+1, T-1]$. These transitions bring the slot to state $T-i$, so the rental agreement lasts at least i periods. Given the initiation of a rental, the slot transitions from state T to $T-1$ with certainty.

By the renewal reward theorem, the profit rate with $\boldsymbol{p}$ converges almost surely to the ratio of expected profit per renewal cycle $R_T^a(\boldsymbol{p})$ to the expected cycle length $\mathbb{E}(\tau(\boldsymbol{p}))+1/\alpha$. To maximize the profit rate over a long run, we can consider

$$\max_{\boldsymbol{p}:p_j\in[s,sj]} \frac{R_T^a(\boldsymbol{p})}{\mathbb{E}(\tau(\boldsymbol{p}))+1/\alpha}.$$

The profit rate maximization problem can be recast for the fixed idleness rate γ after noting the definition of idleness in terms of expected rental and idleness durations: $\gamma = 1/\alpha/(\mathbb{E}(\tau(\boldsymbol{p}))+1/\alpha)$, which leads to an alternative representation of the expected rental duration.

$$\mathbb{E}(\tau(\boldsymbol{p})) = \frac{1-\gamma}{\alpha\gamma},$$

which can be made explicit via (32):

$$1+\sum_{i=1}^{T-2}\rho^{T-1-i}\prod_{j=i+1}^{T-1}F_j(p_j) = \frac{1-\gamma}{\alpha\gamma}.$$

This is the utilization constraint in (7). To this constraint, appending $p_j \in [s, sj]$ for $j \in [1:T-1]$, we arrive at the feasible set $\mathcal{A}_{T-1}^O\big((1-\gamma-\alpha\gamma)/(\alpha\gamma\rho)\big) := \{\boldsymbol{p}: p_k \in [s, ks]\ \text{ for } k \in [1:T-1],\ \mathbb{E}(\tau(\boldsymbol{p})) = (1-\gamma)/(\alpha\gamma)\}$. This set is also defined in the proof of Proposition 4 but without using $\tau(\boldsymbol{p})$.

We return to the maximization of profit rate.

$$\begin{aligned}\max_{\boldsymbol{p}:p_k\in[s,ks]} \frac{R_T^a(\boldsymbol{p})}{\mathbb{E}(\tau(\boldsymbol{p}))+1/\alpha} &= \max_{\gamma} \max_{\boldsymbol{p}\in\mathcal{A}_{T-1}^O((1-\gamma-\alpha\gamma)/(\alpha\gamma\rho))} \frac{R_T^a(\boldsymbol{p})}{(1-\gamma)/(\alpha\gamma)+1/\alpha} \\ &= \max_{\gamma} \alpha\gamma \max_{\boldsymbol{p}\in\mathcal{A}_{T-1}^O((1-\gamma-\alpha\gamma)/(\alpha\gamma\rho))} R_T^a(\boldsymbol{p}).\end{aligned}$$

The inner optimization problem over $\boldsymbol{p}$ is exactly the maximization problem in (7). In conclusion, besides the argument leading to (7) in the main body, we have justified (7) via renewal theory.

C.3. Derivation of (13) from the Utilization Constraint

The equivalence follows after algebraic manipulations of the utilization constraint in (7):

$$
\begin{aligned}
&\sum_{i=1}^{T-2}\rho^{T-2-i}\prod_{j=i+1}^{T-1}F_j(p_j)=\frac{1-\gamma}{\alpha\gamma\rho}-\frac{1}{\rho}\Leftrightarrow\sum_{i=1}^{k-1}\rho^{T-2-i}\prod_{j=i+1}^{T-1}F_j(p_j)+\sum_{i=k}^{T-2}\rho^{T-2-i}\prod_{j=i+1}^{T-1}F_j(p_j)=\frac{1-\gamma}{\alpha\gamma\rho}-\frac{1}{\rho}\\
&\Leftrightarrow\sum_{i=1}^{k-1}\rho^{T-2-i}\prod_{j=i+1}^{T-1}F_j(p_j)=\frac{1-\gamma-\alpha\gamma}{\alpha\gamma\rho}-\sum_{i=k}^{T-2}\rho^{T-2-i}\prod_{j=i+1}^{T-1}F_j(p_j)\\
&\Leftrightarrow\sum_{i=1}^{k-1}\rho^{k-1-i+T-k-1}\prod_{j=k+1}^{T-1}F_j(p_j)\prod_{j=i+1}^{k}F_j(p_j)=\frac{1-\gamma-\alpha\gamma-\alpha\gamma\rho\sum_{i=k}^{T-2}\rho^{T-2-i}\prod_{j=i+1}^{T-1}F_j(p_j)}{\alpha\gamma\rho}\\
&\Leftrightarrow\sum_{i=1}^{k-1}\rho^{k-1-i}\prod_{j=i+1}^{k}F_j(p_j)=\frac{1-\gamma-\alpha\gamma-\alpha\gamma\sum_{i=k}^{T-2}\rho^{T-1-i}\prod_{j=i+1}^{T-1}F_j(p_j)}{\alpha\gamma\rho\rho^{T-k-1}\prod_{j=k+1}^{T-1}F_j(p_j)}\\
&\Leftrightarrow\sum_{i=1}^{k-1}\rho^{k-1-i}\prod_{j=i+1}^{k}F_j(p_j)=\frac{1-\gamma-\alpha\gamma\sum_{i=k}^{T-1}\rho^{T-1-i}\prod_{j=i+1}^{T-1}F_j(p_j)}{\alpha\gamma\rho^{T-k}\prod_{j=k+1}^{T-1}F_j(p_j)}=:a_k(\boldsymbol{p}_{>k}).
\end{aligned}
$$

C.4. Derivations of (16) and (17)

To derive constraint (16) on p_k under fixed $p_{>k}$, we must have the LHS of (15) at $p_j=s$ for $j\le k-1$ to be no larger than its RHS:

$$
\begin{aligned}
&\alpha\gamma\sum_{i=1}^{k-1}\rho^{T-i-1}\prod_{j=i+1}^{T-1}F_j(p_j)\le 1-\gamma-\alpha\gamma\sum_{i=k}^{T-1}\rho^{T-i-1}\prod_{j=i+1}^{T-1}F_j(p_j),\quad p_j=s,\ j\le k-1\\
\Leftrightarrow\ &\alpha\gamma\prod_{j=k}^{T-1}F_j(p_j)\Big(\sum_{i=1}^{k-1}\rho^{T-i-1}\phi_{i+1}^{k-1}\Big)\le 1-\gamma-\alpha\gamma\sum_{i=k}^{T-1}\rho^{T-i-1}\prod_{j=i+1}^{T-1}F_j(p_j)\\
\Leftrightarrow\ &F_k(p_k)\alpha\gamma\rho^{T-k}\prod_{j=k+1}^{T-1}F_j(p_j)\Big(1+\sum_{i=1}^{k-2}\rho^{k-i-1}\phi_{i+1}^{k-1}\Big)\le 1-\gamma-\alpha\gamma\sum_{i=k}^{T-1}\rho^{T-i-1}\prod_{j=i+1}^{T-1}F_j(p_j)\\
\Leftrightarrow\ &F_k(p_k)\le\frac{1-\gamma-\alpha\gamma\sum_{i=k}^{T-1}\rho^{T-i-1}\prod_{j=i+1}^{T-1}F_j(p_j)}{\alpha\gamma\rho^{T-k}\prod_{j=k+1}^{T-1}F_j(p_j)\big(1+\sum_{i=1}^{k-2}\rho^{i}\phi_{k-i}^{k-1}\big)}=\frac{a_k}{1+\sum_{i=1}^{k-2}\rho^{i}\phi_{k-i}^{k-1}}.
\end{aligned}
$$

Similarly for constraint (17), we must have the LHS of (15) when $p_j=js$ for $j\le k-1$ to be no less than its RHS:

$$
\begin{aligned}
&\alpha\gamma\sum_{i=1}^{k-1}\rho^{T-i-1}\prod_{j=i+1}^{T}F_j(p_j)\ge 1-\gamma-\alpha\gamma\sum_{i=k}^{T-1}\rho^{T-i-1}\prod_{j=i+1}^{T}F_j(p_j),\quad p_j=js,\ j\le k-1\\
\Leftrightarrow\ &\alpha\gamma\prod_{j=k}^{T}F_j(p_j)\Big(\sum_{i=1}^{k-1}\rho^{T-i-1}\Phi_{i+1}^{k-1}\Big)\ge 1-\gamma-\alpha\gamma\sum_{i=k}^{T-1}\rho^{T-i-1}\prod_{j=i+1}^{T}F_j(p_j)\\
\Leftrightarrow\ &F_k(p_k)\alpha\gamma\rho^{T-k}\prod_{j=k+1}^{T-1}F_j(p_j)\Big(1+\sum_{i=1}^{k-2}\rho^{k-i-1}\Phi_{i+1}^{k-1}\Big)\ge 1-\gamma-\alpha\gamma\sum_{i=k}^{T-1}\rho^{T-i-1}\prod_{j=i+1}^{T}F_j(p_j)\\
\Leftrightarrow\ &F_k(p_k)\ge\frac{1-\gamma-\alpha\gamma\sum_{i=k}^{T-1}\rho^{T-i-1}\prod_{j=i+1}^{T}F_j(p_j)}{\alpha\gamma\rho^{T-k}\prod_{j=k+1}^{T-1}F_j(p_j)(1+\sum_{i=1}^{k-2}\rho^{k-i-1}\Phi_{i+1}^{k-1})}=\frac{a_k}{1+\sum_{i=1}^{k-2}\rho^{i}\Phi_{k-i}^{k-1}}.
\end{aligned}
$$

D. Analysis of the GMC

The state of the GMC is a T-dimensional vector $\boldsymbol{n}=(n_T,n_{T-1},\cdots,n_1)$ with its i-th to last element denoting the number of slots with rental state i. As the GMC has I inventory slots, $\sum_{i=1}^{T}n_i=I$ for any state $\boldsymbol{n}$. We denote by $\mathcal{N}(I)=\{\boldsymbol{n}:\ n_i\ge 0,\ \sum_{i=1}^{T}n_i=I\}$, the set containing all the possible states of the GMC. The cardinality of $\mathcal{N}(I)$ is $\binom{I+T-1}{I}$, which quickly explodes in both T and I. Our purpose is to calculate $\alpha(I;\boldsymbol{p})$, the probability that a slot initiates an agreement, given it is in the idle state. Naturally, $\alpha(I;\boldsymbol{p})$ is a decreasing function of I for a given demand distribution D for rentals; the larger the inventory is, the more the number of idle slots on average is, and hence the less is the probability that an idle slot initiates an agreement. We first derive the GMC's state transition probabilities and then find the stationary probability of each state.

D.1. GMC's State Transition Probabilities

We consider the arbitrary states $\boldsymbol{n}=(n_T,n_{T-1},\cdots,n_1)$ and $\boldsymbol{n}'=(n'_T,n'_{T-1},\cdots,n'_1)$ and aim to calculate the probability $\omega_{\boldsymbol{nn}'}$ with which the GMC transitions from $\boldsymbol{n}$ to $\boldsymbol{n}'$. Let $C_i^{n_i}$ be the random number of slots, out of the n_i slots in rental state i, that transition to state $i-1$. Because all the n_1 slots with rental state 1 transition to the idle state (rental state T) with certainty, $C_1^{n_1}=0$ wp 1. Moreover, the other non-idle slots either become idle (via a buyout or a return) or their rental states decrease by 1 (via continuing the rental). As such, for all $i\in[2:T]$, to have n'_{i-1} slots with rental state $i-1$ after the $\boldsymbol{n}-\boldsymbol{n}'$ transition, n'_{i-1} slots out of the n_i slots with rental state i must transition to rental state $i-1$, i.e, $C_i^{n_i}=n'_{i-1}$, and the remaining $n_i-n'_{i-1}$ slots must become idle. Note that $n_i\geq n'_{i-1}$ for $i\in[2:T]$, is a necessary condition for $\boldsymbol{n}$ and $\boldsymbol{n}'$ to communicate. The transition under study occurs if and only if $C_i^{n_i}=n'_{i-1}$ for $i\in[2:T]$ and $C_1^{n_i}=0$:

$$\mathrm{P}(C_1^{n_1}=0,C_i^{n_i}=n'_{i-1}\text{ for }i\in[2:T])=\mathrm{P}(C_1^{n_1}=0)\prod_{i=2}^{T}\mathrm{P}(C_i^{n_i}=n'_{i-1})=\prod_{i=2}^{T}\mathrm{P}(C_i^{n_i}=n'_{i-1}),\tag{40}$$

where the first and second equalities hold as non-idle slots evolve independently and $C_1^{n_1}=0$ wp 1, respectively.

Next, we calculate each of the probabilities in the final product in (40) for two cases of $i\in[2:T-1]$ and $i=T$. We denote the Binomial random variable with n trials and success probability q by $X(n,q)$. For $i\in[2:T-1]$, each non-idle slot transitions from rental state i to $i-1$ wp $\rho F_i(p_i)$ and transitions to the idle rental state wp $1-\rho+\rho\bar{F}_i(p_i)=1-\rho F_i(p_i)$. As such, $C_i^{n_i}$ has Binomial distribution with n_i trials and success probability of $\rho F_i(p_i)$:

$$\mathrm{P}(C_i^{n_i}=n'_{i-1})=\mathrm{P}(X(n_i,\rho F_i(p_i))=n'_{i-1})\quad\text{for}\quad i\in[2:T-1].\tag{41}$$

The event $C_T^{n_T}=n'_{T-1}$ involves the random demand D since $C_T^{n_T}$ is the number of idle slots that become non-idle and initiate a rental agreement in the transition; if $n'_{T-1}=n_T$, i.e., if all the n_T idle slots initiate an agreement, then the event $C_T^{n_T}=n'_{T-1}$ occurs if and only if $D\geq n_T=n'_{T-1}$. On the other hand, if $n'_{T-1}<n_T$, i.e., if some of the idle slots initiate a rental agreement, then the event $C_T^{n_T}=n'_{T-1}$ occurs if and only if $D=n'_{T-1}$. Hence,

$$\mathrm{P}(C_T^{n_T}=n'_{T-1})=\mathrm{P}(D\geq n'_{T-1})\mathbb{1}_{n'_{T-1}=n_T}+\mathrm{P}(D=n'_{T-1})\mathbb{1}_{n'_{T-1}<n_T}.\tag{42}$$

Inserting the probabilities in (41) and (42) into (40), we get the probability $\omega_{\boldsymbol{nn}'}(\boldsymbol{p},I)$ of the GMC transitioning from state $\boldsymbol{n}=(n_T,\cdots,n_1)$ to $\boldsymbol{n}'=(n'_T,\cdots,n'_1)$ as

$$\omega_{\boldsymbol{nn}'}(\boldsymbol{p},I)=\Big(\mathrm{P}(D\geq n'_{T-1})\mathbb{1}_{n'_{T-1}=n_T}+\mathrm{P}(D=n'_{T-1})\mathbb{1}_{n'_{T-1}<n_T}\Big)\prod_{i=2}^{T-1}\mathrm{P}(X(n_i,\rho F_i(p_i))=n'_{i-1})$$
$$\text{for }\boldsymbol{n},\boldsymbol{n}'\in\mathcal{N}(I)\text{ with }n_i\geq n'_{i-1}\text{ for }i\in[2:T].\tag{43}$$

D.2. The GMC's Agreement Initiation Probability and Utilization Rate

Next, we calculate $\alpha(I;\boldsymbol{p})$, which is identical for all the I slots due to symmetry. So wlog, we focus on slot 1 for derivation of $\alpha(I;\boldsymbol{p})$. Let $\mathcal{N}_m=\{\boldsymbol{n}:n_T=m\}$ be the class of the GMC states with $m\le I$ idle slots. Suppose that the GMC is in state $\boldsymbol{n}\in\mathcal{N}_m$. Because each slot is equally likely to be one of the idle slots, slot 1 is idle wp m/I. Conditional on slot 1 being idle and realized demand being d, slot 1 initiates an agreement wp $\min\{d,m\}/m$. Accordingly, the probability of slot 1 initiating an agreement is $\sum_{d=1}^{m-1}(d/m)\mathrm{P}(D=d)+\mathrm{P}(D\ge m)$. These probabilities are required for derivation of $\alpha(I;\boldsymbol{p})$ through the manipulations below:

$$
\begin{aligned}
\alpha(I;\boldsymbol{p}) &= \mathrm{P}(\text{slot 1 initiates an agreement} \mid \text{slot 1 is idle})\\
&= \sum_{m=1}^{I}\mathrm{P}(\text{slot 1 initiates an agreement} \mid \text{slot 1 is idle, } m \text{ idle slots})\mathrm{P}(m \text{ idle slots} \mid \text{slot 1 is idle})\\
&= \sum_{m=1}^{I}\Big(\sum_{d=1}^{m-1}(d/m)\mathrm{P}(D=d)+\mathrm{P}(D\ge m)\Big)\frac{\mathrm{P}(m \text{ idle slots})\mathrm{P}(\text{slot 1 is idle} \mid m \text{ idle slots})}{\sum_{i=1}^{I}\mathrm{P}(i \text{ idle slots})\mathrm{P}(\text{slot 1 is idle} \mid i \text{ idle slots})}\\
&= \sum_{m=1}^{I}\Big(\sum_{d=1}^{m-1}(d/m)\mathrm{P}(D=d)+\mathrm{P}(D\ge m)\Big)\frac{\sum_{\boldsymbol{n}\in\mathcal{N}_m}\pi_{\boldsymbol{n}}(I;\boldsymbol{p})(m/I)}{\sum_{i=1}^{I}\sum_{\boldsymbol{n}\in\mathcal{N}_i}\pi_{\boldsymbol{n}}(I;\boldsymbol{p})(i/I)}\\
&= \frac{\sum_{m=1}^{I}\Big(\sum_{d=1}^{m-1}d\mathrm{P}(D=d)+m\mathrm{P}(D\ge m)\Big)\sum_{\boldsymbol{n}\in\mathcal{N}_m}\pi_{\boldsymbol{n}}(I;\boldsymbol{p})}{\sum_{i=1}^{I}i\sum_{\boldsymbol{n}\in\mathcal{N}_i}\pi_{\boldsymbol{n}}(I;\boldsymbol{p})},
\end{aligned}
\tag{44}
$$

where $\pi_{\boldsymbol{n}}(I;\boldsymbol{p})$ is the stationary probability of state $\boldsymbol{n}$ of GMC with I slots under price path $\boldsymbol{p}$. As evident in (44), $\alpha(I;\boldsymbol{p})$ is not slot specific and same for all slots. We have this symmetry as m/I is the steady-state probability for slot 1 being idle, conditional on m idle slots.

D.3. The GMC's Profit Rate

Using the stationary probabilities for a GMC with I slots each offering agreements of term T with the price path $\boldsymbol{p}$, we formulate the profit rate:

$$
\mathcal{R}_T(\boldsymbol{p},I)=\sum_{\boldsymbol{n}\in\mathcal{N}(I)}\pi_{\boldsymbol{n}}(\boldsymbol{p},I)\sum_{\boldsymbol{n}'\in\mathcal{N}(I)}w_{\boldsymbol{n}\boldsymbol{n}'}(\boldsymbol{p},I)r_{\boldsymbol{n}\boldsymbol{n}'}(\boldsymbol{p},I),
$$

where $r_{\boldsymbol{n}\boldsymbol{n}'}(\boldsymbol{p},I)$ is the expected profit for the $\boldsymbol{n}-\boldsymbol{n}'$ transition.

In the $\boldsymbol{n}-\boldsymbol{n}'$ transition, the number of slots out of the n_i slots that transition from rental state i to $i-1$ is n'_{i-1} for $i\in[2:T]$. All these transitions generate s in profit. The rest of the slots, i.e., $n_i-n'_{i-1}$ slots, transition to the idle state. The profits associated with these transitions depend on the number of buyouts. Let $B_i\le n_i-n'_{i-1}$ be the random number of slots out of the $n_i-n'_{i-1}$ slots that return to the idle state from rental state i via buyout. Each of these B_i slots generates the revenue p_i and incurs the cost c. So, conditional on B_i buyouts, the profit obtained from transitions from state i is $B_i(p_i-c)+n'_{i-1}s$. The event $B_i=j\in[0:n_i-n'_{i-1}]$ occurs if j slots become idle with a buyout —each occurring wp $\rho\bar{F}_i(p_i)$— and $n_i-n'_{i-1}-j$ slots become idle with a return —each occurring wp $1-\rho$. Burrowing notation of Appendix D.1,

$$
\begin{aligned}
\mathrm{P}(B_i=j|C_i^{n_i}=n'_{i-1}) &= \frac{\mathrm{P}(C_i^{n_i}=n'_{i-1},B_i=j)}{\mathrm{P}(C_i^{n_i}=n'_{i-1})}=\frac{\binom{n_i}{n'_{i-1}}\binom{n_i-n'_{i-1}}{j}(\rho F_i(p_i))^{n'_{i-1}}(\rho\bar{F}_i(p_i))^{j}(1-\rho)^{n_i-n'_{i-1}-j}}{\binom{n_i}{n'_{i-1}}(\rho F_i(p_i))^{n'_{i-1}}(1-\rho F_i(p_i))^{n_i-n'_{i-1}}}\\
&= \binom{n_i-n'_{i-1}}{j}\left(\frac{\rho\bar{F}_i(p_i)}{1-\rho F_i(p_i)}\right)^{j}\left(\frac{1-\rho}{1-\rho F_i(p_i)}\right)^{n_i-n'_{i-1}-j}
\end{aligned}
$$

$$= \mathrm{P}\Big(X(n_i - n'_{i-1}, \frac{\rho \bar{F}_i(p_i)}{1-\rho F_i(p_i)}) = j\Big) \quad \text{for} \quad i \in [2:T-1],$$

and accordingly for the expected profit $r^i_{\boldsymbol{nn'}}(p_i)$ associated with slots leaving state i, we have

$$r^i_{\boldsymbol{nn'}}(p_i) = n'_{i-1}s + \frac{(n_i - n'_{i-1})\rho \bar{F}_i(p_i)}{1-\rho F_i(p_i)}(p_i - c) \quad \text{for} \quad i \in [2:T-1].$$

Each slot that is in rental state 1 transitions to the idle state with certainty, either via sales wp ρ and the profit $s-c$ or via item return wp $(1-\rho)$ and no revenue. The profit associated with slots leaving state 1 is $r^1_{\boldsymbol{nn'}} = n_1\rho(s-c)$. Finally, the profit from the n'_{T-1} slots that initiate an agreement is $r^T_{\boldsymbol{nn'}} = n'_{T-1}s$. In conclusion, we find $r_{\boldsymbol{nn'}}(\boldsymbol{p}, I) = r^T_{\boldsymbol{nn'}} + \sum_{i=2}^{T-1} r^i_{\boldsymbol{nn'}}(p_i) + r^1_{\boldsymbol{nn'}}$ for communicating states $\boldsymbol{n}, \boldsymbol{n'} \in \mathcal{N}(I)$.

E. The Independence Approximation: Supplementary Discussion and Evaluation

E.1. Approximation of Agreement Initiation Probability under an Arbitrary Price Path

In (21), we approximate the agreement initiation probability under a fixed utilization and direct our attention to buyout price paths $\boldsymbol{p}$ that yield the utilization. However, we can approximate this probability outside the context of fixed utilization for an arbitrary buyout price path $\boldsymbol{p}$. Under the independence approximation, the stationary probability of any slot being idle is $\pi_T(\alpha; \boldsymbol{p})$, obtained from Lemma 1, identical to and independent of other slots. Conditional on slot 1 being idle, there are m total idle slots when $m-1$ of the $I-1$ other independent slots are idle. This probability is equal to $\mathrm{P}\big(X(I-1, \pi_T(\alpha; \boldsymbol{p})) = m-1\big)$, which we use to find α:

$$\begin{aligned}
\alpha &= \mathrm{P}(\text{slot 1 initiates an agreement} \mid \text{slot 1 is idle}) \\
&= \sum_{m=1}^{I} \mathrm{P}(\text{slot 1 initiates an agreement} \mid \text{slot 1 is idle, } m \text{ idle slots})\mathrm{P}(m \text{ idle slots} \mid \text{slot 1 is idle}) \\
&= \sum_{m=1}^{I} \Big(\sum_{d=1}^{m-1} (d/m)\mathrm{P}(D=d) + \mathrm{P}(D \geq m)\Big)\binom{I-1}{m-1}\pi_T(\boldsymbol{p};\alpha)^{m-1}(1-\pi_T(\boldsymbol{p};\alpha))^{I-m}. \qquad (45)
\end{aligned}$$

We define $\alpha^{\mathcal{I}}(I; \boldsymbol{p})$ to be the solution in α to (45), which serves as our approximation for the agreement initiation probability under price path $\boldsymbol{p}$. In the special case of $I=1$, from (45) and (44), $\alpha^{\mathcal{I}}(1; \boldsymbol{p}) = \alpha(1; \boldsymbol{p}) = \mathrm{P}(D \geq 1)$, as expected. When we fix idleness at γ as in §6.1, we let γ be $\pi_T(\boldsymbol{p}; \alpha^{\mathcal{I}}(I; \boldsymbol{p}))$, and with a slight abuse of notation we write $\alpha^{\mathcal{I}}(I, \gamma)$ for $\alpha^{\mathcal{I}}(I; \boldsymbol{p})$. Substituting γ for $\pi_T(\boldsymbol{p}; \alpha^{\mathcal{I}}(I; \boldsymbol{p}))$ in (45) yields (21).

E.2. Evaluation of the Independence Approximation

For the calculations of this section, we consider exponentially distributed valuations with mean \$120 and Poisson distributed demand D with mean μ, i.e., $F_k(p) = F(p) = 1 - e^{-p/120}$ for $k \geq 2$ and $\mathrm{P}(D=d) = \mu^d e^{-\mu}/(d!)$. Also, we fix the rental fee at \$60 per period and the replacement cost at \$80.

We first evaluate the performance of the independence approximation for a GMC with $T=6$. To do so, for a common price path $\boldsymbol{p}$, we obtain $\alpha(I; \boldsymbol{p})$ from (44) and $\alpha^{\mathcal{I}}(I; \boldsymbol{p})$ from (45) and then determine the absolute approximation error $|\alpha(I; \boldsymbol{p}) - \alpha^{\mathcal{I}}(I; \boldsymbol{p})|$. We calculate the error for each possible combination of $\mu \in \{1, 2, 3\}$ and $I \in \{1, 3, 6, 9, 12\}$. For each combination, we set $\boldsymbol{p}$ equal to the approximate optimal buyout prices (see the discussion surrounding (23) on how we obtain these prices) for calculation of $\alpha(I; \boldsymbol{p})$ and $\alpha^{\mathcal{I}}(I; \boldsymbol{p})$.

Table 2 reports the absolute approximation errors for each combination of μ and I as well as the utilization $1-\gamma$ of the GMC at the considered price path. Except for two values, the utilizations are between 45% and

Table 2 **Rounded independence approximation errors for $T=6$ and $s=60$.**

	$\mu=1$				$\mu=2$				$\mu=3$			
I	$1-\gamma$	α	$\alpha^{\mathcal{I}}$	$\lvert$error$\rvert$	$1-\gamma$	α	$\alpha^{\mathcal{I}}$	$\lvert$error$\rvert$	$1-\gamma$	α	$\alpha^{\mathcal{I}}$	$\lvert$error$\rvert$
1	0.65	0.63	0.63	.000	0.71	0.83	0.86	.000	0.73	0.95	0.95	.000
3	0.58	0.49	0.51	.015	0.69	0.79	0.79	.005	0.72	0.91	0.91	.001
6	0.45	0.31	0.33	.012	0.64	0.65	0.67	.020	0.70	0.84	0.84	.007
9	0.33	0.18	0.19	.001	0.57	0.50	0.53	.024	0.67	0.74	0.76	.019
12	0.24	0.12	0.12	.000	0.47	0.36	0.39	.009	0.62	0.63	0.66	.027

75%, in line with the industry figures. The average absolute approximation error is 0.009 while across all instances, the error is less than 3 percentage points.

For completeness, we carry out the above evaluation for a GMC with $T=12$ under each possible combination of $\mu \in \{1,2,3\}$ and $I \in [1:6]$. We do not consider larger values of I as the transition probability matrix becomes prohibitively large rendering stationary probability calculations intractable. In line with the case of $T=6$, the average approximation absolute error is small at 0.007 for the 18 instances considered.

To round out the evaluation of the independence approximation, for the GMC's in Table 2 with $I=3$, we determine the percentage decrease in the profit rate because of deviating from the optimal to the approximately optimal buyout prices. We obtain the optimal profit rates by searching over the price grid formed by discretizing the interval for each buyout price in increments of 5. For example, the considered values for p_4 are $\{60, 65, \cdots, 4\times 60\}$. The percent decrease in the profit rate for $\mu=1$, $\mu=2$ and $\mu=3$ are all extremely small respectively at 0.06%, 0.09% and 0.07%. Judging from different types of numerical experiments above, the independence approximation sufficiently captures the GMC dynamics.

E.3. The Inventory-only Optimization Problem

Under the independence approximation, to find the optimal inventory for a fixed buyout price path $\hat{\boldsymbol{p}}$, we solve $\max_I\{IR_T(\hat{\boldsymbol{p}};\alpha) - wI : \alpha \text{ solves } (45)\}$. This entails finding the agreement initiation probability from (45) under $\hat{\boldsymbol{p}}$ for each value of I.

F. Mathematica Code

For Remark 1, we present the code for evaluating $R_4(\boldsymbol{p})$ and its second derivatives wrt p_2 and p_3:

```
(*Specifying valuation distribution*)

F[p_, \[Lambda]_] := CDF[ExponentialDistribution[1/\[Lambda]], p];
Fbar[p_, \[Lambda]_] := 1 - F[p, \[Lambda]];

(*Specifying functions of stationary probabilities*)

Subscript[\[Pi], 4][p_, \[Alpha]_, \[Rho]_, \[Lambda]_] := 1/(
1 + \[Alpha] + \[Alpha] \!\(
\*UnderoverscriptBox[\(\[Sum]\), \(i = 1\), \(2\)]\((
\*SuperscriptBox[\(\[Rho]\), \(4 - i - 1\)] \(
\*UnderoverscriptBox[\(\[Product]\), \(j = i + 1\), \(3\)]F[
```

```
p[\([\)\(j\)\(]\)], \[Lambda]]\))\)\));
Subscript[\[Pi], 3][
p_, \[Alpha]_, \[Rho]_, \[Lambda]_] := \[Alpha] Subscript[\[Pi],
4][p, \[Alpha], \[Rho], \[Lambda]] ;
Subscript[\[Pi], 2][
p_, \[Alpha]_, \[Rho]_, \[Lambda]_] := \[Alpha] Subscript[\[Pi],
4][p, \[Alpha], \[Rho], \[Lambda]] \[Rho] \!\(
\*UnderoverscriptBox[\(\[Product]\), \(j = 3\), \(3\)]\(F[
p[\([j]\)], \[Lambda]]\)\);
Subscript[\[Pi], 1][
p_, \[Alpha]_, \[Rho]_, \[Lambda]_] := \[Alpha] Subscript[\[Pi],
4][p, \[Alpha], \[Rho], \[Lambda]] \[Rho]^2 \!\(
\*UnderoverscriptBox[\(\[Product]\), \(j = 2\), \(3\)]\(F[
p[\([j]\)], \[Lambda]]\)\);

(*Specifying the profit rate function for T=4*)

Subscript[R, 4][p_, \[Alpha]_, \[Rho]_, \[Lambda]_, s_,
c_] := \[Alpha] Subscript[\[Pi], 4][
p, \[Alpha], \[Rho], \[Lambda]] s + \[Rho] Subscript[\[Pi], 3][
p, \[Alpha], \[Rho], \[Lambda]] (F[p[[3]], \[Lambda]] s
+
Fbar[p[[3]], \[Lambda]] (p[[3]] - c)) + \[Rho] Subscript[\[Pi],
2][p, \[Alpha], \[Rho], \[Lambda]] (F[p[[2]], \[Lambda]] s
+
Fbar[p[[2]], \[Lambda]] (p[[2]] - c)) + \[Rho] Subscript[\[Pi],
1][p, \[Alpha], \[Rho], \[Lambda]] (s - c);

(*Specifying the cross  derivate function wrt Subscript[p, 2] and Subscript[p, 3] and \
evaluating it*)

Subscript[DR, 4][s_, p2_, p3_, \[Alpha]_, \[Rho]_, \[Lambda]_, c_] :=
D[D[Subscript[R,
4][{s, Subscript[p, 2], Subscript[p,
3]}, \[Alpha], \[Rho], \[Lambda], s, c], Subscript[p, 2]],
Subscript[p, 3]] /. {Subscript[p, 2] -> p2,
Subscript[p, 3] -> p3};

Subscript[DR, 4][300, 400, 500, 0.8, 0.92^3, 120, 300]
```