$\boxed{\text{Hw: P.9 3-4, P.11 1, P.20 5}}$

Geometric Representation

$(\mathbb{C}, +, \text{scalar multiplication})$ is a 2-dimensional real vector space. If $c \in \mathbb{C}$, $z \to cz$ is a linear map, as is $z \to \bar{z}$.

Exercise: every $\mathbb{R}$-linear map is of the form $z \to az + b\bar{z}$ for some $a, b \in \mathbb{C}$

$\times$ is best interpreted using polar coordinates

$(r, \sin\theta) \to r(\cos\theta + i\sin\theta)$

$r \geq 0$ is the modulus $r = |z|$

θ defined $\pm 2n\pi$ is the argument

Then

$$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1) r_2 (\cos\theta_2 + i\sin\theta_2)$$
$$= r_1 r_2 (\cos\theta_1\cos\theta_2 - \cos\theta_2\cos\theta_1 + i(\sin\theta_1\cos\theta_2 + \cos\theta_2\sin\theta_1)$$
$$= r_1 r_2 (\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$$

adds arguments, multiplies moduli.

This gives $z^n = r^n(\cos n\theta + i\sin n\theta)$, so every $c \in \mathbb{C}^\times$ has n distinct roots, with arguments differing by $\frac{2\pi k}{n}$, $k \in \mathbb{Z}$, i.e., $z^n = (\omega z)^n$ iff $\omega^n = 1$.

Also implies $\arg(z) = -\arg(z^{-1})$
$\text{mod}(z^{-1}) = (\text{mod}(z))^{-1}$

Analytic Geometry

Many elementary curves can be expressed as relations between z and $\bar{z}$

E.G. circles: $|z-c|^2 = r^2$

lines: $z = a + bt$ $a, b \in \mathbb{C}$ t real

$a' + b't$ gives the same line if b' and $a'-a$ are real multiples of b.

parallel lines if $b' = kb$

(signed) angle of intersection is $\arg\left(\frac{b}{b'}\right)$
(oriented)

other conics: ellipses, parabolas and hyperbolas can be described as the locus of points such that the ratio k of the distance to a fixed point (the focus) to the distance to a fixed line (the directrix) is a constant $<1, =1, >1$ respectively

The function "signed distance to a line through the origin" is $\mathbb{R}$-linear. For the line containing 0 and $\ell \neq 0$ we want $d_\ell(z) = az + b\bar{z}$ s.t. $d_\ell(\ell) = 0$, $d_\ell(i\ell) = |\ell|$.

i.e. $a\ell + b\bar{\ell} = 0$

$ai\ell - bi\bar{\ell} = |\ell|$

the first equation gives $a = \frac{-b\bar{\ell}}{\ell}$

the second then gives $\frac{-b\bar{\ell}}{\ell} i\ell - bi\bar{\ell} = |\ell| \Leftrightarrow -2bi\bar{\ell} = |\ell|$

$\Leftrightarrow b = \frac{-|\ell|}{2i\bar{\ell}}$ which implies $a = \frac{|\ell|}{2i\ell}$

Thus we get $d_\ell(z) = |\ell|\left(\frac{z}{2i\ell} - \frac{\bar{z}}{2i\bar{\ell}}\right)$.
For lines not through the origin we can add a real constant x indicating the signed distance of ℓ from the origin.

If f is the focus, we get

$$\left(|\ell|\left(\frac{z}{2i\ell} - \frac{\bar{z}}{2i\bar{\ell}}\right) + x\right)^2 = k^2|z-f|^2$$

as the desired conic equation.

The spherical representation

Often we want to look at what happens "at infinity" and it would be nice if there were a point there to talk about. As far as algebraic or metric properties of $\mathbb{C}$ are concerned this is nonsense, but topologically it makes sense.

Consider the map from the unit sphere in the x, y, w coordinate space to the $w=0$ plane given by $f((x,y,w))$ is the intersection of the line containing $(0,0,1)$ and (x,y,w) with the $w=0$ plane. This map is bijective where defined, i.e. $w \neq 1$, is continuous and has continuous inverse.
The map is given by

$$f((x,y,w)) = \frac{x+yi}{1-w}$$

We get $|z|^2 = |f((x,y,w))|^2 = \frac{x^2+y^2}{(1-w)^2} = \frac{1-w^2}{(1-w)^2} = \frac{1+w}{1-w}$

and $|z|^2 - 1 = \frac{2w}{1-w}$ and $|z|^2+1 = \frac{2}{1-w}$ $\Rightarrow$

$w = \frac{|z|^2-1}{|z|^2+1}$

Also, $x = (1-w)\cdot\left(\frac{z+\bar{z}}{2}\right) = \frac{z+\bar{z}}{|z|^2+1}$

$y = (1-w)\cdot\left(\frac{z-\bar{z}}{2i}\right) = \frac{z-|z|}{i(|z|^2+1)}$

We can complete the correspondence by considering $(0,0,1)$ as the image of "the point at infinity".

This is called the ~~Riemann~~ sphere

The map is stereographic projection.

(so is the inverse map)

not the only way to "add infinity" but the most convenient when we talk about limits.

Clearly, the image of any line on the plane is a circle. More generally, consider the circle cut by the plane perpendicular to unit vector (a_1, a_2, a_3) and distance $a_0 < 1$ from the origin, given by $a_1x + a_2y + a_3w = a_0$.

The image is given by points satisfying

$$\frac{a_1(z+\bar{z}) - a_2 i(z-\bar{z}) + a_3(|z|^2-1)}{|z|^2+1} = a_0 \quad \Leftrightarrow$$

$$(a_3 - a_0)|z|^2 + (a_1 - a_2 i)z + (a_1 + a_2 i)\bar{z} - a_3 - a_0 = 0$$

get straight line if $a_3 = a_0$, otherwise:

$$\left|z + \frac{a_1+a_2 i}{a_3-a_0}\right|^2 = \frac{a_3+a_0}{a_3-a_0} + \frac{a_1^2+a_2^2}{(a_3-a_0)^2} = \frac{a_3^2 - a_0^2 + a_1^2 + a_2^2}{(a_3-a_0)^2} = \frac{1-a_0^2}{(a_3-a_0)^2}$$

The fact that three points determine a circle or line implies that lines and circles on the plane are in bijective correspondence with circles on the sphere.

II Complex functions

must be well defined
single-valued

Want to consider functions

$$\mathbb{R} \to \mathbb{C}, \quad \mathbb{C} \to \mathbb{R} \quad \text{or} \quad \mathbb{C} \to \mathbb{C}$$

In any case, these are maps between metric spaces, so we have notions of limits and continuity

e.g. $\lim_{x \to a} f(x) = A$ iff $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t.

$$d(x, x') < \delta \Rightarrow d(f(x), f(x')) < \varepsilon$$

Standard rules for interchanging sums, products, quotients with limits still hold.

$$\text{Since } \lim_{x \to a} f(x) = A \iff \lim_{x \to a} \overline{f(x)} = \overline{A},$$

we get $\lim_{x \to a} \mathrm{Re}(f(x)) = \mathrm{Re}(A)$ and

$\quad$ " Im " $= \mathrm{Im}$ "

Furthermore, can define the derivative of f at a

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$