because there are infinitely many different recurrence relations satisfied by any sequence.)

7. Show that the sequence \( \{a_n\} \) is a solution of the recurrence relation \( a_n = a_{n-1} + 2a_{n-2} + 2n - 9 \) if

- a) \( a_n = -n + 2 \)
- b) \( a_n = 5(-1)^n - n + 2 \)
- c) \( a_n = 3(-1)^n + 2n - n + 2 \)
- d) \( a_n = 7 \cdot 2^n - n + 2 \)

8. Find the solution to each of these recurrence relations with the given initial conditions. Use an iterative approach such as that used in Example 5.

- a) \( a_n = -a_{n-1}, a_0 = 5 \)
- b) \( a_n = a_{n-1} + 3, a_0 = 1 \)
- c) \( a_n = a_{n-1} - n, a_0 = 4 \)
- d) \( a_n = 2a_{n-1} - 3, a_0 = -1 \)
- e) \( a_n = (n + 1)a_{n-1}, a_0 = 2 \)
- f) \( a_n = 2na_{n-1}, a_0 = 3 \)
- g) \( a_n = -a_{n-1} + n - 1, a_0 = 7 \)

9. Find the solution to each of these recurrence relations and initial conditions. Use an iterative approach such as that used in Example 5.

- a) \( a_n = 3a_{n-1}, a_0 = 2 \)
- b) \( a_n = a_{n-1} + 2, a_0 = 3 \)
- c) \( a_n = a_{n-1} + n, a_0 = 1 \)
- d) \( a_n = a_{n-1} + 2n + 3, a_0 = 4 \)
- e) \( a_n = 2a_{n-1} - 1, a_0 = 1 \)
- f) \( a_n = 3a_{n-1} + 1, a_0 = 1 \)
- g) \( a_n = na_{n-1}, a_0 = 5 \)
- h) \( a_n = 2na_{n-1}, a_0 = 1 \)

10. A person deposits $1000 in an account that yields 9% interest compounded annually.

- a) Set up a recurrence relation for the amount in the account at the end of \( n \) years.
- b) Find an explicit formula for the amount in the account at the end of \( n \) years.
- c) How much money will the account contain after 100 years?

11. Suppose that the number of bacteria in a colony triples every hour.

- a) Set up a recurrence relation for the number of bacteria after \( n \) hours have elapsed.
- b) If 100 bacteria are used to begin a new colony, how many bacteria will be in the colony in 10 hours?

12. Assume that the population of the world in 2002 was 6.2 billion and is growing at the rate of 1.3% a year.

- a) Set up a recurrence relation for the population of the world \( n \) years after 2002.
- b) Find an explicit formula for the population of the world \( n \) years after 2002.
- c) What will the population of the world be in 2022?

13. A factory makes custom sports cars at an increasing rate. In the first month only one car is made, in the second month two cars are made, and so on, with \( n \) cars made in the \( n \)th month.

- a) Set up a recurrence relation for the number of cars produced in the first \( n \) months by this factory.
- b) How many cars are produced in the first year?
- c) Find an explicit formula for the number of cars produced in the first \( n \) months by this factory.

14. An employee joined a company in 1999 with a starting salary of $50,000. Every year this employee receives a raise of $1000 plus 5% of the salary of the previous year.

- a) Set up a recurrence relation for the salary of this employee \( n \) years after 1999.
- b) What will the salary of this employee be in 2007?
- c) Find an explicit formula for the salary of this employee \( n \) years after 1999.

15. Find a recurrence relation for the balance \( B(k) \) owed at the end of \( k \) months on a loan of $5000 at a rate of 7% if a payment of $100 is made each month. [Hint: Express \( B(k) \) in terms of \( B(k-1) \); the monthly interest is \( 0.07/12 \)B(k-1).]

16. a) Find a recurrence relation for the balance \( B(k) \) owed at the end of \( k \) months on a loan at a rate of \( r \) if a payment \( P \) is made on the loan each month. [Hint: Express \( B(k) \) in terms of \( B(k-1) \) and note that the monthly interest rate is \( r/12 \).]

- b) Determine what the monthly payment \( P \) should be so that the loan is paid off after \( T \) months.

- Use mathematical induction to verify the formula derived in Example 5 for the number of moves required to complete the Tower of Hanoi puzzle.

18. a) Find a recurrence relation for the number of permutations of a set with \( n \) elements.

- b) Use this recurrence relation to find the number of permutations of a set with \( n \) elements using iteration.

19. A vending machine dispensing books of stamps accepts only dollar coins, $1 bills, and $5 bills.

- a) Find a recurrence relation for the number of ways to deposit \( n \) dollars in the vending machine, where the order in which the coins and bills are deposited matters.

- b) What are the initial conditions?

- c) How many ways are there to deposit $10 for a book of stamps?

20. A country uses as currency coins with values of 1 peso, 2 pesos, 5 pesos, and 10 pesos and bills with values of 5 pesos, 10 pesos, 20 pesos, 50 pesos, and 100 pesos. Find a recurrence relation for the number of ways to pay a bill of \( n \) pesos if the order in which the coins and bills are paid matters.

21. How many ways are there to pay a bill of 17 pesos using the currency described in Exercise 20, where the order in which coins and bills are paid matters?
22. a) Find a recurrence relation for the number of strictly increasing sequences of positive integers that have 1 as their first term and \( n \) as their last term, where \( n \) is a positive integer. That is, sequences \( a_1, a_2, \ldots, a_k \), where \( a_1 = 1, \ a_k = n, \) and \( a_j < a_{j+1} \) for \( j = 1, 2, \ldots, k-1 \).
   
   b) What are the initial conditions?
   
   c) How many sequences of the type described in (a) are there when \( n \) is a positive integer with \( n \geq 2 \)?

23. a) Find a recurrence relation for the number of bit strings of length \( n \) that contain a pair of consecutive 0s.
   
   b) What are the initial conditions?
   
   c) How many bit strings of length seven contain two consecutive 0s?

24. a) Find a recurrence relation for the number of bit strings of length \( n \) that contain three consecutive 0s.
   
   b) What are the initial conditions?
   
   c) How many bit strings of length seven contain three consecutive 0s?

25. a) Find a recurrence relation for the number of bit strings of length \( n \) that do not contain three consecutive 0s.
   
   b) What are the initial conditions?
   
   c) How many bit strings of length seven do not contain three consecutive 0s?

26. a) Find a recurrence relation for the number of bit strings that contain the string 01.
   
   b) What are the initial conditions?
   
   c) How many bit strings of length seven contain the string 01?

27. a) Find a recurrence relation for the number of ways to climb \( n \) stairs if the person climbing the stairs can take one stair or two stairs at a time.
   
   b) What are the initial conditions?
   
   c) How many ways can this person climb a flight of eight stairs?

28. a) Find a recurrence relation for the number of ways to climb \( n \) stairs if the person climbing the stairs can take one, two, or three stairs at a time.
   
   b) What are the initial conditions?
   
   c) How many ways can this person climb a flight of eight stairs?

29. a) Find a recurrence relation for the number of ternary strings that do not contain two consecutive 0s.
   
   b) What are the initial conditions?
   
   c) How many ternary strings of length six do not contain two consecutive 0s?

30. a) Find a recurrence relation for the number of ternary strings that contain two consecutive 0s.
   
   b) What are the initial conditions?
   
   c) How many ternary strings of length six contain two consecutive 0s?

31. a) Find a recurrence relation for the number of ternary strings that do not contain two consecutive 0s or two consecutive 1s.

32. a) Find a recurrence relation for the number of ternary strings that contain either two consecutive 0s or two consecutive 1s.
   
   b) What are the initial conditions?
   
   c) How many ternary strings of length six contain two consecutive 0s or two consecutive 1s?

33. a) Find a recurrence relation for the number of ternary strings that do not contain consecutive symbols that are the same.
   
   b) What are the initial conditions?
   
   c) How many ternary strings of length six do not contain consecutive symbols that are the same?

34. a) Find a recurrence relation for the number of ternary strings that contain two consecutive symbols that are the same.
   
   b) What are the initial conditions?
   
   c) How many ternary strings of length six contain consecutive symbols that are the same?

35. Messages are transmitted over a communications channel using two signals. The transmit signal requires 1 microsecond, and the transmit signal requires 2 microseconds.
   
   a) Find a recurrence relation for the number of different messages consisting of sequences of these two signals, where each signal in the message is immediately followed by the next signal, that can be sent in \( n \) microseconds.
   
   b) What are the initial conditions?
   
   c) How many different messages can be sent in 10 microseconds using these two signals?

36. A bus driver pays all tolls, using only nickels and dimes, by throwing one coin at a time into the mechanical toll collector.
   
   a) Find a recurrence relation for the number of different ways the bus driver can pay a toll of \( n \) cents (where the order in which the coins are used matters).
   
   b) In how many different ways can the driver pay a toll of 45 cents?

37. a) Find the recurrence relation satisfied by \( R_n \), where \( R_n \) is the number of regions that a plane is divided into by \( n \) lines, if no two of the lines are parallel and no three of the lines go through the same point.
   
   b) Find \( R_n \) using iteration.

38. a) Find the recurrence relation satisfied by \( S_n \), where \( S_n \) is the number of regions into which the surface of a sphere is divided by \( n \) great circles (which are the intersections of the sphere and planes passing through the center of the sphere), if no three of the great circles go through the same point.
   
   b) Find \( S_n \) using iteration.

39. a) Find the recurrence relation satisfied by \( S_n \), where \( S_n \) is the number of regions into which three-dimensional space is divided by \( n \) planes if every three of the planes
7.2 Solving Linear Recurrence Relations

is a particular solution. Hence, all solutions of the original recurrence relation \( a_n = a_{n-1} + n \) are
given by \( a_n = a_n^{(k)} + a_n^{(p)} = c + n(n + 1)/2 \). Because \( a_1 = 1 \), we have \( 1 = a_1 = c + 1 \cdot 2/2 = c + 1 \), so \( c = 0 \). It follows that \( a_n = n(n + 1)/2 \). (This is the same formula given in Table 2 in Section 2.4 and derived previously.)

**Exercises**

1. Determine which of these are linear homogeneous recurrence relations with constant coefficients. Also, find the degree of those that are.
   a) \( a_n = 3a_{n-1} + 4a_{n-2} + 5a_{n-3} \)
   b) \( a_n = 2a_{n-1} + a_{n-2} \)
   c) \( a_n = a_{n-1} + a_{n-4} \)
   d) \( a_n = a_{n-1} + 2 \)
   e) \( a_n = a_{n-1}^2 + a_{n-2} \)
   f) \( a_n = a_{n-2} \)
   g) \( a_n = a_{n-1} + n \)

2. Determine which of these are linear homogeneous recurrence relations with constant coefficients. Also, find the degree of those that are.
   a) \( a_n = 3a_{n-2} \)
   b) \( a_n = 3 \)
   c) \( a_n = a_{n-1}^2 \)
   d) \( a_n = a_{n-1} + 2a_{n-3} \)
   e) \( a_n = a_{n-1}/n \)
   f) \( a_n = a_{n-1} + a_{n-2} + n + 3 \)
   g) \( a_n = 4a_{n-2} + 5a_{n-4} + 9a_{n-7} \)

3. Solve these recurrence relations together with the initial conditions.
   a) \( a_n = 2a_{n-1} \) for \( n \geq 1, a_0 = 3 \)
   b) \( a_n = a_{n-1} \) for \( n \geq 1, a_0 = 2 \)
   c) \( a_n = 5a_{n-1} - 6a_{n-2} \) for \( n \geq 2, a_0 = 1, a_1 = 0 \)
   d) \( a_n = 4a_{n-1} - 4a_{n-2} \) for \( n \geq 2, a_0 = 0, a_1 = 8 \)
   e) \( a_n = -a_{n-1} - 4a_{n-2} \) for \( n \geq 2, a_0 = 0, a_1 = 1 \)
   f) \( a_n = 4a_{n-2} \) for \( n \geq 2, a_0 = 0, a_1 = 4 \)
   g) \( a_n = a_{n-2}/4 \) for \( n \geq 2, a_0 = 1, a_1 = 0 \)

4. Solve these recurrence relations together with the initial conditions.
   a) \( a_n = a_{n-1} + 6a_{n-2} \) for \( n \geq 2, a_0 = 3, a_1 = 6 \)
   b) \( a_n = 7a_{n-1} - 10a_{n-2} \) for \( n \geq 2, a_0 = 2, a_1 = 1 \)
   c) \( a_n = 6a_{n-1} - 8a_{n-2} \) for \( n \geq 2, a_0 = 4, a_1 = 10 \)
   d) \( a_n = 2a_{n-1} - a_{n-2} \) for \( n \geq 2, a_0 = 4, a_1 = 4 \)
   e) \( a_n = a_{n-2} \) for \( n \geq 2, a_0 = 5, a_1 = -1 \)
   f) \( a_n = -6a_{n-1} - 9a_{n-2} \) for \( n \geq 2, a_0 = 3, a_1 = -3 \)
   g) \( a_n = -4a_{n-1} + 5a_{n-2} \) for \( n \geq 0, a_0 = 2, a_1 = 8 \)

5. How many different messages can be transmitted in \( n \) microseconds using the two signals described in Exercise 35 in Section 7.1?

6. How many different messages can be transmitted in \( n \) microseconds using three different signals if one signal requires 1 microsecond for transmittal, the other two signals require 2 microseconds each for transmittal, and a signal in a message is followed immediately by the next signal?

7. In how many ways can a \( 2 \times n \) rectangular checkerboard be tiled using 1 \( \times 2 \) and 2 \( \times 2 \) pieces?

8. A model for the number of lobsters caught per year is based on the assumption that the number of lobsters caught in a year is the average of the number caught in the two previous years.
   a) Find a recurrence relation for \( L_n \), where \( L_n \) is the number of lobsters caught in year \( n \), under the assumption for this model.
   b) Find \( L_n \) if 100,000 lobsters were caught in year 1 and 300,000 were caught in year 2.

9. A deposit of $100,000 is made to an investment fund at the beginning of a year. On the last day of each year two dividends are awarded. The first dividend is 20% of the amount in the account during that year. The second dividend is 45% of the amount in the account in the previous year.
   a) Find a recurrence relation for \( P_n \), where \( P_n \) is the amount in the account at the end of \( n \) years if no money is ever withdrawn.
   b) How much is in the account after \( n \) years if no money has been withdrawn?


11. The **Lucas numbers** satisfy the recurrence relation

\[
L_n = L_{n-1} + L_{n-2},
\]

and the initial conditions \( L_0 = 2 \) and \( L_1 = 1 \).

a) Show that \( L_n = f_{n+1} + f_{n-1} \) for \( n = 2, 3, \ldots \), where \( f_n \) is the \( n \)th Fibonacci number.

b) Find an explicit formula for the Lucas numbers.

12. Find the solution to \( a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3} \) for \( n = 3, 4, 5, \ldots \) with \( a_0 = 3, a_1 = 1, a_2 = 10 \).

13. Find the solution to \( a_n = 7a_{n-2} + 6a_{n-3} \) with \( a_0 = 9, a_1 = 10, \) and \( a_2 = 32 \).

14. Find the solution to \( a_n = 5a_{n-2} - 4a_{n-4} \) with \( a_0 = 3, a_1 = 2, a_2 = 6, \) and \( a_3 = 8 \).

15. Find the solution to \( a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3} \) with \( a_0 = 7, a_1 = -4, \) and \( a_2 = 8 \).

*16. Prove Theorem 3.

17. Prove this identity relating the Fibonacci numbers and the binomial coefficients:

\[
f_{n+1} = C(n, 0) + C(n - 1, 1) + \cdots + C(n - k, k),
\]

where \( n \) is a positive integer and \( k = \lfloor n/2 \rfloor \). [Hint: Let \( a_n = C(n, 0) + C(n - 1, 1) + \cdots + C(n - k, k) \). Show that the sequence \( \{a_n\} \) satisfies the same recurrence relation and initial conditions satisfied by the sequence of Fibonacci numbers.]
18. Solve the recurrence relation \( a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3} \) with \( a_0 = -5, a_1 = 4, \) and \( a_2 = 88. \)

19. Solve the recurrence relation \( a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3} \) with \( a_0 = 5, a_1 = -9, \) and \( a_2 = 15. \)

20. Find the general form of the solutions of the recurrence relation \( a_n = 8a_{n-2} - 16a_{n-4}. \)

21. What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has roots \( 1, 1, 1, -2, -2, -2, 3, 3, -4? \)

22. What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has the roots \( -1, -1, -1, 2, 2, 5, 5, 7? \)

23. Consider the nonhomogeneous linear recurrence relation \( a_n = 3a_{n-1} + 2^n. \)
   a) Show that \( a_n = -2^n + 1 \) is a solution of this recurrence relation.
   b) Use Theorem 5 to find all solutions of this recurrence relation.
   c) Find the solution with \( a_0 = 1. \)

24. Consider the nonhomogeneous linear recurrence relation \( a_n = 2a_{n-1} + 2^n. \)
   a) Show that \( a_n = n2^n \) is a solution of this recurrence relation.
   b) Use Theorem 5 to find all solutions of this recurrence relation.
   c) Find the solution with \( a_0 = 2. \)

25. a) Determine values of the constants \( A \) and \( B \) such that \( a_n = An + B \) is a solution of recurrence relation \( a_n = 2a_{n-1} + n + 5. \)
   b) Use Theorem 5 to find all solutions of this recurrence relation.
   c) Find the solution of this recurrence relation with \( a_0 = 4. \)

26. What is the general form of the particular solution guaranteed to exist by Theorem 6 of the linear nonhomogeneous recurrence relation \( a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3} + F(n) \) if
   a) \( F(n) = n^2? \)
   b) \( F(n) = 2^n? \)
   c) \( F(n) = n2^n? \)
   d) \( F(n) = (-2)^n? \)
   e) \( F(n) = n^22^n? \)
   f) \( F(n) = n3(-2)^n? \)
   g) \( F(n) = 3^n? \)

27. What is the general form of the particular solution guaranteed to exist by Theorem 6 of the linear nonhomogeneous recurrence relation \( a_n = 8a_{n-2} - 16a_{n-4} + F(n) \) if
   a) \( F(n) = n^3? \)
   b) \( F(n) = (-2)^n? \)
   c) \( F(n) = n^2? \)
   d) \( F(n) = n^22^n? \)
   e) \( F(n) = (n^2 - 2)(-2)^n? \)
   f) \( F(n) = n^22^n? \)
   g) \( F(n) = 2^n? \)

28. a) Find all solutions of the recurrence relation \( a_n = 2a_{n-1} + 2n^2. \)
   b) Find the solution of the recurrence relation in part (a) with initial condition \( a_1 = 4. \)

29. a) Find all solutions of the recurrence relation \( a_n = 2a_{n-1} + 3^n. \)
   b) Find the solution of the recurrence relation in part (a) with initial condition \( a_1 = 5. \)

30. a) Find all solutions of the recurrence relation \( a_n = -5a_{n-1} - 6a_{n-2} + 42 \cdot 4^n. \)
   b) Find the solution of this recurrence relation with \( a_1 = 56 \) and \( a_2 = 278. \)

31. Find all solutions of the recurrence relation \( a_n = 5a_{n-1} - 6a_{n-2} + 2^n + 3n. [Hint: Look for a particular solution of the form \( qn2^n + p_1n + p_2, \) where \( q, p_1, \) and \( p_2 \) are constants.]
   b) Find the solution of the recurrence relation \( a_n = 2a_{n-1} + 3 \cdot 2^n. \)

32. Find all solutions of the recurrence relation \( a_n = 4a_{n-1} - 4a_{n-2} + (n + 1)2^n. \)

33. Find all solutions of the recurrence relation \( a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3} + 3n4^n \) with \( a_0 = -2, a_1 = 0, \) and \( a_2 = 5. \)

34. Find the solution of the recurrence relation \( a_n = 4a_{n-1} - 3a_{n-2} + 2^n + n + 3 \) with \( a_0 = 1 \) and \( a_1 = 4. \)

35. Let \( a_n \) be the sum of the first \( n \) perfect squares, that is, \( a_n = \sum_{k=1}^{n} k^2. \) Show that the sequence \( \{a_n\} \) satisfies the linear nonhomogeneous recurrence relation \( a_n = a_{n-1} + n^2 \) and the initial condition \( a_1 = 1. \) Use Theorem 6 to determine a formula for \( a_n \) by solving this recurrence relation.

36. Let \( a_n \) be the sum of the first \( n \) triangular numbers, that is, \( a_n = \sum_{k=1}^{n} t_k, \) where \( t_k = k(k + 1)/2. \) Show that the sequence \( \{a_n\} \) satisfies the linear nonhomogeneous recurrence relation \( a_n = a_{n-1} + n(n + 1)/2 \) and the initial condition \( a_1 = 1. \) Use Theorem 6 to determine a formula for \( a_n \) by solving this recurrence relation.

37. a) Find the characteristic roots of the linear homogeneous recurrence relation \( a_n = 2a_{n-1} - 2a_{n-2}. \) (Note: These are complex numbers.)
   b) Find the solution of the recurrence relation in part (a) with \( a_0 = 1 \) and \( a_1 = 2. \)

39. a) Find the characteristic roots of the linear homogeneous recurrence relation \( a_n = a_{n-4}. \) (Note: These include complex numbers.)
   b) Find the solution of the recurrence relation in part (a) with \( a_0 = 1, a_1 = 0, a_2 = -1, \) and \( a_3 = 1. \)

40. Solve the simultaneous recurrence relations
   \[ a_n = 3a_{n-1} + 2b_{n-1}, \]
   \[ b_n = a_{n-1} + 2b_{n-1}, \]
   with \( a_0 = 1 \) and \( b_0 = 2. \)

41. a) Use the formula found in Example 4 for \( f_n, \) the \( n \)th Fibonacci number, to show that \( f_n \) is the integer closest to
   \[ \frac{1}{\sqrt{5}} \left( 1 + \sqrt{5} \right)^n. \]
where \( f(2) = 1 \), exceeds the number of comparisons needed to solve the closest-pair problem for \( n \) points. By the Master Theorem (Theorem 2), it follows that \( f(n) = O(n \log n) \). The two sorts of points by their \( x \) coordinates and by their \( y \) coordinates each can be done using \( O(n \log n) \) comparisons, by using the merge sort, and the sorted subsets of these coordinates at each of the \( O(\log n) \) steps of the algorithm can be done using \( O(n) \) comparisons each. Thus, we find that the closest-pair problem can be solved using \( O(n \log n) \) comparisons.

### Exercises

1. How many comparisons are needed for a binary search in a set of 64 elements?

2. How many comparisons are needed to locate the maximum and minimum elements in a sequence with 128 elements using the algorithm referred to in Example 2?

3. Multiply \((1110)_2\) and \((1010)_2\) using the fast multiplication algorithm.

4. Express the fast multiplication algorithm in pseudocode.

5. Determine a value for the constant \( C \) in Example 4 and use it to estimate the number of bit operations needed to multiply two 64-bit integers using the fast multiplication algorithm.

6. How many operations are needed to multiply two \( 32 \times 32 \) matrices using the algorithm referred to in Example 4?

7. Suppose that \( f(n) = f(n / 3) + 1 \) when \( n \) is divisible by 3, and \( f(1) = 1 \). Find
   a) \( f(9) \).
   b) \( f(27) \).
   c) \( f(729) \).

8. Suppose that \( f(n) = 2f(n / 2) + 3 \) when \( n \) is even, and \( f(1) = 5 \). Find
   a) \( f(2) \).
   b) \( f(8) \).
   c) \( f(64) \).
   d) \( f(1024) \).

9. Suppose that \( f(n) = f(n / 5) + 3n^2 \) when \( n \) is divisible by 5, and \( f(1) = 4 \). Find
   a) \( f(5) \).
   b) \( f(125) \).
   c) \( f(3125) \).

10. Find \( f(n) \) when \( n = 2^k \), where \( f \) satisfies the recurrence relation \( f(n) = f(n / 2) + 1 \) with \( f(1) = 1 \).

11. Estimate the size of \( f \) in Exercise 10 if \( f \) is an increasing function.

12. Find \( f(n) \) when \( n = 3^k \), where \( f \) satisfies the recurrence relation \( f(n) = 2f(n / 3) + 4 \) with \( f(1) = 1 \).

13. Estimate the size of \( f \) in Exercise 12 if \( f \) is an increasing function.

14. Suppose that there are \( n = 2^k \) teams in an elimination tournament, where there are \( n / 2 \) games in the first round, with the \( n / 2 = 2^{k-1} \) winners playing in the second round, and so on. Develop a recurrence relation for the number of rounds in the tournament.

15. How many rounds are in the elimination tournament described in Exercise 14 when there are 32 teams?

16. Solve the recurrence relation for the number of rounds in the tournament described in Exercise 14.

17. Suppose that the votes of \( n \) people for different candidates (where there can be more than two candidates) for a particular office are the elements of a sequence. A person wins the election if this person receives a majority of the votes.

   a) Devise a divide-and-conquer algorithm that determines whether a candidate received a majority and, if so, determine who this candidate is. [Hint: Assume that \( n \) is even and split the sequence of votes into two sequences, each with \( n / 2 \) elements. Note that a candidate could not have received a majority of votes without receiving a majority of votes in at least one of the two halves.]

   b) Use the Master Theorem to estimate the number of comparisons needed by the algorithm you devised in part (a).

18. Suppose that each person in a group of \( n \) people votes for exactly two people from a slate of candidates to fill two positions on a committee. The top two finishers both win positions as long as each receives more than \( n / 2 \) votes.

   a) Devise a divide-and-conquer algorithm that determines whether the two candidates who received the most votes each received at least \( n / 2 \) votes and, if so, determine who these two candidates are.

   b) Use the Master Theorem to estimate the number of comparisons needed by the algorithm you devised in part (a).

19. a) Set up a divide-and-conquer recurrence relation for the number of multiplications required to compute \( x^n \), where \( x \) is a real number and \( n \) is a positive integer, using the recursive algorithm from Exercise 26 in Section 4.4.

   b) Use the recurrence relation you found in part (a) to construct a big-\( O \) estimate for the number of multiplications used to compute \( x^n \) using the recursive algorithm.

20. a) Set up a divide-and-conquer recurrence relation for the number of modular multiplications required to compute \( a^n \mod m \), where \( a, m, \) and \( n \) are positive integers, using the recursive algorithms from Example 3 in Section 4.4.

   b) Use the recurrence relation you found in part (a) to construct a big-\( O \) estimate for the number of modular multiplications used to compute \( a^n \mod m \) using the recursive algorithm.
21. Suppose that the function \( f \) satisfies the recurrence relation 
\[ f(n) = 2f(\sqrt{n}) + 1 \] 
whenever \( n \) is a perfect square greater than 1 and \( f(2) = 1 \).

\( \text{a)} \) Find \( f(16) \).
\( \text{b)} \) Find a big-\( O \) estimate for \( f(n) \). [\text{Hint: Make the substitution} \( m = \log n \).]

22. Suppose that the function \( f \) satisfies the recurrence relation 
\[ f(n) = 2f(\sqrt{n}) + \log n \] 
whenever \( n \) is a perfect square greater than 1 and \( f(2) = 1 \).

\( \text{a)} \) Find \( f(16) \).
\( \text{b)} \) Find a big-\( O \) estimate for \( f(n) \). [\text{Hint: Make the substitution} \( m = \log n \).]

**23.** This exercise deals with the problem of finding the largest sum of consecutive terms of a sequence of \( n \) real numbers. When all terms are positive, the sum of all terms provides the answer, but the situation is more complicated when some terms are negative. For example, the maximum sum of consecutive terms of the sequence \(-2, 3, -1, 6, -7, 4\) is \(3 + (-1) + 6 = 8\). (This exercise is based on [Be86].)

\( \text{a)} \) Use pseudocode to describe an algorithm that solves this problem by finding the sums of consecutive terms starting with the first term, the sums of consecutive terms starting with the second term, and so on, keeping track of the maximum sum found so far as the algorithm proceeds.

\( \text{b)} \) Determine the computational complexity of the algorithm in part (a) in terms of the number of sums computed and the number of comparisons made.

\( \text{c)} \) Devise a divide-and-conquer algorithm to solve this problem. [\text{Hint: Assume that there are an even number of terms in the sequence and split the sequence into two halves. Explain how to handle the case when the maximum sum of consecutive terms includes terms in both halves.}]

\( \text{d)} \) Use the algorithm from part (c) to find the maximum sum of consecutive terms of each of the sequences: \(-2, 4, -1, 3, 5, -6, 1, 2, 4, 1, -3, 7, -1, -5, 3, -2; \text{ and } -1, 6, 3, -4, -5, 8, -1, 7\).

\( \text{e)} \) Find a recurrence relation for the number of sums and comparisons used by the divide-and-conquer algorithm from part (c).

\( \text{f)} \) Use the Master Theorem to estimate the computational complexity of the divide-and-conquer algorithm. How does it compare in terms of computational complexity with the algorithm from part (a)?

24. Apply the algorithm described in Example 12 for finding the closest pair of points, using the Euclidean distance between points, to find the closest pair of the points \((1, 3), (1, 7), (2, 4), (2, 9), (3, 1), (3, 5), (4, 3), \text{ and } (4, 7)\).

25. Apply the algorithm described in Example 12 for finding the closest pair of points, using the Euclidean distance between points, to find the closest pair of the points \((1, 2), (1, 6), (2, 4), (2, 8), (3, 1), (3, 6), (3, 10), (4, 3), (5, 1), (5, 5), (5, 9), (6, 7), (7, 1), (7, 4), (7, 9), \text{ and } (8, 6)\).

26. Use pseudocode to describe the recursive algorithm for solving the closest-pair problem as described in Example 12.

27. Construct a variation of the algorithm described in Example 12 along with justifications of the steps used by the algorithm to find the smallest distance between two points if the distance between two points is defined to be 
\[ d((x_i, y_i), (x_j, y_j)) = \max(|x_i - x_j|, |y_i - y_j|) \]

28. Suppose someone picks a number \( x \) from a set of \( n \) numbers. A second person tries to guess the number by successively selecting subsets of the \( n \) numbers and asking the first person whether \( x \) is in each set. The first person answers either "yes" or "no." When the first person answers each query truthfully, we can find \( x \) using \( \log n \) queries by successively splitting the sets used in each query in half. Ulam's problem, proposed by Stanislaw Ulam in 1976, asks for the number of queries required to find \( x \), supposing that the first person is allowed to lie exactly once.

\( \text{a)} \) Show that by asking each question twice, given a number \( x \) and a set with \( n \) elements, and asking one more question when we find the lie, Ulam's problem can be solved using \( 2 \log n + 1 \) queries.

\( \text{b)} \) Show that by dividing the initial set of \( n \) elements into four parts, each with \( n/4 \) elements, \( 1/4 \) of the elements can be eliminated using two queries. [\text{Hint: Use two queries, where each of the queries asks whether the element is in the union of two of the subsets with \( n/4 \) elements and where one of the subsets of \( n/4 \) elements is used in both queries.}]

\( \text{c)} \) Show from part (b) that if \( f(n) \) equals the number of queries used to solve Ulam's problem using the method from part (b) and \( n \) is divisible by 4, then \( f(n) = f(3n/4) + 2 \).

\( \text{d)} \) Solve the recurrence relation in part (c) for \( f(n) \).

\( \text{e)} \) Is the naive way to solve Ulam's problem by asking each question twice or the divide-and-conquer method based on part (b) more efficient? The most efficient way to solve Ulam's problem has been determined by A. Pelc [Pe87].

In Exercises 29–33, assume that \( f \) is an increasing function satisfying the recurrence relation 
\[ f(n) = af(n/b) + cn^d \]
where \( a \geq 1, b \) is an integer greater than 1, and \( c \) and \( d \) are positive real numbers. These exercises supply a proof of Theorem 2.

\( \text{*29.} \) Show that if \( a = b^d \) and \( n \) is a power of \( b \), then \( f(n) = O((1)n^d + cn^d \log_b n) \).

\( \text{30.} \) Use Exercise 29 to show that if \( a = b^d \), then \( f(n) \) is \( O(n^d \log n) \).

\( \text{*31.} \) Show that if \( a \neq b^d \) and \( n \) is a power of \( b \), then 
\[ f(n) = C_1n^d + C_2n^d \log_a n, \]
where \( C_1 = b^d c/(b^d - a) \) and \( C_2 = f(1) + b^d c/(a - b^d) \).

\( \text{32.} \) Use Exercise 31 to show that if \( a < b^d \), then \( f(n) \) is \( O(n^d) \).

\( \text{33.} \) Use Exercise 31 to show that if \( a > b^d \), then \( f(n) \) is \( O(n^{d \log_a c}) \).
times by the expression on the right-hand side of this equation. Our goal is to evaluate this quantity. By Corollary 2 of Section 5.4, we have

\[ C(r, 0) - C(r, 1) + C(r, 2) - \cdots + (-1)^r C(r, r) = 0. \]

Hence,

\[ 1 = C(r, 0) = C(r, 1) - C(r, 2) + \cdots + (-1)^{r+1} C(r, r). \]

Therefore, each element in the union is counted exactly once by the expression on the right-hand side of the equation. This proves the principle of inclusion–exclusion.

The inclusion–exclusion principle gives a formula for the number of elements in the union of \( n \) sets for every positive integer \( n \). There are terms in this formula for the number of elements in the intersection of every nonempty subset of the collection of the \( n \) sets. Hence, there are \( 2^n - 1 \) terms in this formula.

**EXAMPLE 5**

Give a formula for the number of elements in the union of four sets.

**Solution:** The inclusion–exclusion principle shows that

\[
|A_1 \cup A_2 \cup A_3 \cup A_4| = |A_1| + |A_2| + |A_3| + |A_4| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4| + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4| - |A_1 \cap A_2 \cap A_3 \cap A_4|.
\]

Note that this formula contains 15 different terms, one for each nonempty subset of \( \{A_1, A_2, A_3, A_4\} \).

**Exercises**

1. How many elements are in \( A_1 \cup A_2 \) if there are 12 elements in \( A_1 \), 18 elements in \( A_2 \), and
   a) \( A_1 \cap A_2 = \emptyset \)?
   b) \( |A_1 \cap A_2| = 1? \)
   c) \( |A_1 \cap A_2| = 6? \)
   d) \( A_1 \subseteq A_2 \)?

2. There are 345 students at a college who have taken a course in calculus, 212 who have taken a course in discrete mathematics, and 188 who have taken courses in both calculus and discrete mathematics. How many students have taken a course in either calculus or discrete mathematics?

3. A survey of households in the United States reveals that 96% have at least one television set, 98% have telephone service, and 95% have telephone service and at least one television set. What percentage of households in the United States have neither telephone service nor a television set?

4. A marketing report concerning personal computers states that 650,000 owners will buy a modem for their machines next year and 1,250,000 will buy at least one software package. If the report states that 1,450,000 owners will buy either a modem or at least one software package, how many will buy both a modem and at least one software package?

5. Find the number of elements in \( A_1 \cup A_2 \cup A_3 \) if there are 100 elements in each set and if
   a) the sets are pairwise disjoint.
   b) there are 50 common elements in each pair of sets and no elements in all three sets.
   c) there are 50 common elements in each pair of sets and 25 elements in all three sets.
   d) the sets are equal.

6. Find the number of elements in \( A_1 \cup A_2 \cup A_3 \) if there are 100 elements in \( A_1 \), 1000 in \( A_2 \), and 10,000 in \( A_3 \) if
   a) \( A_1 \subseteq A_2 \) and \( A_2 \subseteq A_3 \).
   b) the sets are pairwise disjoint.
   c) there are two elements common to each pair of sets and one element in all three sets.

7. There are 2504 computer science students at a school. Of these, 1876 have taken a course in Pascal, 999 have taken a course in Fortran, and 345 have taken a course in C.
Further, 876 have taken courses in both Pascal and Fortran, 231 have taken courses in both Fortran and C, and 290 have taken courses in both Pascal and C. If 189 of these students have taken courses in Fortran, Pascal, and C, how many of these 2504 students have not taken a course in any of these three programming languages?

8. In a survey of 270 college students, it is found that 64 like brussels sprouts, 94 like broccoli, 58 like cauliflower, 26 like both brussels sprouts and broccoli, 28 like both brussels sprouts and cauliflower, 22 like both broccoli and cauliflower, and 14 like all three vegetables. How many of the 270 students do not like any of these vegetables?

9. How many students are enrolled in a course either in calculus, discrete mathematics, data structures, or programming languages at a school if there are 507, 292, 312, and 344 students in these courses, respectively; 14 in both calculus and data structures; 213 in both calculus and programming languages; 211 in both discrete mathematics and data structures; 43 in both discrete mathematics and programming languages; and no student may take calculus and discrete mathematics, or data structures and programming languages, concurrently?

10. Find the number of positive integers not exceeding 100 that are not divisible by 5 or by 7.

11. Find the number of positive integers not exceeding 100 that are either odd or the square of an integer.

12. Find the number of positive integers not exceeding 1000 that are either the square or the cube of an integer.

13. How many bit strings of length eight do not contain six consecutive 0s?

14. How many permutations of the 26 letters of the English alphabet do not contain any of the strings fish, rat, or bird?

15. How many permutations of the 10 digits either begin with the 3 digits 987, contain the digits 45 in the fifth and sixth positions, or end with the 3 digits 123?

16. How many elements are in the union of four sets if each of the sets has 100 elements, each pair of the sets shares 50 elements, each three of the sets share 25 elements, and there are 5 elements in all four sets?

17. How many elements are in the union of four sets if the sets have 50, 60, 70, and 80 elements, respectively, each pair of the sets has 5 elements in common, each triple of the sets has 1 common element, and no element is in all four sets?

18. How many terms are there in the formula for the number of elements in the union of 10 sets given by the principle of inclusion–exclusion?

19. Write out the explicit formula given by the principle of inclusion–exclusion for the number of elements in the union of five sets.

20. How many elements are in the union of five sets if the sets contain 10,000 elements each, each pair of sets has 1000 common elements, each triple of sets has 100 common elements, every four of the sets have 10 common elements, and there is 1 element in all five sets?

21. Write out the explicit formula given by the principle of inclusion–exclusion for the number of elements in the union of six sets when it is known that no three of these sets have a common intersection.


23. Let $E_1$, $E_2$, and $E_3$ be three events from a sample space $S$. Find a formula for the probability of $E_1 \cup E_2 \cup E_3$.

24. Find the probability that when a fair coin is flipped five times tails comes up exactly three times, the first and last flips come up tails, or the second and fourth flips come up heads.

25. Find the probability that when four numbers from 1 to 100, inclusive, are picked at random with no repetitions allowed, either all are odd, all are divisible by 3, or all are divisible by 5.

26. Find a formula for the probability of the union of four events in a sample space if no three of them can occur at the same time.

27. Find a formula for the probability of the union of five events in a sample space if no four of them can occur at the same time.

28. Find a formula for the probability of the union of $n$ events in a sample space when no two of these events can occur at the same time.

29. Find a formula for the probability of the union of $n$ events in a sample space.

7.6 Applications of Inclusion–Exclusion

**Introduction**

Many counting problems can be solved using the principle of inclusion–exclusion. For instance, we can use this principle to find the number of primes less than a positive integer. Many problems can be solved by counting the number of onto functions from one finite set to another. The inclusion–exclusion principle can be used to find the number of such functions. The famous hatcheck problem can be solved using the principle of inclusion–exclusion. This problem asks for the probability that no person is given the correct hat back by a hatcheck person who gives the hats back randomly.
TABLE 2 The Probability of a Derangement.

<table>
<thead>
<tr>
<th>n</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_n/n!$</td>
<td>0.50000</td>
<td>0.33333</td>
<td>0.37500</td>
<td>0.36667</td>
<td>0.36806</td>
<td>0.36786</td>
</tr>
</tbody>
</table>

Consequently, inserting these quantities into our formula for $D_n$ gives

$$D_n = n! - C(n, 1)(n - 1)! + C(n, 2)(n - 2)! - \cdots + (-1)^nC(n, n)(n - n)!$$

$$= n! - \frac{n!}{1!(n - 1)!}(n - 1)! + \frac{n!}{2!(n - 2)!}(n - 2)! - \cdots + (-1)^n\frac{n!}{n!0!}0!.$$

Simplifying this expression gives

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n\frac{1}{n!}\right].$$

It is now simple to find $D_n$ for a given positive integer $n$. For instance, using Theorem 2, it follows that

$$D_3 = 3!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right] = 6\left(1 - 1 + \frac{1}{2} - \frac{1}{6}\right) = 2,$$

as we have previously remarked.

The solution of the problem in Example 4 can now be given.

**Solution:** The probability that no one receives the correct hat is $D_n/n!$. By Theorem 2, this probability is

$$\frac{D_n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n\frac{1}{n!}.$$

The values of this probability for $2 \leq n \leq 7$ are displayed in Table 2.

Using methods from calculus it can be shown that

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n\frac{1}{n!} + \cdots \sim 0.368.$$

Because this is an alternating series with terms tending to zero, it follows that as $n$ grows without bound, the probability that no one receives the correct hat converges to $e^{-1} \sim 0.368$. In fact, this probability can be shown to be within $1/(n + 1)!$ of $e^{-1}$.

**Exercises**

1. Suppose that in a bushel of 100 apples there are 20 that have worms in them and 15 that have bruises. Only those apples with neither worms nor bruises can be sold. If there are 10 bruised apples that have worms in them, how many of the 100 apples can be sold?

2. Of 1000 applicants for a mountain-climbing trip in the Himalayas, 450 get altitude sickness, 622 are not in good enough shape, and 30 have allergies. An applicant qualifies if and only if this applicant does not get altitude sickness, is in good shape, and does not have allergies. If there are 111 applicants who get altitude sickness and are not in good enough shape, 14 who get altitude sickness and have allergies, 18 who are not in good enough shape and have allergies, and 9 who get altitude sickness, are
not in good enough shape, and have allergies, how many applicants qualify?

3. How many solutions does the equation \( x_1 + x_2 + x_3 = 13 \)
   have where \( x_1, x_2, \) and \( x_3 \) are nonnegative integers less than 6?

4. Find the number of solutions of the equation \( x_1 + x_2 + x_3 + x_4 = 17 \),
   where \( x_i, i = 1, 2, 3, 4, \) are nonnegative integers such that \( x_1 \leq 3, x_2 \leq 4, x_3 \leq 5, \) and \( x_4 \leq 8. \)

5. Find the number of the primes less than 200 using the principle
   of inclusion–exclusion.

6. An integer is called \textbf{squarefree} if it is not divisible by
   the square of a positive integer greater than 1. Find the
   number of squarefree positive integers less than 100.

7. How many positive integers less than 10,000 are not the
   second or higher power of an integer?

8. How many onto functions are there from a set with seven
   elements to one with five elements?

9. How many ways are there to distribute six different toys
   to three different children such that each child gets at least
   one toy?

10. In how many ways can eight distinct balls be distributed
    into three distinct urns if each urn must contain at least
    one ball?

11. In how many ways can seven different jobs be assigned to
    four different employees so that each employee is assigned
    at least one job and the most difficult job is assigned to
    the best employee?

12. List all the derangements of \( \{1, 2, 3, 4, \} \).

13. How many derangements are there of a set with seven
    elements?

14. What is the probability that none of 10 people receives
    the correct hat if a hatcheck person hands their hats back
    randomly?

15. A machine that inserts letters into envelopes goes haywire
    and inserts letters randomly into envelopes. What is the
    probability that in a group of 100 letters
    a) no letter is put into the correct envelope?
    b) exactly one letter is put into the correct envelope?
    c) exactly 98 letters are put into the correct envelopes?
    d) exactly 99 letters are put into the correct envelopes?
    e) all letters are put into the correct envelopes?

16. A group of \( n \) students is assigned seats for each of two
    classes in the same classroom. How many ways can these
    seats be assigned if no student is assigned the same seat
    for both classes?

17. How many ways can the digits 0, 1, 2, 3, 4, 5, 6, 7,
    8, 9 be arranged so that no even digit is in its original
    position?

18. Use a combinatorial argument to show that the sequence
    \( \{ D_n \} \), where \( D_n \) denotes the number of derangements
    of \( n \) objects, satisfies the recurrence relation
    \[
    D_n = (n - 1)(D_{n-1} + D_{n-2})
    \]
    for \( n \geq 2. [\text{Hint: } \text{Note that there are } n - 1 \text{ choices for the}
    \]
    first element \( k \) of a derangement. Consider separately
    the derangements that start with \( k \) that do and do not have \( 1 \)
    in the \( k \)th position.]

19. Use Exercise 18 to show that
    \[
    D_n = n D_{n-1} + (-1)^n
    \]
    for \( n \geq 1. \)

20. Use Exercise 19 to find an explicit formula for \( D_n \).

21. For which positive integers \( n \) is \( D_n \), the number of
derangements of \( n \) objects, even?

22. Suppose that \( p \) and \( q \) are distinct primes. Use the
    principle of inclusion–exclusion to find \( \phi(pq) \), the number
    of positive integers not exceeding \( pq \) that are relatively
    prime to \( pq \).

23. Use the principle of inclusion–exclusion to derive a forrmula
    for \( \phi(n) \) when the prime factorization of \( n \) is
    \[
    n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}.
    \]

24. Show that if \( n \) is a positive integer, then
    \[
    n! = C(n, 0) D_n + C(n, 1) D_{n-1} + \cdots + C(n, n-1) D_1 + C(n, n) D_0,
    \]
    where \( D_k \) is the number of derangements of \( k \) objects.

25. How many derangements of \( \{1, 2, 3, 4, 5, 6 \} \)
    begin with the integers 1, 2, and 3, in some order?

26. How many derangements of \( \{1, 2, 3, 4, 5, 6 \} \)
    end with the integers 1, 2, and 3, in some order?

27. Prove Theorem 1.

**Key Terms and Results**

**TERMS**

\textbf{recurrence relation}: a formula expressing terms of a sequence, except for some initial terms, as a function of one or more previous terms of the sequence

\textbf{initial conditions for a recurrence relation}: the values of the terms of a sequence satisfying the recurrence relation before this relation takes effect

\textbf{linear homogeneous recurrence relation with constant coefficients}: a recurrence relation that expresses the terms of a sequence, except initial terms, as a linear combination of previous terms