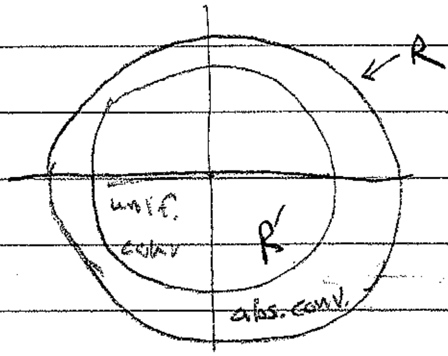
Last time:

$f = \sum a_n x^n$

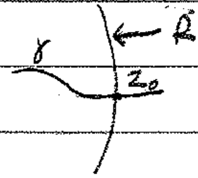


terms unbounded.

What about $|z| = R$?

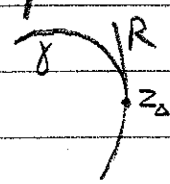
(Depends, $\sum z^n \to \infty$ on all $|z|=1$, $\sum \frac{1}{n} z^n$ converges on $|z|=1, z \neq 1$)

Suppose:



Expect, if $f(z_0)$ converges, $f(z) \to f(z_0)$ as $z \to z_0$ along γ.

Possible problem:

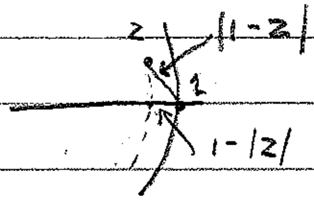


f may diverge on $|z| = R$ near z_0

<u>Abel's Theorem:</u>

Suppose $f = \sum a_n z^n$ has Hadamard radius 1, $f(1)$ converges.

Then, as $z \to 1$ s.t. $\frac{|1-z|}{1-|z|}$ is bounded, $f(z) \to f(1)$.



Then $R_j = 0$ iff

1) $Q_j' = 0$ and

2) $j = \deg(F_k)$ or $Q_j = 0$.

Thus some F_k must be of the form

$$c f^m = 0 \qquad c \in \mathbb{C}^\times \qquad \square$$

Can get non algebraic functions via power series (at least up to radius of convergence).

If $f' = f$, $f(0) = 0$, we get the zero power series, with $R = \infty$.

~~Setting $z=0$ gives $a_k = \frac{f^{(k)}(0)}{k!}$~~
~~so if f has a power series development~~
~~it must be the Taylor expansion.~~
~~(and is unique).~~

~~Exponential and trigonometric functions~~

Define $f(x)=e^x : \mathbb{C} \longrightarrow \mathbb{C}$ s.t.
1) $f'(x)=f(x)$
2) $f'(0)=1$

Then, if it has a power series expansion:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots$$
$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots$$

Thus $a_0 = 1$, $a_1 = a_0 = 1$, $a_2 = \frac{a_1}{2}$, ... $a_k = \frac{1}{k!}$

Thus $e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^k}{k!} + \ldots$

if this converges.

Since we can choose n large enough s.t. $\frac{|x|^2}{(n-k)(k+1)} < \frac{1}{4}$

for all $k \in 0, \ldots, n-1$,

$\Sigma \frac{1}{2^n}$ eventually majorizes $\frac{|x|^n}{n!}$.

Thus e^x converges on the whole plane.

Addition: If $a, b \in \mathbb{C}$ let $c = a+b$.

Then $e^a e^b = e^a e^{c-a} = (e^z e^{c-z})\big|_a$

But $D(e^z e^{c-z}) = e^z e^{c-z} - e^z e^{c-z} = 0$

This implies that $e^z e^{c-z}$ is constant

since if $Re(e^{z_1} e^{c-z_1}) \neq Re(e^{z_2} e^{c-z_2})$ then

the derivative is nonzero somewhere

on the line segment $tz_1+(1-t)z_2$, $0\le t\le 1$

by the intermediate value theorem.

(similarly for Im)

Thus $e^a e^b = e^0 e^{c-0} = e^c = e^{b+a}$.

Note that $e^z e^{-z} = e^0 = 1 \Rightarrow e$ is never 0.

We can similarly define sin, cos s.t.

$\sin(0)=0$, $\cos(0)=1$

$\sin'(z)=\cos(z)$

$\cos'(z)=-\sin(z)$

We get

$$\cos(z) = 1-\frac{z^2}{2!}+\frac{z^4}{4!}$$

$$\sin(z) = x-\frac{z^3}{3!}+\frac{z^5}{5!}-\dots$$

since clearly $\cos(z) = \dfrac{e^{iz}+e^{-iz}}{2}$

$$\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$$

these series converge on all of z.

These formulas also imply

$$\cos(z)+i\sin(z) = e^{iz}$$

$$\cos^2(z)+\sin^2(z) = \frac{1}{2}+\frac{1}{2} = 1$$

and by addition of exponents:

$$\cos(z_1+z_2) = \cos(z_1)\cos(z_2)-\sin(z_1)\sin(z_2)$$

$$\sin(z_1+z_2) = \sin z_1 \cos z_2 + \sin z_2 \cos z_1$$

Periodicity:

$f: \mathbb{C} \to \mathbb{C}$ has period c if $f(z+c) = f(z)$ $\forall z \in \mathbb{C}$.

Claim: e^z has a period.

By addition, $e^{z+(a+bi)} = e^{a+bi} e^z$, so $a+bi$ is period iff $e^{a+bi} = 1$.

Since $|e^{a+bi}| = |e^a||e^{bi}|$, must have $a=0$.

$e^{bi} = \cos b + i \sin b$ and $\cos^2 b + \sin^2 b = 1$,

so need $b \neq 0$ s.t. $\sin b = 0$.

$\sin' 0 = \cos 0 = 1$, so there is a unique smallest period b.

Since $\sin b = \sin \frac{b}{2} \cos \frac{b}{2} + \cos \frac{b}{2} \sin \frac{b}{2}$

$\cos \frac{b}{2} = 0$ is the least positive zero for cos,

$\cos^2 x + \sin^2 x = 1 \Rightarrow \cos x \leq 1$ $\forall x \in \mathbb{R}$.

$\int_0^x$ on both sides gives

$\sin x \leq x$

again

$-\cos x \leq \frac{x^2}{2} - 1$

$-\sin x \leq \frac{x^3}{6} - x$

$\cos x \leq \frac{x^4}{24} - \frac{x^2}{2} + 1$

Thus $\cos \sqrt{3} \leq \frac{9}{24} - \frac{3}{2} + 1 < 0$.

Thus cos, sin, exp share a period smaller than $2\sqrt{3}$,

since $\cos' x = -\sin x \leq \frac{x^3}{6} - x$ is negative on $0 < x < \sqrt{3}$,

cos is strictly decreasing, so there is only one period in $0 < x < 2\sqrt{3}$. Call it 2π.

Functions so far:

$f(z) = c$

$f(z) = z$

by $\times$, $+$ rules:

$f(z) = P(z)$

by $\div$ rule:

$f = P/Q \iff Qf = P$

More generally could do Algebraic Functions

$$P_n f^n + P_{n-1} f^{n-1} + \dots + P_0 = 0$$

e.g. $f^2(z) - z = 0$ $(\sqrt{\ })$

or solve diff. eq. s

simplest $f' - P = 0$

next $f' - f = 0$ — Algebraic?

Thm: $f' = f$, f algebraic $\Rightarrow f = 0$.

Pf: Suppose f satisfies

$F_0 = P_n f^n + \dots + P_0 = 0$, some $P_0, \dots, P_n$

Given F_k, define $F_{k+1} = \deg_f F_k \cdot F_k - F_k'$.

$F_k = 0 \Rightarrow F_{k+1} = 0$.

Suppose $F_k = Q_m f^m + \dots + Q_0$

Then $F_k' = (Q_m' + m Q_m) f^m + (Q_{m-1}' + m-1\, Q_{m-1}) f^{m-1} + \dots + Q_0'$

Let $F_{k+1} = R_m f^m + \dots + R_0$

Logarithm

inverse function for e^z.
$\log(0)$ d.n.e.
by periodicity, log is only unique up to $\pm 2\pi i n$, i.e.
$e^{\log x + 2\pi i n} = e^{\log x} e^{2\pi i n} = x$.

If $w \neq 0$,
$$e^{x+iy} = w \Rightarrow |e^x| = |w|$$
$$e^{iy} = \frac{w}{|w|}$$ unique up to $2\pi i$
$$\log w = \log |w| + i \arg(w)$$

Can define a^b as $e^{(\log a) b}$.
By convention, if $a \in \mathbb{R}$, assume $\log a \in \mathbb{R}$.
In general, a^b is unique up to $\mathbb{Z}$ multiples of $e^{2\pi i b}$.

We can define inverse trig functions in terms of the log:
$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = w \Leftrightarrow$$
$$(e^{iz})^2 - 2we^{iz} + 1 = 0$$
$$e^{iz} = \frac{2w \pm \sqrt{4w^2 - 4}}{2} = w \pm \sqrt{w^2 - 1}$$
$$iz = \log(w \pm \sqrt{w^2-1})$$
$$= \pm \log(w + \sqrt{w^2-1}) \quad \left(\text{since } (w + \sqrt{w^2-1})^{-1} = w - \sqrt{w^2-1}\right)$$

$$\arccos z = \pm i \log(w + \sqrt{w^2-1})$$
$$\arcsin z = \frac{\pi}{2} + \arccos z$$