Syllabus

Book

We'll study analytic functions in one complex variable

These are functions $f: \mathbb{C} \to \mathbb{C}$ which are (equivalently)

1) complex differentiable
2) smooth
3) conformal
4) given locally by power series

The course will concern

1) Foundational work
2) Integration on planar regions (zeros and poles)
3) Infinite sums and products of functions
4) conformal mappings
5) Elliptic (doubly periodic) functions
6) global analytic functions

Readings and homework

Chapter I: Complex Numbers

Will assume: Some construction of the reals $\mathbb{R}$

Properties of $\mathbb{R}$

0) $\mathbb{R}$ is a set.

1) $(\mathbb{R}, +)$ is a commutative (abelian) group
   i.e. $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying:
   commutative $a+b = b+a$
   associative $(a+b)+c = a+(b+c)$
   with
   identity 0: $0+a = a+0 = a$ (unique)
   inverses: $a + a^{-1} = 0$

2) $\times : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ s.t.
   $(\mathbb{R}^{\times}, \times)$ is a commutative group $= \mathbb{R} \setminus \{0\}$

3) distributive: $(a+b)c = ac + bc$

A *field* is a set S and ops $+, \times$ satisfying (1), (2), (3)

4) $(\mathbb{R}, +, \times, <)$ is an ordered field.
   $<$ is a relation $\mathbb{R} \times \mathbb{R} \to \{\text{True}, \text{False}\}$
   s.t.
   a) total order: exactly one of $a<b$, $b<a$, $a=b$ holds
   2) $a<b \Rightarrow a+c < b+c$
   3) $a>0$ and $b>0 \Rightarrow ab>0$ $(1>0, -1<0, \cancel{(-a)}(-b)>0)$

5) $\mathbb{R}$ is archimedian
   $\forall a,b>0 \; \exists n \in \mathbb{N}$ s.t. $\overbrace{a+\dots+a}^{n \text{ times}} > b$

6) $\mathbb{R}$ is Dedekind-complete i.e.
   Every set with an upper bound has a least upper bound.

One can show that any two fields satisfying 4-6 are isomorphic.

<u>Def</u>: A field F is algebraically closed if for every polynomial $P(x) = a_n x^n + \dots + a_1 x + a_0$ with coefficients in F $\exists x_0 \in F$ s.t. $P(x_0) = 0$.

7) $\mathbb{R}$ is not algebraically closed, e.g. $x^2 - n = 0$ has solutions for fixed n only if $n \geq 0$.

<u>Complex Numbers</u>:

Suppose F is a field containing $\mathbb{R}$ and some i s.t. $i^2 = -1$. Then $a + bi \in F$ $\forall a, b \in \mathbb{R}$.

If $a + bi = c + di$, then $(a-c) = (b-d)i$.

Since $((b-d)i)^2 = (b-d)^2 \cdot (-1) \in \mathbb{R}$ and ≤ 0, $b - d = 0$ and $a - c = 0$. Thus $a = c$, $b = d$.

Let $\mathbb{C} = \{f \in F \mid f = a + bi \text{ for some } a, b \in \mathbb{R}\}$.

Then each $c \in \mathbb{C}$ has a unique expression as $a + bi$.

Furthermore for each $c = a + bi \in \mathbb{C}$, one has a conjugate $\bar{c} = a - bi$, and $c\bar{c} = (a+bi)(a-bi) = a^2 + b^2 \in \mathbb{R}$, with $a^2 + b^2 = 0$ iff $c = 0$.

Thus $c^{-1} = \frac{\bar{c}}{c\bar{c}} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i \in \mathbb{C}$ if $c \neq 0$.

Thus $\mathbb{C}$ is a <u>subfield</u> of F.

Clearly it is the minimal subfield containing $\mathbb{R}$ and i.

Given another field F containing a copy $\mathbb{R}'$ of $\mathbb{R}$ and an i' s.t. $(i')^2 = -1$, there is a minimal subfield $\mathbb{C}'$ containing $\mathbb{R}'$ and i'.

The map sending
$$\begin{aligned} \mathbb{R} &\to \mathbb{R}' \\ i &\to i' \end{aligned}$$
is easily seen to be a field isomorphism.

One can define $\mathbb{C}$ as
$$\mathbb{C} = \left(\{a+bi \mid a, b \in \mathbb{R}\}, \begin{aligned} &(a+bi)+(c+di) = (a+c)+(b+d)i, \\ &(a+bi)(c+di) = (ac-bd)+(bc+ad)i \end{aligned} \right)$$
which, reusing the above arguments, is clearly a field isomorphic to $\mathbb{C}$. (with $c^{-1} = \frac{\bar{c}}{c\bar{c}}$)

$\mathbb{C}$ is the unique* minimal field containing $\mathbb{R}$ and a root of x^2+1.

Later, we'll prove the Fundamental Theorem of Algebra which says that $\mathbb{C}$ is the algebraic closure of $\mathbb{R}$, i.e. the minimal field containing $\mathbb{R}$ and a root of every polynomial in $\mathbb{R}$. (we already showed minimality)

Conjugation, absolute value

$\mathbb{C}$ comes with the following natural and useful maps:

1) conjugation $(\bar{\ })$: $\mathbb{C} \to \mathbb{C}$ $\overline{a+bi} = a-bi$

- This is an <u>involution</u>: $\bar{\bar{c}} = c$.
- $\overline{c+d} = \bar{c} + \bar{d}$, $\overline{cd} = \bar{c}\bar{d}$ (implies $\overline{c^{-1}} = \bar{c}^{-1}$)

2) Re, Im: $\mathbb{C} \to \mathbb{R}$. Let $c = a+bi$.

- $Re(c) = \frac{c+\bar{c}}{2} = a$
- $Im(c) = \frac{c-\bar{c}}{2i} = b$

3) Absolute value $|\ |: \mathbb{C} \to \mathbb{R}$

- $|c| = \sqrt{c\bar{c}} = \sqrt{a^2+b^2}$ = "length" of $a+bi$, where $\sqrt{\ } > 0$.
- $|cd| = \sqrt{cd\overline{cd}} = |c||d|$
- $|\bar{c}| = \sqrt{\bar{c}\bar{\bar{c}}} = |c|$
- $|c^{-1}| = \left|\frac{\bar{c}}{c\bar{c}}\right| = \left|\frac{1}{c\bar{c}}\bar{c}\right| = \left|\frac{1}{c\bar{c}}\right||\bar{c}| = \frac{1}{|c|^2}|c| = \frac{1}{|c|}$
- $|c+d|^2 = (c+d)(\bar{c}+\bar{d}) = c\bar{c} + d\bar{d} + c\bar{d} + \bar{c}d$
  $= |c|^2 + |d|^2 + 2Re(c\bar{d})$
- $|c-d|^2 = |c|^2 + |d|^2 - 2Re(c\bar{d})$ similarly.

<u>Inequalities</u>:

- $-|c| = -\sqrt{a^2+b^2} \leq -\sqrt{a^2} \leq \overset{Re(c)}{\overset{\shortparallel}{a}} \leq \sqrt{a^2} \leq \sqrt{a^2+b^2} = |c|$.
- similarly, $-|c| \leq Im(c) \leq |c|$.
- This implies $2Re(c\bar{d}) \leq 2|c\bar{d}| = 2|c||d|$, so
  $|c+d|^2 \leq |c|^2 + |d|^2 + 2|c||d| = (|c|+|d|)^2$, so
  $|c+d| \leq |c| + |d|$. This is the <u>triangle inequality</u>.

4) a <u>metric</u> $d: \mathbb{C}\times\mathbb{C} \to \mathbb{R}$

$d(c,d) = |d-c|$. It satisfies

1) $d(c,d) \geq 0$ with $=$ only when $c=d$
2) $d(c,d) = d(d,c)$
3) $d(c,e) = |e-c| = |e-d+d-c| \leq |e-d| + |d-c|$
   $= d(c,d) + d(d,e)$

Thus $\mathbb{C}$ is a metric space

Another common inequality (though less immediately intuitive, and less easy to prove), is Cauchy's inequality:

$$\left|\sum_{i=1}^{n} a_i b_i\right|^2 \le \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 \qquad |a\cdot b|^2 \ge \|a\|^2\|b\|^2$$

Pf: Let $a=(a_1,\dots,a_n)$ $b=(b_1,\dots,b_n)$ be n-dimensional vectors over $\mathbb{C}$.

Let $a\cdot b = \sum_i a_i b_i$

Define a norm $\|a\| = \sqrt{\sum |a_i|^2}$ $\left(=\sqrt{a\cdot\bar{a}}\right)$

Then the above equation becomes

$$|a\cdot b|^2 \le \|a\|^2\|b\|^2 \iff$$

$$\|a\|^2 - \frac{|a\cdot b|^2}{\|b\|^2} \ge 0$$

But $\|a\|^2 - \frac{|a\cdot b|^2}{\|b\|^2} = \|a^2\| + \left|\frac{a\cdot b}{\|b\|^2}\right|^2\|b^2\| - 2\frac{|a\cdot b|^2}{\|b\|^2}$

$$= \|a\|^2 + \left|\frac{a\cdot b}{\|b\|^2}\right|^2\|b^2\| - 2\,\mathrm{Re}\left[\overline{\left(\frac{a\cdot b}{\|b\|^2}\right)}\; a\cdot b\right]$$

$$= \left\|a - \frac{a\cdot b}{\|b\|^2}\, b\right\|^2 \ge 0$$

(the formula for $\|a-b\|$ comes componentwise from that for $|a_i - b_i|$

Note: $\|a-b\|^2 = \sum |a_i - b_i|^2 = \sum\left(|a_i|^2 + |b_i|^2 - 2\,\mathrm{Re}(a_i b_i)\right)$

$$= \|a\|^2 + \|b\|^2 - 2\,\mathrm{Re}(a\cdot b)$$