The fact that three points determine a circle or line implies that lines and circles on the plane are in bijective correspondence with circles on the sphere.

II Complex functions

Want to consider functions

$$\mathbb{R} \to \mathbb{C}, \quad \mathbb{C} \to \mathbb{R} \quad \text{or} \quad \mathbb{C} \to \mathbb{C}$$

must be well defined single-valued

In any case, these are maps between metric spaces, so we have notions of limits and continuity

e.g. $\lim_{x \to a} f(x) = A$ iff $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t.

$$d(x, x') < \delta \Rightarrow d(f(x), f(x')) < \varepsilon$$

Standard rules for interchanging sums, products, quotients with limits still hold.

Since $\lim_{x \to a} f(x) = A \Leftrightarrow \lim_{x \to a} \overline{f(x)} = \overline{A}$,

we get $\lim_{x \to a} \mathrm{Re}(f(x)) = \mathrm{Re}(A)$ and

" Im " = Im "

Furthermore, can define the derivative of f at a

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Behavior here depends on whether dom(f), ran(f) are $\mathbb{R}$ or $\mathbb{C}$

e.g. $z \to \mathrm{Re}(z)$ is differentiable when considered as a function $\mathbb{R}^2 \to \mathbb{R}$.
however,

$$\frac{\mathrm{Re}(z+h) - \mathrm{Re}(h)}{h} \text{ is } \begin{array}{l} 1 \text{ for real } h \\ 0 \text{ for imaginary } h. \end{array}$$

For any $f: \mathbb{C} \to \mathbb{R}$ the difference quotient is real for real h, imaginary for imaginary h. Thus we have proven:

<u>Prop</u>: If $f: \mathbb{C} \to \mathbb{R}$, then $f'(x) = 0$ or d.n.e.

Functions $f: \mathbb{R} \to \mathbb{C}$ may, if desired, be broken up into functions u and $v: \mathbb{R} \to \mathbb{R}$
i.e. $f(t) = u(t) + iv(t)$

Analytic Functions

<u>Def</u>: An <u>analytic function</u> (or <u>holomorphic</u>) $f: \mathbb{C} \to \mathbb{C}$ is a function s.t. $f'(z)$ exists wherever f is defined.

If f, g are analytic then so are $f+g$, fg, f/g by limit rules.

If $f'(z)$ exists, then f is continuous at z, since $\lim_{h \to 0} f(z+h) - f(z) = \lim_{h \to 0} h \frac{f(z+h)-f(z)}{h}$

$= \lim_{h \to 0}(h) \cdot \lim_{h \to 0} \frac{f(z+h)-f(z)}{h} = 0 \cdot f'(z) = 0.$

If we restrict to $h \in \mathbb{R}$ we get

$f'(z) = \lim_{h \to 0} \frac{f(x+h+yi) - f(x+iy)}{h} = \frac{df}{dx} = \frac{du}{dx} + i\frac{dv}{dx}$

With restriction $h \in i\mathbb{R}$ we get

$f'(z) = -i\frac{df}{dy} = \frac{dv}{dy} - i\frac{du}{dy}$

Thus we must have $\frac{du}{dx} = \frac{dv}{dy}$ and $\frac{dv}{dx} = -\frac{du}{dy}$

These are the Cauchy-Riemann equations.

In this case, the jacobian $\frac{du}{dx}\frac{dv}{dy} - \frac{dv}{dx}\frac{du}{dy}$

$= \left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dx}\right)^2 = |f'(z)|^2.$