Problem 3.1.1 Solution

The CDF of $X$ is

$$F_X(x) = \begin{cases} 
0 & x < -1 \\
(x + 1)/2 & -1 \leq x < 1 \\
1 & x \geq 1
\end{cases} \quad (1)$$

Each question can be answered by expressing the requested probability in terms of $F_X(x)$.

(a)\[ P \{ X > 1/2 \} = 1 - P \{ X \leq 1/2 \} = 1 - F_X(1/2) = 1 - \frac{3}{4} = \frac{1}{4} \quad (2)\]

(b) This is a little trickier than it should be. Being careful, we can write

$$P \{-1/2 \leq X < 3/4\} = P \{-1/2 < X \leq 3/4\} + P \{X = -1/2\} - P \{X = 3/4\} \quad (3)$$

Since the CDF of $X$ is a continuous function, the probability that $X$ takes on any specific value is zero. This implies $P[X = 3/4] = 0$ and $P[X = -1/2] = 0$. (If this is not clear at this point, it will become clear in Section 3.6.) Thus,

$$P \{-1/2 \leq X < 3/4\} = P \{-1/2 < X \leq 3/4\} = F_X(3/4) - F_X(-1/2) = 5/8 \quad (4)$$

(c)

$$P \{X \leq 1/2\} = P \{-1/2 \leq X \leq 1/2\} = P \{X \leq 1/2\} - P \{X < -1/2\} \quad (5)$$

Note that $P[X \leq 1/2] = F_X(1/2) = 3/4$. Since the probability that $P[X = -1/2] = 0$, $P[X < -1/2] = P[X \leq -1/2]$. Hence $P[X < -1/2] = F_X(-1/2) = 1/4$. This implies

$$P \{X \leq 1/2\} = P \{X \leq 1/2\} - P \{X < -1/2\} = 3/4 - 1/4 = 1/2 \quad (6)$$

(d) Since $F_X(1) = 1$, we must have $a \leq 1$. For $a \leq 1$, we need to satisfy

$$P \{X \leq a\} = F_X(a) = \frac{a + 1}{2} = 0.8 \quad (7)$$

Thus $a = 0.6$. 
Problem 3.2.4 Solution

For \( x < 0 \), \( F_X(x) = 0 \). For \( x \geq 0 \),

\[
F_X(x) = \int_0^x f_X(y) \, dy = \int_0^x \sigma^2 y e^{-\sigma^2 y^2/2} \, dy = -e^{-\sigma^2 x^2/2} \bigg|_0^x = 1 - e^{-\sigma^2 x^2/2} \tag{1}
\]

A complete expression for the CDF of \( X \) is

\[
F_X(x) = \begin{cases} 
0 & x < 0 \\
1 - e^{-\sigma^2 x^2/2} & x \geq 0 
\end{cases} \tag{2}
\]
Problem 3.3.1 Solution

\[ f_X(x) = \begin{cases} 
1/4 & -1 \leq x \leq 3 \\
0 & \text{otherwise}
\end{cases} \]  \hspace{1cm} (1)

We recognize that \(X\) is a uniform random variable from \([-1,3]\).

(a) \(E[X] = 1\) and \(\text{Var}[X] = \frac{(3+1)^2}{12} = 4/3\).

(b) The new random variable \(Y\) is defined as \(Y = h(X) = X^2\). Therefore

\[ h(E[X]) = h(1) = 1 \]  \hspace{1cm} (2)

and

\[ E[h(X)] = E[X^2] = \text{Var}[X] + E[X]^2 = 4/3 + 1 = 7/3 \]  \hspace{1cm} (3)

Finally

\[ E[Y] = E[h(X)] = E[X^2] = 7/3 \]  \hspace{1cm} (4)

\[ \text{Var}[Y] = E[X^4] - E[X^2]^2 = \int_{-1}^{3} \frac{x^4}{4} \, dx - \frac{49}{9} = \frac{61}{5} - \frac{49}{9} \]  \hspace{1cm} (5)
Problem 3.3.7 Solution

To find the moments, we first find the PDF of \( U \) by taking the derivative of \( F_U(u) \). The CDF and corresponding PDF are

\[
F_U(u) = \begin{cases} 
0 & u < -5 \\
(u+5)/8 & -5 \leq u < -3 \\
1/4 & -3 \leq u < 3 \\
1/4 + 3(u-3)/8 & 3 \leq u < 5 \\
1 & u \geq 5.
\end{cases}
\]

\[
f_U(u) = \begin{cases} 
0 & u < -5 \\
1/8 & -5 \leq u < -3 \\
3/8 & -3 \leq u < 3 \\
3/8 & 3 \leq u < 5 \\
0 & u \geq 5.
\end{cases}
\] (1)

(a) The expected value of \( U \) is

\[
E[U] = \int_{-\infty}^{\infty} u f_U(u) \, du = \int_{-\infty}^{-3} \frac{u}{8} \, du + \int_{3}^{\infty} \frac{3u}{8} \, du
\]

\[
= \left[ \frac{u^2}{16} \right]_{-3}^{3} + \left[ \frac{3u^2}{16} \right]_{3}^{\infty}
\]

\[
= \frac{9}{16} + \frac{27}{16} = 2
\] (2)

(b) The second moment of \( U \) is

\[
E[U^2] = \int_{-\infty}^{\infty} u^2 f_U(u) \, du = \int_{-\infty}^{-3} \frac{u^2}{8} \, du + \int_{3}^{\infty} \frac{3u^2}{8} \, du
\]

\[
= \left[ \frac{u^3}{24} \right]_{-3}^{3} + \left[ \frac{u^3}{8} \right]_{3}^{\infty}
\]

\[
= \frac{3}{24} + \frac{27}{8} = \frac{49}{3}
\] (3)

The variance of \( U \) is \( \text{Var}[U] = E[U^2] - (E[U])^2 = \frac{37}{3} \).

(c) Note that \( 2^U = e^{(\ln 2)U} \). This implies that

\[
\int 2^u \, du = \int e^{(\ln 2)u} \, du = \frac{1}{\ln 2} e^{(\ln 2)u} = \frac{2^u}{\ln 2}
\] (4)

The expected value of \( 2^U \) is then

\[
E[2^U] = \int_{-\infty}^{\infty} 2^u f_U(u) \, du = \int_{-\infty}^{-3} \frac{2^u}{8} \, du + \int_{3}^{\infty} \frac{3 \cdot 2^u}{8} \, du
\]

\[
= \left[ \frac{2^u}{8 \ln 2} \right]_{-3}^{3} + \left[ \frac{3 \cdot 2^u}{8 \ln 2} \right]_{3}^{\infty}
\]

\[
= \frac{2307}{256 \ln 2} = 13.001
\] (5)
Problem 3.4.1 Solution

The reflect power $Y$ has an exponential ($\lambda = 1/P_0$) PDF. From Theorem 3.8, $E[Y] = P_0$. The probability that an aircraft is correctly identified is

$$P \left[ Y > P_0 \right] = \int_{P_0}^{\infty} \frac{1}{P_0} e^{-y/P_0} \, dy = e^{-1}. \tag{1}$$

Fortunately, real radar systems offer better performance.
Problem 3.4.10 Solution

The integral $I_1$ is

$$I_1 = \int_0^\infty \lambda e^{-\lambda x} \, dx = -e^{-\lambda x} \bigg|_0^\infty = 1 \tag{1}$$

For $n > 1$, we have

$$I_n = \int_0^\infty \frac{\lambda^{n-1}x^{n-1}}{(n-1)!} \lambda e^{-\lambda x} \, dx \tag{2}$$

We define $u$ and $dv$ as shown above in order to use the integration by parts formula $\int u \, dv = uv - \int v \, du$. Since

$$du = \frac{\lambda^{n-1}x^{n-1}}{(n-2)!} \, dx \quad v = -e^{-\lambda x} \tag{3}$$

we can write

$$I_n = uv \bigg|_0^\infty - \int_0^\infty v \, du \tag{4}$$

$$= -\frac{\lambda^{n-1}x^{n-1}}{(n-1)!} e^{-\lambda x} \bigg|_0^\infty + \int_0^\infty \frac{\lambda^{n-1}x^{n-1}}{(n-2)!} e^{-\lambda x} \, dx \tag{5}$$

$$= 0 + I_{n-1} \tag{6}$$

Hence, $I_n = 1$ for all $n \geq 1$. 
Problem 3.4.5 Solution

(a) The PDF of a continuous uniform random variable distributed from $[-5, 5]$ is

$$f_X(x) = \begin{cases} 
1/10 & -5 \leq x \leq 5 \\
0 & \text{otherwise}
\end{cases} \tag{1}$$

(b) For $x < -5$, $F_X(x) = 0$. For $x \geq 5$, $F_X(x) = 1$. For $-5 \leq x \leq 5$, the CDF is

$$F_X(x) = \int_{-5}^{x} f_X(\tau) d\tau = \frac{x + 5}{10} \tag{2}$$

The complete expression for the CDF of $X$ is

$$F_X(x) = \begin{cases} 
0 & x < -5 \\
(x + 5)/10 & -5 \leq x \leq 5 \\
1 & x > 5
\end{cases} \tag{3}$$

(c) the expected value of $X$ is

$$\int_{-5}^{5} \frac{x}{10} dx = \frac{x^2}{20} \bigg|_{-5}^{5} = 0 \tag{4}$$

Another way to obtain this answer is to use Theorem 3.6 which says the expected value of $X$ is

$$E[X] = \frac{5 + (-5)}{2} = 0 \tag{5}$$

(d) The fifth moment of $X$ is

$$\int_{-5}^{5} \frac{x^5}{10} dx = \frac{x^6}{60} \bigg|_{-5}^{5} = 0 \tag{6}$$

(e) The expected value of $e^X$ is

$$\int_{-5}^{5} \frac{e^x}{10} dx = \frac{e^5 - e^{-5}}{10} = 14.84 \tag{7}$$
Problem 3.5.10 Solution

This problem is mostly calculus and only a little probability. From the problem statement, the SNR $Y$ is an exponential $(1/y)$ random variable with PDF

$$f_Y(y) = \begin{cases} \frac{1}{y} e^{-y/y} & y \geq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

Thus, from the problem statement, the BER is

$$P_e = E[P_e(Y)] = \int_{-\infty}^{\infty} Q(\sqrt{2y}) f_Y(y) \, dy = \int_{0}^{\infty} Q(\sqrt{2y}) \frac{y}{y} e^{-y/y} \, dy \quad (2)$$

Like most integrals with exponential factors, it’s a good idea to try integration by parts. Before doing so, we recall that if $X$ is a Gaussian $(0, 1)$ random variable with CDF $F_X(x)$, then

$$Q(x) = 1 - F_X(x) \quad (3)$$

It follows that $Q(x)$ has derivative

$$Q'(x) = \frac{d}{dx} Q(x) = -\frac{d}{dx} F_X(x) = -f_X(x) = -\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad (4)$$

To solve the integral (2), we use the integration by parts formula $\int_a^b u \, dv = uv|_a^b - \int_a^b v \, du$, where

$$u = Q(\sqrt{2y}) \quad dv = \frac{1}{y} e^{-y/y} \, dy \quad (5)$$

$$du = Q'(\sqrt{2y}) \frac{1}{\sqrt{2y}} = -\frac{e^{-y}}{2\sqrt{\pi} y} \quad v = -e^{-y/y} \quad (6)$$

From integration by parts, it follows that

$$P_e = uv|_0^{\infty} - \int_0^{\infty} v \, du \quad (7)$$

$$= -Q(\sqrt{2y}) e^{-y/y} \bigg|_0^{\infty} - \int_0^{\infty} \frac{1}{\sqrt{\pi}} e^{-y/(1+y)} \, dy \quad (8)$$

$$= 0 + Q(0) e^{-0} - \frac{1}{2\sqrt{\pi}} \int_0^{\infty} y^{-1/2} e^{-y/(1+y)} \, dy \quad (9)$$

where $\tilde{y} = y/(1+y)$. Next, recalling that $Q(0) = 1/2$ and making the substitution $t = y/\tilde{y}$, we obtain

$$P_e = \frac{1}{2} - \frac{1}{2\sqrt{\pi}} \int_0^{\infty} t^{-1/2} e^{-t} \, dt \quad (10)$$

From Math Fact B.11, we see that the remaining integral is the $\Gamma(z)$ function evaluated $z = 1/2$.

Since $\Gamma(1/2) = \sqrt{\pi}$,

$$P_e = \frac{1}{2} - \frac{1}{2\sqrt{\pi}} \Gamma(1/2) = \frac{1}{2} \left[ 1 - \sqrt{\frac{y}{1+y}} \right] = \frac{1}{2} \left[ 1 - \sqrt{\frac{y}{1+y}} \right] \quad (11)$$
Problem 3.5.3 Solution

$X$ is a Gaussian random variable with zero mean but unknown variance. We do know, however, that

$$P [|X| \leq 10] = 0.1$$ \hfill (1)

We can find the variance $\text{Var}[X]$ by expanding the above probability in terms of the $\Phi(\cdot)$ function.

$$P [-10 \leq X \leq 10] = F_X (10) - F_X (-10) = 2\Phi \left( \frac{10}{\sigma_X} \right) - 1$$ \hfill (2)

This implies $\Phi(10/\sigma_X) = 0.55$. Using Table 3.1 for the Gaussian CDF, we find that $10/\sigma_X = 0.15$ or $\sigma_X = 66.6$. 
Problem 3.6.1 Solution

(a) Using the given CDF

\[ P[X < -1] = F_X(-1^-) = 0 \]  \hspace{1cm} (1)
\[ P[X \leq -1] = F_X(-1) = -1/3 + 1/3 = 0 \]  \hspace{1cm} (2)

Where \( F_X(-1^-) \) denotes the limiting value of the CDF found by approaching \(-1\) from the left. Likewise, \( F_X(-1^+) \) is interpreted to be the value of the CDF found by approaching \(-1\) from the right. We notice that these two probabilities are the same and therefore the probability that \( X \) is exactly \(-1\) is zero.

(b)

\[ P[X < 0] = F_X(0^-) = 1/3 \]  \hspace{1cm} (3)
\[ P[X \leq 0] = F_X(0) = 2/3 \]  \hspace{1cm} (4)

Here we see that there is a discrete jump at \( X = 0 \). Approached from the left the CDF yields a value of \( 1/3 \) but approached from the right the value is \( 2/3 \). This means that there is a non-zero probability that \( X = 0 \), in fact that probability is the difference of the two values.

\[ P[X = 0] = P[X \leq 0] - P[X < 0] = 2/3 - 1/3 = 1/3 \]  \hspace{1cm} (5)
Problem 3.7.1 Solution
Since $0 \leq X \leq 1$, $Y = X^2$ satisfies $0 \leq Y \leq 1$. We can conclude that $F_Y(y) = 0$ for $y < 0$ and that $F_Y(y) = 1$ for $y \geq 1$. For $0 \leq y < 1$,

$$F_Y(y) = P \left[ X^2 \leq y \right] = P \left[ X \leq \sqrt{y} \right]$$

(1)

Since $f_x(x) = 1$ for $0 \leq x \leq 1$, we see that for $0 \leq y < 1$,

$$P \left[ X \leq \sqrt{y} \right] = \int_0^{\sqrt{y}} dx = \sqrt{y}$$

(2)

Hence, the CDF of $Y$ is

$$F_Y(y) = \begin{cases} 
0 & y < 0 \\
\sqrt{y} & 0 \leq y < 1 \\
1 & y \geq 1 
\end{cases}$$

(3)

By taking the derivative of the CDF, we obtain the PDF

$$f_Y(y) = \begin{cases} 
1/(2\sqrt{y}) & 0 \leq y < 1 \\
0 & \text{otherwise} 
\end{cases}$$

(4)