1/21/09 ①

Last time: Analytic functions

$$f(z) = u(z) + iv(z) \text{ analytic}$$
$$f'(z) = \frac{du}{dx} + i\frac{dv}{dx}$$
$$= -i\left(\frac{du}{dy} + i\frac{dv}{dy}\right) = \frac{dv}{dy} - i\frac{du}{dy}$$
$$\frac{du}{dx} = \frac{dv}{dy} \quad \frac{du}{dy} = -\frac{dv}{dx} \quad \text{— cauchy riemann}$$

First examples: polynomials are analytic since $z \to kz$ is by closure under sums and products.

derivatives follow from $(f \cdot g)' = f'g + g'f$

Note: $z \to \overline{z}$ is not analytic.

(recall Cauchy-Riemann)

Harmonic Functions

Later we'll show that if f is analytic then so is f'.

This means that if $f(z) = u(z) + iv(z)$

$$f'(z) = \frac{du}{dx} - i\frac{du}{dy} = \frac{dv}{dy} + i\frac{dv}{dx}$$ and since f' is analytic

$$f''(z) = \frac{d^2u}{dx^2} - i\frac{d^2v}{dy^2} = -\frac{d^2u}{dy^2} \pm \frac{d^2v}{dx^2}$$

so $\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0$ and $\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = 0$

Note: f' is conjugate to the gradient of u.

The LHS are denoted Δu and Δv respectively and

$\Delta u = 0$ is known as <u>Laplace's Equation</u> for u.

Any two real-valued differentiable functions u, v satisfying Laplace's equation and Cauchy-Riemann are called <u>conjugate harmonic functions</u>.

One Idea: Think of u as the "potential" field of a fluid flow field with vectors given by $v(z) = f'(z) = \frac{du}{dx} + i\frac{du}{dy}$

Laplace's first equation says that if the flow is accellerating in the x direction, it must decellerate proportionally in the y direction.

A flow with this property is <u>incompressible</u>.

The second equation (rewritten $-\frac{d^2u}{dy\,dx} + \frac{d^2u}{dx\,dy}$) says that if the x-ward flow is increasing as we move y-ward, the y-ward flow is proportionally increasing as we move x-ward.

A flow with this property is called <u>irrotational</u>.

$\overline{f'} = \frac{du}{dx} + i\frac{du}{dy}$ is the gradient vector field for u, so u is constant along curves which are at every point perpendicular to the direction of flow.

The Riemann equations imply that $\frac{dv}{dx} + i\frac{dv}{dy} = -i\left(\frac{du}{dx} + i\frac{du}{dy}\right)$, so v is constant along flow lines of the gradient flow for u.

Another Idea: steady state heat equation in homogeneous medium

The flow rate of heat energy is proportional to temperature gradient

$q = -k\left(\frac{du}{dx} + i\frac{du}{dy}\right)$ (u = temperature function)

constant flow → constant temperature

accellerating flow → increasing temperature in 1 dim'n

In 2 dimensions accelleration in x direction may be offset by decelleration in y.

In this case, $\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0$

Many of the results we'll prove have intuitive interpretations as properties of physical systems.

Every harmonic function has a conjugate, computable by integration. It is unique up to addition by a constant.

Computing f from u without integrating

Requirements: $u(z) = u(x,y)$ can be written as a composite of $\mathbb{R}$-valued functions in x and y, each of which extends to a $\mathbb{C}$-valued function. (e.g. rational functions).

To construct $f = u + vi$, define $\bar{f} : \mathbb{C} \to \mathbb{C}$ s.t. $\bar{f}(z) = \overline{f(\bar{z})}$

Then $u(z) = u(x,y) = \frac{1}{2}\left(f(x+iy) + \bar{f}(x-iy)\right)$

$\cancel{v(z) = v(x,y) = \frac{-i}{2}\left(f(x+iy) - \bar{f}(x-yi)\right)}$

Substitute complex values for x, y:

$x = \frac{z}{2} \qquad y = \frac{-iz}{2}$

Then

$u\left(\frac{z}{2}, \frac{-iz}{2}\right) = \frac{1}{2}\, f(z) + \bar{f}(0)$

Since v is unique up to sclar, we can assume $f(0) \in \mathbb{R}$, so $f(0) = u(0,0) + i \cdot 0$.

Thus $\frac{1}{2}(u(0,0)) = \bar{f}(0)$,

Thus $f(z) = 2u\left(\frac{z}{2}, \frac{-iz}{2}\right) - u(0,0)$.

Analytic functions from harmonic conjugates

prop: if u, v are har. conjugates with continuous first order p-derivatives, then $f = u + vi$ is analytic.

Proof:

For $\varepsilon_1 > 0$ given, can choose δ_1 s.t.

$|h| < \delta_1$ and $|k| < \delta_1 \Rightarrow \left| u(x+h, y+k) - u(x,y) - \left(\frac{du}{dx} \cdot h + \frac{du}{dy} \cdot k\right)\right| < (h+k)\varepsilon_1$.

Similarly for ε_2 one can choose δ_2 s.t. for small h, k

$\left| v(x+h, y+k) - v(x,y) - \left(\frac{dv}{dx} \cdot h + \frac{dv}{dy} \cdot k\right)\right| < (h+k)\varepsilon_2$

Then using the Cauchy-Riemann equations

$\dfrac{f(x+h+ik) - f(x)}{h+ik} = \frac{du}{dx} + i\frac{dv}{dx} + \frac{(h+k)}{h+ik}\varepsilon_3$, where

$|Re(\varepsilon_3)| < \varepsilon_1$ and $|Im(\varepsilon_3)| < \varepsilon_2$.

Since $\left|\frac{h+k}{h+ik}\right|^2 = \frac{h^2+k^2+2hk}{h^2+k^2} = 1 + \frac{2hk}{h^2+k^2} < 2$

(by Cauchy's Inequality: $|(h,k) \cdot (k,h)| \le \|(h,k)\| \, \|(k,h)\|$)

is bounded, f is analytic.