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<u>Theorem</u>: Let $S(x) = \sum_n a_n x^n$. Then $\exists R$, $0 \le R \le \infty$ s.t.

1) S converges absolutely on $|x| < R$
2) " " uniformly " $|x| < R'$ if $R' < R$
3) if $|x| > R$, $(a_n x^n)_n$ is unbounded,
4) S is analytic on $|z| < R$ with termwise derivative and same radius.
5) $R = \left(\limsup_{n\to\infty} \sqrt[n]{|a_n|}\right)^{-1}$ ← hadamard's formula.

Pf: Clearly, at most one choice of R works.

(3): if $|x| \ge R(1+\varepsilon_1)$ we can find N s.t. $\forall N_1 > N$ we have $\sup_{n > N_1} \sqrt[n]{|a_n|} > \frac{(1-\varepsilon_2)}{R}$. Thus there are ∞ many terms $n > N$ s.t. $|a_n x^n| = |a_n| R^n (1+\varepsilon_1)^n > \frac{R^n}{R^n} (1+\varepsilon_1)^n (1-\varepsilon_2)^n = (1+\varepsilon_1 - \varepsilon_2 - \varepsilon_1\varepsilon_2)^n \xrightarrow[n\to\infty]{\varepsilon_2 \text{ small}} \infty$

(1 & 2) if $|x| < R' < R$, then $\exists N$ s.t. $n > N \Rightarrow$ $\left(\sup_{n\to\infty} \sqrt[n]{|a_n|}\right)^{-1} k > R'$ for $k < 1$.

Then $|a_n x^n| \le \left(\frac{k}{R'}\right)^n R'^n = k^n$, a geometric series independent of x.

By the Weierstrass M-test, 1 & 2 hold.

(4) The series $\hat{S}(x) = \sum_n n a_{n+1} x^{n}$ converges on $R'' = \left(\limsup_{n\to\infty} \sqrt[n]{|n a_{n+1}|}\right)^{-1}$

since $\sqrt[n]{n a_{n+1}} = \sqrt[n]{n} \cdot \sqrt[n+1]{a_{n+1}} \cdot (a_{n+1})^{\left(\frac{1}{n(n+1)}\right)}$

We have $R'' = R$ if $\lim_{n\to\infty} \sqrt[n]{n} = 1$

if $\sqrt[n]{n} = 1 + \delta_n$, we have $n = (1+\delta_n)^n = 1 + n\delta_n + \frac{n(n-1)}{2}\delta_n^2 + \ldots$

Thus $\delta_n^2 < \frac{2}{n-1}$, so $\sqrt[n]{n} \to 1$, so $R = R''$.

Let $R_n(x) = S(x) - S_n(x)$, $|z_0| < R' < R$, $|z| < R' < R$.

Then $\frac{S(z) - S(z_0)}{z - z_0} - \hat{S}(z_0) =$

$$(*) \quad \left(\frac{S_n(z) - S_n(z_0)}{z - z_0} - \hat{S}_n(z_0)\right) + \left(\hat{S}_n(z_0) - \hat{S}(z_0)\right) + \frac{R_n(z) - R_n(z_0)}{z - z_0}$$

for any n.

For fixed z, z_0 s.t. $|z|<R'$, $|z_0|<R'$ we can write

$$\frac{R_n(z)-R_n(z_0)}{(z-z_0)} = \sum_{k=n+1}^{\infty} a_k\left(z^{k-1}+z^{k-2}z_0+\dots+2z_0^{k-2}+z_0^k\right)$$

since $(z^k-z_0^k) = (z-z_0)(z^{k-1}+z^{k-2}z_0+\dots+2z_0^{k-2}+z_0^k)$

Then $\left|\frac{R_n(z)-R_n(z_0)}{(z-z_0)}\right| \leq \sum_{k>n} k\, a_k R'^{\,k-1}$ for some $R'<R$ if $z_0<R$.

But this series converges, since $\sqrt[k]{k+1}\to 1$ (similar to before), uniformly in z, z_0 provided both have norm $\leq R'$

Thus we can choose large n s.t.

$$\frac{R_n(z)-R_n(z_0)}{z-z_0} < \frac{\varepsilon}{3}$$

and also clearly if n is large enough

$$|\hat{S}_{n-1}(z_0)-\hat{S}(z_0)|<\frac{\varepsilon}{3}.$$

Since $S_n' = \hat{S}_{n-1}$ we can choose δ for each ε s.t. $|z-z_0|<\delta \Rightarrow \left(\frac{S_n(z)-S_n(z_0)}{z-z_0}\right)-\hat{S}_{n-1}(z_0)<\frac{\varepsilon}{3}$

Thus $S'(z_0)=\hat{S}(z_0)$ $\forall z_0\in E$ since $(*)\to 0$ as $z\to z_0$ with any sufficiently large choice of n. $\square$

If $f(z) = a_0+a_1z+a_2z^2+\dots$, then

$f'(z) = a_1+2a_2z+\dots$

$\vdots$

$f^{(k)}(z) = k!\,a_k+\frac{(k+1)!}{1!}a_{k+1}z+\dots\frac{(k+\ell)!}{\ell!}a_{k+\ell}z^\ell+\dots$

are all analytic.

Setting $z=0$ gives $a_k = \frac{f^{(k)}(0)}{k!}$
so if f has a power series development
it must be the Taylor expansion.
(and is unique).

Exponential and trigonometric functions

Define $f(x)=e^x : \mathbb{C} \longrightarrow \mathbb{C}$ s.t.

1) $f'(x) = f(x)$
2) $f(0) = 1$

Then, if it has a power series expansion:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots$$
$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \dots$$

Thus $a_0 = 1,\ a_1 = a_0 = 1,\ a_2 = \frac{a_1}{2}, \dots a_k = \frac{1}{k!}$

Thus $e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^k}{k!} + \dots$
if this converges.
Since we can choose n large enough s.t. $\frac{|x|^2}{(n-k)(k+1)} < \frac{1}{4}$
for all $k \in 0, \dots, n-1$,
$\sum \frac{1}{2^n}$ eventually majorizes $\frac{|x|^n}{n!}$.
Thus e^x converges on the whole plane.

Addition: If $a, b \in \mathbb{C}$ let $c = a+b$.
Then $e^a e^b = e^a e^{c-a} = (e^z e^{c-z})\big|_a$.
But $D(e^z e^{c-z}) = e^z e^{c-z} - e^z e^{c-z} = 0$
This implies that $e^z e^{c-z}$ is constant
since if $Re(e^{z_1} e^{c-z_1}) \neq Re(e^{z_2} e^{c-z_2})$ then
the derivative is nonzero somewhere