Rational functions

By the Fundamental Theorem of Algebra (proven later), every $\mathbb{C}$ polynomial satisfies $P(x)=0$ for some x.
Suppose $P(a)=0$ for a polynomial $P(x)$ of degree n.
Then by division $P(x)=(x-a)P'(x)$
for some $P'(x)$ of degree $n-1$.
Continuing, we get

$$P(x)=(x-a_1)(x-a_2)\cdots(x-a_n)\cdot k$$

Each a_i is called a zero of P.
The number of times it appears is its order.
The degree of P is the sum of the orders of the zeros.
derivative decrements orders of zeros → By convention, the degree of the zero polynomial is $-\infty$.

Rational functions are of the form

$$R(x)=\frac{P(x)}{Q(x)}$$ where P, Q have no common zeros.

R is defined whenever $Q(x)\neq 0$.
Points x_0 s.t. $Q(x_0)=0$ are called poles of R.
The order of a pole x_0 is the order of x_0 in Q.

Taking the derivative

$$R'(x)=\frac{P'(x)Q(x)-P(x)Q'(x)}{Q^2(x)}$$

increments the order of each pole.
decrements the order of each zero.

By convention, we also consider $R(\infty)$, given by $R(\frac{1}{x})$ at $x=0$ as a potential pole or zero.

the rational form $R_1(x)$ for

Suppose $R(x) = \dfrac{a_0 + a_1 x + \dots + a_n x^n}{b_0 + b_1 x + \dots + b_m x^m}$.

Then $\quad R_1(x) = \dfrac{x^m (a_0 x^n + a_1 x^{n-1} + \dots + a_n)}{x^n (b_0 x^m + \dots + b_m)}$

If $m > n$, R_1 has a zero of order $m-n$ at $x=0$
If $m < n$, R_1 " " pole " " $n-m$ " "

Thus, counting $R(\infty)$
$\#$ zeros in $R = \max(m,n)$
$\#$ poles in $R = \max(m,n)$

This number is called the <u>order</u> of R.

<u>Prop</u>: Let R be rational with order n.
Then $\forall c \in \mathbb{C}$, $R(x) = c$ has n roots, counting multiplicity.
<u>Pf</u>: Since $R(x)$ has n poles, so does $R(x) - c$.
Thus $R(x) - c$ has n zeros.

<u>Def</u>: A (Fractional) linear transformation is an order 1 rational function.

Prop: R is invertible iff of order 1

Pf: If order$(R)=1$, then

$$R = \frac{az+b}{cz+d} \quad \text{with } ad-bc \neq 0,$$

Then $R^{-1} = \dfrac{dz-b}{-cz+a}$.

Conversely, if R is invertible, it has one pole and one zero, thus

$$R = \frac{k(z-a)^m}{(z-b)^m} = \left(\sqrt[m]{k}\,\frac{z-a}{z-b}\right)^m, \left(\frac{1}{z-b}\right)^m \text{ or } (z-b)^m.$$

In any case it is the mth power of an invertible function. If $m>1$, R is not invertible.

Prop (partial fraction representation): R is a sum of rational functions with only one pole.

Pf: By induction on # of poles.

Let $R(p)=\infty$, $R(x) = \frac{P(x)}{Q(x)}$

If $p=\infty$, Then divide P by Q to get $R(x) = G(x)+H(x)$, where $G(x)$ is a polynomial, $H(x)$ is finite at ∞.

Since $R = G+H$, G has one pole at ∞, H has $n-1$,

If $p \neq \infty$, Let $R'(\xi) = R\left(p-\frac{1}{\xi}\right)$, so $R(z) = R'\left(\frac{1}{p-z}\right)$.

Then dividing P' by Q' gives $G'(\xi)+H'(\xi) = G'\left(\frac{1}{p-z}\right)+H'\left(\frac{1}{p-z}\right)$.

Then, similarly, $G'\left(\frac{1}{p-z}\right)$ has one pole at p, $H'\left(\frac{1}{p-z}\right)$ has $n-1$ poles.

Note: by adding a constant we can assume that the root of each of these functions is at zero,

Power Series

Power series are the completion of polynomial functions on compact subsets of $\mathbb{C}$.

Def: A sequence $a_1, a_2, \ldots$ $(a_n)_n$ is a Cauchy sequence if $\forall \varepsilon > 0 \ \exists N$ s.t. $m,n > N \Rightarrow |a_m - a_n| < \varepsilon$.

Prop: $(a_n)_n$ a complex sequence is Cauchy iff $(Re(a_n))_n$ and $(Im(a_n))_n$ are.

Pf: $|z| \leq \sqrt{2}\, Max(|Re(z)|, |Im(z)|)$, $|Re(z)| \leq |z|$, $|Im(z)| \leq |z|$

A series $S = \sum_{n=1}^{\infty} a_n$
has an associated sequence $(S_n)_n$ s.t. $S_n = \sum_{k=1}^{n} a_k$.
We say S converges if $(S_n)_n$ does.
S converges absolutely if $\sum_{i=1}^{n} |a_n|$ converges.

Since $|S_n - S_m| = |a_{m+1} + \ldots + a_n| \leq |a_{m+1}| + \ldots + |a_n|$,
abs. convergence $\Rightarrow$ convergence.

Note: $\underline{\lim}\,(a_n) = \lim_{N\to\infty} \inf_{n \in N} (a_n)$ may not exist nor $\overline{\lim}$ (a.k.a. lim inf, lim sup)

Functions: Let $f_n(x)$ be a sequence of functions.
f_n converges ~~pointwise~~ to f at x if $\lim_{n\to\infty} f_n(x) = f(x)$.

f_n converges uniformly on a set E to f if
$\forall \varepsilon > 0 \ \exists N$ s.t. $n > N \Rightarrow (x \in E \Rightarrow |f_n(x) - f(x)| < \varepsilon)$

Prop: Let $f_n \to f$ uniformly on E. If f_n are continuous, so is $f|_E$.

pf: choose N s.t. $|f_n(x) - f(x)| < \frac{\varepsilon}{3} \quad \forall x \in E$
for $x_0 \in E$, choose δ s.t. $|x - x_0| < \delta \Rightarrow |f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}$.
Then $|f(x) - f(x_0)| \le$
$|f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f(x_0) - f_n(x_0)| = \varepsilon$.

Test for uniform convergence (Weierstraus M-test)
A series $\sum a_n$ majorizes $\sum f_n$ if
$\exists M$ s.t. $|f_n(x)| \le M a_n \ \forall n \in \mathbb{N}, \forall x \in E$.
If $\sum a_n$ converges, then $\sum f_n(x)$ converges absolutely, and f_n converges uniformly on E.

Power Series

A power series is of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

with $a_n, z, z_0 \in \mathbb{C}$. We'll assume $z_0 = 0$ since this changes little.

E.G. $S = \sum_n z^n$. Clearly,
S diverges if $|z| > 1$
$s_n = \frac{(z^n - 1)}{z - 1} \longrightarrow \frac{1}{1-z}$ if $|z| < 1$.