Basic Counting Principles

The Product Rule

Suppose that a procedure can be broken into a sequence of two tasks.
If there are $N_1$ ways to do the first task and for each of these ways of doing the first task, there are $N_2$ ways to do the second task, then there are $N_1N_2$ ways to do the procedure.

Example

How many different 8-bit strings of length 7 are there?

Answer

Each of the seven bits can be chosen in two ways (0 or 1).
By the product rule, there are $2^7$ 7-bit strings.

The Sum Rule

If a task can be done either in one of $N_1$ ways or in one of $N_2$ ways, where none of the set of $N_1$ ways
is the same as any of the set of \( n_2 \) ways, then there are \( n_1 + n_2 \) ways to do the task.

**Example:** A student can choose a computer project from one of 3 lists. The 3 lists contain 23, 15 and 19 projects respectively. No project is on more than 1 list. How many possible projects are there to choose from?

**Answer:** By the sum rule, there are \( 23 + 15 + 19 = 57 \) ways to choose a project.

**More Complex Counting Problems**

The problems that need to use both the product rule and the sum rule to solve.
Example: Each user on a computer system has a password.

A password is 6 to 8 characters long, where each character is an uppercase letter or a digit. Each password must contain at least 1 digit. How many possible passwords are there?

Answer:

- $P_6$: number of 6-character passwords
- $P_7$: number of 7-character passwords
- $P_8$: number of 8-character passwords
- $N_6$: number of 6-character strings
- $N_7$: number of 7-character strings
- $N_8$: number of 8-character strings
- $C_6$: number of 6-character strings without a digit
- $C_7$: "7-
- $C_8$: "8-

Number of passwords $= P_6 + P_7 + P_8$

$= (N_6 - C_6) + (N_7 - C_7) + (N_8 - C_8)$

$= (36^6 - 26^6) + (36^7 - 26^7) + (36^8 - 26^8)$

$= 2,684,483,063,360$
The Inclusion-Exclusion Principle

Suppose that a task can be done in \( N_1 \) or in \( N_2 \) ways, but some of the set of \( N_1 \) ways are the same as some of the \( N_2 \) ways. Then the number of different ways of doing the task is

\[ N_1 + N_2 - N_3, \]

where \( N_3 \) is the number of ways to do the task in a way that is both among the set of \( N_1 \) ways and the set of \( N_2 \) ways.

Example: How many bit strings of 8 bits start with a 1 or end with 00?

Answer: \( N_1 \): \( \# \) of 8-bit strings start with 1
\( N_2 \): \( \# \) end with 00
\( N_3 \): \( \# \) start with 1 and end with 00

Total number = \( N_1 + N_2 - N_3 \)

\[ = 2^7 + 2^6 - 2^5 \]

\[ = 160 \]
Inclusion-Exclusion Principle

in Terms of Sets

\[ |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| \]

Example:

350 Students:

220 majored in CS \( (A_1) \)

147 majored in EE \( (A_2) \)

51 majored in both CS and EE \( (A_1 \cap A_2) \)

How many of these students majored 

\textit{neither in CS nor in EE}?

Answer:

The number of students who majored 
\textit{either in CS or in EE} is

\[ |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| \]

\[ = 220 + 147 - 51 \]

\[ = 316 \]

The number of students who majored 
\textit{neither in CS nor in EE} is

\[ 350 - 316 = 34. \]
The Pigeonhole Principle

If \( k \) is a positive integer and \( k+1 \) or more objects are placed into \( k \) boxes, then there is at least one box containing 2 or more of the objects.

**Proof:** Suppose none of the \( k \) boxes contains more than 1 object. Then the total number of objects is at most \( k \), contradicting that there are \( k+1 \) objects.

**Corollary:** A function from a set with \( k+1 \) or more elements to a set with \( k \) elements is not 1-to-1.

**Proof:**

\[
\begin{array}{c}
\text{domain} \\ \geq k+1 \text{ elements}
\end{array} \xrightarrow{+} \begin{array}{c}
\text{codomain} \\ k \text{ elements}
\end{array}
\]

By the pigeonhole principle, there exist \( x_1 \) and \( x_2 \) s.t. \( x_1 \neq x_2 \) and \( f(x_1) = f(x_2) \). Thus, \( f \) is not 1-to-1.
Example Among any group of 367 people, there must be at least two with the same birthday, because there are at most 366 days in a year.

The Generalized Pigeonhole Principle

If \( N \) objects are placed into \( k \) boxes, then there is at least 1 box containing \( \left\lfloor \frac{N}{k} \right\rfloor \) objects.

Proof Suppose none of the boxes contains more than \( \left\lfloor \frac{N}{k} \right\rfloor - 1 \) objects, (1 less \( \left\lfloor \frac{N}{k} \right\rfloor \)) then the total number of objects is at most

\[
k \left( \left\lfloor \frac{N}{k} \right\rfloor - 1 \right) < k \left( \frac{N}{k} + 1 \right) - 1 = N.
\]

A contradiction!
Example

a) How many cards must be selected (randomly) from a standard deck of 52 cards (13 ♦, 13 ♣, 13 ♥, 13 ♠) to guarantee that at least 3 cards of the same suit are chosen?

b) How many must be selected to guarantee that at least 3 hearts are selected?

Answer:

a) Divide 52 cards into 4 boxes. By the generalized pigeonhole principle, at least 1 box contains $\lceil N/4 \rceil$ or more cards. At least 3 cards of one suit are selected if $\lceil N/4 \rceil \geq 3$.

The smallest $N$ s.t. $\lceil N/4 \rceil \geq 3$ is $N = 2 \times 4 + 1$. So 9 cards suffice.

b) We need $3 \times 13 + 3$ cards in the worst case to get 3 hearts.
Example: Within 30 days, a baseball team plays at least 1 game a day, but no more than 45 games in total. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

Proof: \( a_j \): the number of games played on or before the \( j \)th day.

Then \( a_1 < a_2 < \ldots < a_{29} < a_{30} \leq 45 \)

Also \( a_{i+14} < a_{i+14} < \ldots < a_{29+14} < a_{30+14} \leq 59 \).

By the pigeonhole principle, 2 of these 60 numbers are equal.

Because \( a_j, 1 \leq j \leq 30 \), are distinct and \( a_{j+14}, 1 \leq j \leq 30 \) are also distinct, there must be \( i \) and \( j \) s.t. \( a_i = a_{j+14} \).

This means 14 games are played from day \( j+1 \) to day \( i \).

\( a_1, \ldots, a_i, \ldots, a_{29}, a_{i+14}, \ldots, a_{30+14} \ldots \)
Example: Show that among any \( n+1 \) positive integers not exceeding \( 2n \) there must be an integer that divides one of the other integers.

Proof: Any positive integer is a power of an odd integer:
\[
\begin{align*}
2^0 \cdot 1 &= 1 \\
2^1 \cdot 1 &= 2 \\
2^1 \cdot 3 &= 6 \\
2^2 \cdot 5 &= 20 \\
&
\end{align*}
\]

Consider \( n+1 \) positive integers \( a_1, a_2, \ldots, a_{n+1} \), where \( a_j = 2^{k_j} q_j \), \( 1 \leq j \leq n+1 \)

- \( k_j \): nonnegative integer
- \( q_j \): odd integer.

There are \( n \) odd integers \( a_j \) less than \( 2n \).

By the pigeonhole principle 2 of \( q_j \)'s are the same, say \( q_x = q_y = q \). Then
\[
ax = 2^{k_x} q 
\]
and
\[
y = 2^{k_y} q,
\]
\( ax \) divides \( ay \) if \( k_x < k_y \), while \( ay \) divides \( ax \) if \( k_x > k_y \).
Permutations

A permutation of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of \( r \) elements of a set is called an \( r \)-permutation.

Example: \( S = \{1, 2, 3\} \)

\( (3, 1, 2) \) is a permutation of \( S \).

\( (3, 2) \) is a 2-permutation of \( S \).

The number of \( r \)-permutations of a set with \( n \) elements is denoted by \( \text{P}(n, r) \).

\[
\text{P}(n, r) = n(n-1)(n-2) \cdots (n-r+1) = \frac{n!}{(n-r)!}
\]

for \( 0 \leq r \leq n \).

Example

How many permutations of \( \{A, B, \ldots, G, H\} \) contain the string \( ABC \).

Answer: It is the number of permutations of \( \{A, B, D, E, F, G, H\} \), with \( n = 6 = r \).

There are \( 6! = 720 \) such permutations.
A r-combination of a set of n elements is an unordered selection of r elements from the set. The number of such r-combinations is denoted by \( C(n, r) \).

\( C(n, r) \) is also denoted by \( \binom{n}{r} \), and it is called a binomial coefficient.

For \( 0 \leq r \leq n \),

\[
C(n, r) = \frac{n!}{r! (n-r)!}
\]

Fact: \( C(n, r) = C(n, n-r) \).

Example: Department A has \( a \) people, and department B has \( b \) people. How many ways to select \( i \) people from dept. A and \( j \) people from dept. B to form a committee (of \( i+j \) people)?

Answer: \( C(a, i) \cdot C(b, j) \).
The binomial theorem

Let \( x \) and \( y \) be variables, let \( n \) be a nonnegative integer. Then,

\[
(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j
\]

\[
= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n} y^n
\]

**Corollary:** \( \sum_{k=0}^{n} \binom{n}{k} = 2^n \)

**Proof:** Let \( x = y = 1 \). Then

\[
2^n = (x+y)^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}
\]

**Useful identities**

\[
\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}
\]

\[
\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k} \binom{n}{k}
\]

\[
\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2
\]

\[
\binom{n+1}{r+1} = \sum_{j=0}^{r} \binom{j}{r}
\]
Theorem: The number of \( r \)-permutations of a set of \( n \) elements with repetition allowed is \( n^r \).

**Combinations with Repetition**

Theorem: There are

\[
C(n+r-1, r) = C(n+r-1, n-1)
\]

\( r \)-combinations from a set of \( n \) elements if repetition of elements is allowed.

Example: How many different ways to choose 6 cookies from 4 different kinds of cookies.

Answer: \( n = 4, \ r = 6 \), with repetition allowed.

\[
C(n+r-1, r) = C(9, 6) = C(9, 3) = 84.
\]
Example: How many solutions does
\[ x_1 + x_2 + x_3 = 11 \]
have, where \( x_1, x_2, x_3 \) are nonnegative integers?

Answer: Let \( S = \{ a, b, c \} \)

\[ a \text{ correspond value 1} \]
\[ b \text{ correspond value 2} \]
\[ c \text{ correspond value 3} \]

Let \( n = 3, r = 11 \). The number of ways is the same as the number of ways to choose 11 elements from \( S \) with repetition allowed. That is,

\[ C(n + r - 1, r) = C(13, 11) = C(13, 2) = 78 \]

Example: How many solutions of \( x_1 + x_2 + x_3 = 11 \)
where \( x_1 \geq 1, x_2 \geq 2, x_3 \geq 3 \) are nonnegative integers.

Answer: First, choose 1 \( a \), 2 \( b \), and 3 \( c \). Then, choose additional elements.

\[ C(n + r - 1, r) = C(3 + 5 - 1, 5) = C(7, 5) = 21. \]
Permutations with Indistinguishable Objects

The number of different permutations of \( n \) objects, where there are \( n_1 \) indistinguishable objects of type 1, \( n_2 \) of type 2, \ldots, \( n_k \) of type \( n_k \), is

\[
\frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_k!}
\]

Example: How many different strings can be made by reordering the letters of SUCCESS?

\[
\begin{array}{c}
3 \ Ss \\
2 \ Cs \\
1 \ U \\
1 \ E
\end{array}
\]

\[
\frac{7!}{3! \cdot 2! \cdot 1! \cdot 1!} = 420
\]

Distributing Objects into Boxes

Objects: distinguishable (labeled) and indistinguishable (unlabeled)

Boxes: distinguishable (labeled) and indistinguishable (unlabeled)
1. Distinguishable Objects and Distinguishable Boxes

The number of ways to distribute \( n \) distinguishable objects into \( k \) distinguishable boxes so that \( n_i \) objects are placed into box \( i \), \( i = 1, 2, \ldots, k \), is

\[
\frac{n!}{n_1! \cdot n_2! \cdots n_k!}
\]

Example

How many ways to distribute hands of 5 cards to each of 4 players from the standard deck of 52 cards?

Answer:

\[
\begin{array}{c}
\text{player 1} & \text{player 2} & \text{player 3} & \text{player 4} & \text{rest} \\
(\text{box 1}) & (\text{box 2}) & (\text{box 3}) & (\text{box 4}) & (\text{box 5}) \\
n_1 = 5 & n_2 = 5 & n_3 = 5 & n_4 = 5 & n_5 = 32 \\
(\text{N = 52})
\end{array}
\]

\[
\frac{52!}{5! \cdot 5! \cdot 5! \cdot 5! \cdot 32!}
\]
Indistinguishable Objects and Distinctable Boxes

\[ n \text{ objects} \]
\[ k \text{ boxes} \]

The number of such distributions is equal to the number of \( n \)-combinations for a set of \( k \) elements when repetitions are allowed:

\[ C(k+n-1, n) = C(k+n-1, k-1) \]

Example:

How many ways are there to place 10 indistinguishable balls into 8 distinguishable bins.

\[ \begin{array}{ccc}
\downarrow & \downarrow & \downarrow \\
\cdot & \cdot & \cdot \\
n & n & n \\
n & n & n \\
n & n & n \\
n & n & n \\
n & n & n \\
n & n & n \\
n & n & n \\
\end{array} \]

Answer: \[ C(8 + 10 - 1, 10) = C(17, 10) = \frac{17!}{10! \cdot 8!} \]

Consider: Bin_1, Bin_2, \ldots, Bin_8

How many ways of selecting bin indices (10 combinations) from a set of 8 bin indices with repetitions allowed?
(3) Distinguishable Objects and Indistinguishable Boxes

No simple closed formula for the number of such distributions.

(4) Indistinguishable Objects and Indistinguishable Boxes

No simple closed formula for the number of such distributions.