Problem 4.1.1 Solution

(a) The probability $P[X \leq 2, Y \leq 3]$ can be found by evaluating the joint CDF $F_{X,Y}(x, y)$ at $x = 2$ and $y = 3$. This yields

$$P[X \leq 2, Y \leq 3] = F_{X,Y}(2, 3) = (1 - e^{-2})(1 - e^{-3})$$

(b) To find the marginal CDF of $X$, $F_X(x)$, we simply evaluate the joint CDF at $y = \infty$.

$$F_X(x) = F_{X,Y}(x, \infty) = \begin{cases} 1 - e^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(c) Likewise for the marginal CDF of $Y$, we evaluate the joint CDF at $X = \infty$.

$$F_Y(y) = F_{X,Y}(\infty, y) = \begin{cases} 1 - e^{-y} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$
Problem 4.2.1 Solution

In this problem, it is helpful to label points with nonzero probability on the $X$, $Y$ plane:

(a) We must choose $c$ so the PMF sums to one:

$$\sum_{x=1,3,4} \sum_{y=1,3} P_{X,Y}(x, y) = c \sum_{x=1,3,4} x \sum_{y=1,3} y = c[1(1+3) + 2(1+3) + 4(1+3)] = 28c \quad (1)$$

Thus $c = 1/28$.

(b) The event $\{Y < X\}$ has probability

$$P[Y < X] = \sum_{x=1,2,4} \sum_{y=2} P_{X,Y}(x, y) = \frac{1(0) + 2(1) + 4(1+3)}{28} = \frac{18}{28} \quad (2)$$

(c) The event $\{Y > X\}$ has probability

$$P[Y > X] = \sum_{x=1,2,4} \sum_{y>x} P_{X,Y}(x, y) = \frac{1(3) + 2(3) + 4(0)}{28} = \frac{9}{28} \quad (3)$$

(d) There are two ways to solve this part. The direct way is to calculate

$$P[Y = X] = \sum_{x=1,2,4} \sum_{y=x} P_{X,Y}(x, y) = \frac{1(1) + 2(0)}{28} = \frac{1}{28} \quad (4)$$

The indirect way is to use the previous results and the observation that

$$P[Y = X] = 1 - P[Y < X] - P[Y > X] = (1 - 18/28 - 9/28) = 1/28 \quad (5)$$

(e)

$$P[Y = 3] = \sum_{x=1,2,4} P_{X,Y}(x, 3) = \frac{(1)(3) + (2)(3) + (4)(3)}{28} = \frac{21}{28} = \frac{3}{4} \quad (6)$$
Problem 4.2.2 Solution
On the $X$, $Y$ plane, the joint PMF is

(a) To find $c$, we sum the PMF over all possible values of $X$ and $Y$. We choose $c$ so the sum equals one.

$$
\sum_{x} \sum_{y} P_{X,Y}(x, y) = \sum_{x=-1,0,2} \sum_{y=-1,0,1} c|x + y| = 6c + 2c + 6c = 14c \quad (1)
$$

Thus $c = 1/14$.

(b)

$$
P[Y < X] = P_{X,Y}(0, -1) + P_{X,Y}(2, -1) + P_{X,Y}(2, 0) + P_{X,Y}(2, 1) \quad (2)
$$

$$
= c + c + 2c + 3c = 7c = 1/2 \quad (3)
$$

(c)

$$
P[Y > X] = P_{X,Y}(-2, -1) + P_{X,Y}(-2, 0) + P_{X,Y}(-2, 1) + P_{X,Y}(0, 1) \quad (4)
$$

$$
= 3c + 2c + c + c = 7c = 1/2 \quad (5)
$$

(d) From the sketch of $P_{X,Y}(x, y)$ given above, $P[X = Y] = 0$.

(e)

$$
P[X < 1] = P_{X,Y}(-2, -1) + P_{X,Y}(-2, 0) + P_{X,Y}(-2, 1) + P_{X,Y}(0, -1) + P_{X,Y}(0, 1) \quad (6)
$$

$$
= 8c = 8/14 \quad (7)
$$
Problem 4.2.6 Solution

As the problem statement indicates, $Y = y < n$ if and only if

$A$: the first $y$ tests are acceptable, and

$B$: test $y + 1$ is a rejection.

Thus $P[Y = y] = P[AB]$. Note that $Y \leq X$ since the number of acceptable tests before the first failure cannot exceed the number of acceptable circuits. Moreover, given the occurrence of $AB$, the event $X = x < n$ occurs if and only if there are $x - y$ acceptable circuits in the remaining $n - y - 1$ tests. Since events $A$, $B$ and $C$ depend on disjoint sets of tests, they are independent events. Thus, for $0 \leq y \leq x < n$,

$$P_{X,Y}(x, y) = P[X = x, Y = y]$$

$$= P[ABC]$$

$$= P[A]P[B]P[C]$$

$$= \frac{p^y (1 - p)^{n - y - 1}}{P[A]} \frac{x - y}{P[B]} \frac{n - y - 1 - (x - y)}{P[C]}$$

$$= \left( \frac{n - y - 1}{x - y} \right) p^x (1 - p)^{n - x}$$

Note that the remaining case, $y = x = n$ occurs when all $n$ tests are acceptable and thus $P_{X,Y}(n, n) = p^n$. 

Problem 4.3.2 Solution

On the $X$, $Y$ plane, the joint PMF is

$$
P_{X,Y}(x, y)
$$

The PMF sums to one when $c = 1/14$.

(a) The marginal PMFs of $X$ and $Y$ are

$$
P_X(x) = \sum_{y=-1,0,1} P_{X,Y}(x, y) = \begin{cases} 
6/14 & x = -2, 2 \\
2/14 & x = 0 \\
0 & \text{otherwise}
\end{cases} \quad (1)
$$

$$
P_Y(y) = \sum_{x=-2,0,2} P_{X,Y}(x, y) = \begin{cases} 
5/14 & y = -1, 1 \\
4/14 & y = 0 \\
0 & \text{otherwise}
\end{cases} \quad (2)
$$

(b) The expected values of $X$ and $Y$ are

$$
E[X] = \sum_{x=-2,0,2} x P_X(x) = -2(6/14) + 2(6/14) = 0 \quad (3)
$$

$$
E[Y] = \sum_{y=-1,0,1} y P_Y(y) = -1(5/14) + 1(5/14) = 0 \quad (4)
$$

(c) Since $X$ and $Y$ both have zero mean, the variances are

$$
\text{Var}[X] = E[X^2] = \sum_{x=-2,0,2} x^2 P_X(x) = (-2)^2(6/14) + 2^2(6/14) = 24/7 \quad (5)
$$

$$
\text{Var}[Y] = E[Y^2] = \sum_{y=-1,0,1} y^2 P_Y(y) = (-1)^2(5/14) + 1^2(5/14) = 5/7 \quad (6)
$$

The standard deviations are $\sigma_X = \sqrt{24/7}$ and $\sigma_Y = \sqrt{5/7}$.
Problem 4.4.1 Solution

(a) The joint PDF of \( X \) and \( Y \) is

\[
 f_{X,Y}(x, y) = \begin{cases} 
  c & x + y \leq 1, x, y \geq 0 \\
  0 & \text{otherwise} 
\end{cases}
\]  
(1)

To find the constant \( c \) we integrate over the region shown. This gives

\[
 \int_0^1 \int_0^{1-x} c \, dy \, dx = cx - \frac{cx}{2} \Big|_0^1 = \frac{c}{2} = 1
\]  
(2)

Therefore \( c = 2 \).

(b) To find the \( P[X \leq Y] \) we look to integrate over the area indicated by the graph

\[
 P[X \leq Y] = \int_0^{1/2} \int_x^{1-x} dy \, dx
\]  
(3)

\[
 = \int_0^{1/2} (2 - 4x) \, dx
\]  
(4)

\[
 = \frac{1}{2}
\]  
(5)

(c) The probability \( P[X + Y \leq 1/2] \) can be seen in the figure. Here we can set up the following integrals

\[
P[X + Y \leq 1/2] = \int_0^{1/2} \int_0^{1/2-x} 2 \, dy \, dx
\]  
(6)

\[
= \int_0^{1/2} (1 - 2x) \, dx
\]  
(7)

\[
= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}
\]  
(8)
Problem 4.5.4 Solution
The joint PDF of \( X \) and \( Y \) and the region of nonzero probability are

\[
f_{X,Y}(x, y) = \begin{cases} 
5x^2/2 & -1 \leq x \leq 1, 0 \leq y \leq x^2 \\
0 & \text{otherwise}
\end{cases} \quad (1)
\]

We can find the appropriate marginal PDFs by integrating the joint PDF.

(a) The marginal PDF of \( X \) is

\[
f_X(x) = \int_0^{x^2} \frac{5x^2}{2} dy = \begin{cases} 
5x^4/2 & -1 \leq x \leq 1 \\
0 & \text{otherwise}
\end{cases} \quad (2)
\]

(b) Note that \( f_Y(y) = 0 \) for \( y > 1 \) or \( y < 0 \). For \( 0 \leq y \leq 1 \),

\[
f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dx \quad (3)
\]

\[
= \int_{-1}^{-\sqrt{y}} \frac{5x^2}{2} \, dx + \int_{\sqrt{y}}^{1} \frac{5x^2}{2} \, dx \quad (4)
\]

\[
= 5(1 - y^{3/2})/3 \quad (5)
\]

The complete expression for the marginal CDF of \( Y \) is

\[
f_Y(y) = \begin{cases} 
5(1 - y^{3/2})/3 & 0 \leq y \leq 1 \\
0 & \text{otherwise}
\end{cases} \quad (6)
\]
Problem 4.6.1 Solution
In this problem, it is helpful to label points $X$, $Y$ with nonzero probability along with the corresponding values of $W = X - Y$. From the statement of Problem 4.6.1, we have

(a) To find the PMF of $W$, we simply add the probabilities associated with each possible value of $W$.

\[
P_W(-2) = P_{X,Y}(1, 3) = 3/28 \quad P_W(-1) = P_{X,Y}(2, 3) = 6/28
\]

\[
P_W(0) = P_{X,Y}(1, 1) = 1/28 \quad P_W(1) = P_{X,Y}(2, 1) + P_{X,Y}(4, 3) = 14/28
\]

\[
P_W(3) = P_{X,Y}(4, 1) = 4/28
\]

For all other values of $w$, $P_W(w) = 0$.

(b) The expected value of $W$ is

\[
E[W] = \sum_w w P_W(w) = -2(3/28) + -1(6/28) + 0(1/28) + 1(14/28) + 3(4/28) = 1/2
\]

(c) \[
P[W > 0] = P_W(1) + P_W(3) = 18/28
\]
Problem 4.6.10 Solution

The position of the mobile phone is equally likely to be anywhere in the area of a circle with radius 16 km. Let $X$ and $Y$ denote the position of the mobile. Since we are given that the cell has a radius of 4 km, we will measure $X$ and $Y$ in kilometers. Assuming the base station is at the origin of the $X, Y$ plane, the joint PDF of $X$ and $Y$ is

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{16\pi} & x^2 + y^2 \leq 16 \\ 0 & \text{otherwise} \end{cases}$$

(1)

Since the radial distance of the mobile from the base station is $R = \sqrt{x^2 + y^2}$, the CDF of $R$ is

$$F_R(r) = P[R \leq r] = P[x^2 + y^2 \leq r]$$

(2)

By changing to polar coordinates, we see that for $0 \leq r \leq 4$ km,

$$F_R(r) = \int_0^{2\pi} \int_0^r \frac{r'}{16\pi} \, dr' \, d\theta' = r^2/16$$

(3)

So

$$F_R(r) = \begin{cases} 0 & r < 0 \\ r^2/16 & 0 \leq r < 4 \\ 1 & r \geq 4 \end{cases}$$

(4)

Then by taking the derivative with respect to $r$ we arrive at the PDF

$$f_R(r) = \begin{cases} r/8 & 0 \leq r \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

(5)
**Problem 4.7.8 Solution**

The joint PDF of $X$ and $Y$ is

$$f_{X,Y}(x, y) = \begin{cases} 
(x + y)/3 & 0 \leq x \leq 1, 0 \leq y \leq 2 \\
0 & \text{otherwise}
\end{cases} \quad (1)$$

Before calculating moments, we first find the marginal PDFs of $X$ and $Y$. For $0 \leq x \leq 1$,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dy = \int_{0}^{2} \frac{x + y}{3} \, dy = \frac{xy}{3} + \frac{y^2}{6} \bigg|_{y=0}^{y=2} = \frac{2x + 2}{3} \quad (2)$$

For $0 \leq y \leq 2$,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dx = \int_{0}^{1} \left( \frac{x}{3} + \frac{y}{3} \right) \, dx = \frac{x^2}{6} + \frac{xy}{3} \bigg|_{x=0}^{x=1} = \frac{2y + 1}{6} \quad (3)$$

Complete expressions for the marginal PDFs are

$$f_X(x) = \begin{cases} 
\frac{2x+2}{3} & 0 \leq x \leq 1 \\
0 & \text{otherwise}
\end{cases} \quad f_Y(y) = \begin{cases} 
\frac{2y+1}{6} & 0 \leq y \leq 2 \\
0 & \text{otherwise}
\end{cases} \quad (4)$$

(a) The expected value of $X$ is

$$E[X] = \int_{-\infty}^{\infty} xf_X(x) \, dx = \int_{0}^{1} x \frac{2x+2}{3} \, dx = \frac{2x^3}{9} + \frac{x^2}{3} \bigg|_{x=0}^{x=1} = \frac{5}{9} \quad (5)$$

The second moment of $X$ is

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) \, dx = \int_{0}^{1} x^2 \frac{2x+2}{3} \, dx = \frac{x^4}{6} + \frac{2x^3}{9} \bigg|_{x=0}^{x=1} = \frac{7}{18} \quad (6)$$

The variance of $X$ is $\text{Var}[X] = E[X^2] - (E[X])^2 = 7/18 - (5/9)^2 = 13/162$.

(b) The expected value of $Y$ is

$$E[Y] = \int_{-\infty}^{\infty} yf_Y(y) \, dy = \int_{0}^{2} y \frac{2y+1}{6} \, dy = \frac{y^2}{12} + \frac{y^3}{9} \bigg|_{y=0}^{y=2} = \frac{11}{9} \quad (7)$$

The second moment of $Y$ is

$$E[Y^2] = \int_{-\infty}^{\infty} y^2 f_Y(y) \, dy = \int_{0}^{2} y^2 \frac{2y+1}{6} \, dy = \frac{y^3}{18} + \frac{y^4}{12} \bigg|_{y=0}^{y=2} = \frac{16}{9} \quad (8)$$

The variance of $Y$ is $\text{Var}[Y] = E[Y^2] - (E[Y])^2 = 23/81$. 

(c) The correlation of $X$ and $Y$ is

$$ E[XY] = \int_{0}^{1} \int_{0}^{2} xyf_{X,Y}(x,y) \, dx \, dy $$

$$ = \int_{0}^{1} x \int_{x/3}^{2} yf_{X,Y}(y) \, dy \, dx $$

$$ = \int_{0}^{1} \left( \frac{x^2 y^2}{6} + \frac{y^3}{9} \right) \bigg|_{y=0}^{y=\frac{x}{3}} \, dx $$

$$ = \int_{0}^{1} \left( \frac{2x^3}{9} + \frac{8x}{9} \right) \, dx = \left[ \frac{2x^3}{9} + \frac{8x}{9} \right]_{0}^{1} = \frac{2}{3} $$


(d) The expected value of $X$ and $Y$ is

$$ E[X + Y] = E[X] + E[Y] = \frac{5}{9} + \frac{11}{9} = \frac{16}{9} $$

(e) By Theorem 4.15,

$$ \text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y] + 2\text{Cov}[X, Y] = \frac{13}{162} + \frac{23}{81} - \frac{2}{81} = \frac{55}{162} $$
Problem 4.8.3 Solution

Given the event $A = \{X + Y \leq 1\}$, we wish to find $f_{X,Y|A}(x,y)$. First we find

$$P[A] = \int_0^1 \int_0^{1-x} 6e^{-(a+x+y)} \, dy \, dx = 1 - 3e^{-2} + 2e^{-3} \tag{1}$$

So then

$$f_{X,Y|A}(x,y) = \begin{cases} \frac{6e^{-(2x+y)}}{1-3e^{-2}+2e^{-3}} & x + y \leq 1, \ x \geq 0, \ y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$
Problem 4.9.13 Solution

The key to solving this problem is to find the joint PMF of M and N. Note that \( N \geq M \). For \( n > m \), the joint event \( \{ M = m, N = n \} \) has probability

\[
P[M = m, N = n] = \frac{m - 1 \text{ calls}}{\text{calls}} \frac{n - m - 1 \text{ calls}}{\text{calls}}
\]

\[
= (1 - p)^{m-1} p (1 - p)^{n-m-1} p
\]

\[
= (1 - p)^{n-2} p^2
\]

(1)

(2)

(3)

A complete expression for the joint PMF of M and N is

\[
P_{M,N}(m, n) = \begin{cases} (1 - p)^{n-2} p^2 & m = 1, 2, \ldots, n - 1; \ n = m + 1, m + 2, \ldots \\ 0 & \text{otherwise} \end{cases}
\]

(4)

For \( n = 2, 3, \ldots \), the marginal PMF of N satisfies

\[
P_N(n) = \sum_{m=1}^{n-1} (1 - p)^{m-2} p^2 = (n - 1)(1 - p)^{n-2} p^2
\]

(5)

Similarly, for \( m = 1, 2, \ldots \), the marginal PMF of M satisfies

\[
P_M(m) = \sum_{n=m+1}^{\infty} (1 - p)^{n-2} p^2
\]

\[
= p^2 [(1 - p)^{m-1} + (1 - p)^m + \cdots]
\]

\[
= (1 - p)^{m-1} p
\]

(6)

(7)

(8)

The complete expressions for the marginal PMF's are

\[
P_M(m) = \begin{cases} (1 - p)^{m-1} p & m = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}
\]

(9)

\[
P_N(n) = \begin{cases} (n - 1)(1 - p)^{n-2} p^2 & n = 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}
\]

(10)

Not surprisingly, if we view each voice call as a successful Bernoulli trial, M has a geometric PMF since it is the number of trials up to and including the first success. Also, N has a Pascal PMF since it is the number of trials required to see 2 successes. The conditional PMF's are now easy to find.

\[
P_{\text{N|M}}(n|m) = \frac{P_{M,N}(m, n)}{P_M(m)} = \begin{cases} (1 - p)^{n-m-1} p & n = m + 1, m + 2, \ldots \\ 0 & \text{otherwise} \end{cases}
\]

(11)

The interpretation of the conditional PMF of N given M is that given \( M = m \), \( N = m + M' \) where \( N' \) has a geometric PMF with mean \( 1/p \). The conditional PMF of M given N is

\[
P_{\text{M|N}}(m|n) = \frac{P_{M,N}(m, n)}{P_N(n)} = \begin{cases} 1/(n - 1) & m = 1, \ldots, n - 1 \\ 0 & \text{otherwise} \end{cases}
\]

(12)

Given that call \( N = n \) was the second voice call, the first voice call is equally likely to occur in any of the previous \( n - 1 \) calls.