Week 8: Differential Equation

Welcome to the Weekly Review for MATH 2414. This week’s review talks about Differential Equations. We would like to thank Patrick Bourque and the Spring 2015 MATH 2414 students for allowing us to film the Weekly Reviews.

The following problems are presented in the Week 8 videos. Thank you!

Part A: Differential Equations

1. Find values of $r$ so that $y = e^{rx}$ is a solution of $y''' - 6y'' + 11y' - 6y = 0$

2. Find values of $n$ so that $y = x^n$ is a solution of $x^2y'' + 7xy' + 8y = 0$
Part B: Application Problems

1. Assume that we have a cup of coffee with temperature $T$ in a room of constant temperature $M$. The cup of coffee will experience a change in temperature proportional to the difference in the temperature of the cup of coffee and the room $(M - T)$. This gives the differential equation:

$$\frac{dT}{dt} = k(M - T)$$

Solve this differential equation for the temperature of the coffee $T(t)$ as a function of time.

2. Assume $P$ represents the population of a bacteria in a petri dish. The differential equation that models the size of a population of species in an environment is given by the following differential equation:

$$\frac{dP}{dt} = kP$$

Solve this differential equation for the population $P(t)$ as a function of time and then show:

$$\lim_{t \to \infty} P(t) = \infty$$
3. Assume $P$ represents the population of a bacteria in a petri dish and $M$ is the carrying capacity of the environment: that is, $M$ is the maximum population of the species that can fit in the environment. The differential equation that models the size of a population of species in an environment of fixed size is given by the following differential equation:

$$\frac{dP}{dt} = kP(M - P)$$

Solve this differential equation for the population $P(t)$ as a function of time and then show:

$$\lim_{t \to \infty} P(t) = M$$
4. Another type of population model is the Gopertz growth model. The model assumes the population will increase at a rate proportional to the size of the population. That means the population will increase at a rate of \( kP(t) \). The Gopertz growth model also takes into account the maximum population a species can have in an environment of fixed size and resources. Instead of using \( (M - P(t)) \) as a factor, the Gopertz growth model uses \( \ln \left( \frac{M}{P(t)} \right) \) as a factor, with \( M \) being the maximum population. The Gopertz growth model is

\[
\frac{dP}{dt} = kP(\ln(\frac{M}{P})) \quad k > 0
\]

Solve this differential equation for the population \( P(t) \) as a function of time and then show:

\[
\lim_{t \to \infty} P(t) = M
\]
5. Differential equations can also be used to model the genetic change or evolution of a species. A commonly used hybrid selection model is

\[ \frac{dy}{dt} = ky(1 - y)(a - by) \]

Where \( y \) represents the portion of a population that has a certain characteristic and \( a, b, \) and \( k \) are constants and \( t \) is time measured in generations. Solve the hybrid selection model when \( a = 2 \) and when \( b = 1 \).
6. If $r$ is the population of rabbits and $w$ is the population of wolves in a field, how could their populations change?

7. If the following represent a predator-prey model, which variable $x$ and $y$ represents the predator?

\[
\frac{dx}{dt} = 0.01x + 0.02xy \quad \frac{dy}{dt} = -0.05y + 0.1xy
\]
Part C: Solving Differential Equations

1. Solve:
   (a) \( \frac{dy}{dx} = \sqrt{16x^2y - 4x^2y^2} \) when \( y(2) = 1 \)

   (b) \( xdy = (y^2 + 4y + 5)\sqrt{x^3 - 1} \, dx \) when \( y(1) = -2 \)