(Frobenius' formula) This is of course just the beginning. The next step would be to compute characters

Prop: $\chi_\lambda(C_i) = \left[\Delta(x) \cdot \prod_j P_j(x)^{i_j}\right]_{(\ell_1, \ldots, \ell_k)}$

where

λ = a partition

C_i = a conj. class

$P_j(\bar{x}) = x_1^j + x_2^j + \ldots + x_k^j$ $\quad k = |\{\text{rows in } \lambda\}|$ "power sums"

$\Delta(\bar{x}) = \prod_{i<j\le k} (x_i - x_j)$ "discriminant"

$[f(\bar{x})]_{(\ell_1, \ldots, \ell_k)}$ = coefficient of $x_1^{\ell_1} x_2^{\ell_2} \ldots x_k^{\ell_k}$ in the formal power series f

$\ell_i = \lambda_i + k - i$

$i_j = |\{\text{cycles of length } j \text{ in } C_i\}|$

e.g. $\chi_{(3,2)}((123)(45)) = \left[(x_1 - x_2)(x_1^2 + x_2^2)(x_1^3 + x_2^3)\right]_{(4,2)} = 1$

The symmetric polynomials of degree d in k variables form a vector space with basis elts indexed by partitions of d with at most k elements. One basis for this vector space is given by the Schur Polynomials:

$$S_\lambda = \frac{|x_j^{\lambda_i + k - i}|}{\Delta(\bar{x})}$$

det of a $k \times k$ matrix.

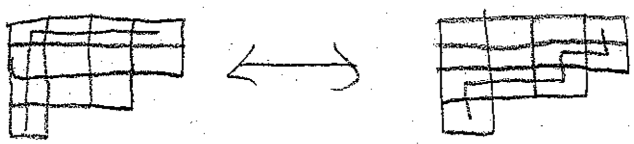
Frobenius' formula can be re-stated as a relation between Schur Polynomials and power sum polynomials:

$$\prod_j P_j(x)^{i_j} = \sum \chi_\lambda(C_i)\, S_\lambda$$

Frobenius' formula is in general hard to compute. The <u>Murnaghan-Nakayama</u> rule gives a nice inductive way to compute them.

<u>Def</u>: Any hook in a diagram D_λ has an associated <u>skew hook</u> which is the lowermost path between the ends of the hook.

e.g.



The Murnaghan-Nakayama rule is

1) Write $g \in S_d$ as an m-cycle and disjoint permutation $h \in S_{d-m}$
2) $\chi_\lambda(g) = \sum (-1)^{r(\mu)} \chi_\mu(h)$, summing over all partitions μ given by removing a skew hook of length m from λ, where $r(\mu)$ is # of rows in the skew hook -1.

Young diagrams can be used to prove a great many things about symmetric groups.

For instance, the fact

$$\sum_\lambda \dim(V_\lambda)^2 = n!$$

has a purely combinatorial proof via the Robinson-Schensted algorithm, which gives a bijection:

Prop: pairs of std. Young tableau of same shape $\overset{1:1}{\leftrightarrow}$ elts of S_n

pf: Given a permutation $\begin{matrix} 1 & 2 & \cdots & n \\ x_1 & x_2 & \cdots & x_n \end{matrix}$, construct T_I, T_R as follows:

For each $i \in 1, \ldots n$.

1) insert x_i into T_I at row $j=1$

This procedure is defined as follows:

if x_i is bigger than all elts of T_I in row j, add x_i to the end of row j and stop.

otherwise,

Let x_j be the least elt of row j greater than x_i. Replace x_j with x_i, and insert x_j into T_I at row $j+1$.

2) at the position of the last element inserted place i into T_R.

One can check that

1) this produces two std tableaux of the same shape

2) The process gives a bijection.

Example $\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{matrix}$

	T_I	T_R
step 1	$\begin{array}{\|c\|}\hline 3 \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline 1 \\ \hline\end{array}$
step 2	$\begin{array}{\|c\|}\hline 2 \\ \hline 3 \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline 1 \\ \hline 2 \\ \hline\end{array}$
step 3	$\begin{array}{\|c\|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$
step 4	$\begin{array}{\|c\|c\|}\hline 1 & 5 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$
step 5	$\begin{array}{\|c\|c\|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array}$

Another example of where Young tableau are useful is the branching rule, which tells how to induce or restrict an irrep into the next larger or smaller symmetric group. To wit:

$$\operatorname{res}(V_\lambda) = \bigoplus V_{\lambda^-}$$

$$\operatorname{Ind}(V_\lambda) = \bigoplus V_{\lambda^+}$$

where λ^- ranges over all partitions given by removing a single box from the end of a row and column of λ, and λ^+ ranges similarly over partitions given by adding a box.

e.g. $\mathrm{Ind}(V_{2,1}) \cong V_{3,1} \oplus V_{2,2} \oplus V_{2,1,1}$

One last thing:

The symmetric groups provide a way to generate representations of arbitrary groups.

Let $\rho: G \to \mathrm{Aut}(V)$. Then $\rho^{\otimes n}$ acts on $V \otimes \cdots \otimes V$, the n-fold tensor product. We have

$$\begin{tikzcd} & G \arrow[r] & \mathrm{Aut}(V) \\ & \times & \otimes \\ G \arrow[uur] \arrow[r] \arrow[ddr] & \vdots & \vdots \\ & \times & \otimes \\ & G \arrow[r] & \mathrm{Aut}(V) \end{tikzcd}$$

S_n also acts on $V^{\otimes n}$, by permuting the factors. Therefore, so does $\mathbb{C}[S_n]$. The action of S_n commutes with the action of G. Therefore $c_\lambda(V^{\otimes n})$ is a subrepresentation of $V^{\otimes n}$.

For the case $n=2$, this gives the symmetric and alternating squares.

$c_\lambda(V^{\otimes n})$ will be zero if $\dim(V) <$ # rows in λ.
For instance, $Alt^2(\mathbb{C}) = 0$.

Suppose $G = GL(V)$, acting on V in the usual way. Then

__Thm:__ 1) $V^{\otimes n} \cong \bigoplus_\lambda (c_\lambda V^{\otimes n})^{\oplus m_\lambda}$ where $m_\lambda = \dim(V_\lambda)$ for the irrep V_λ of S_n

2) each $c_\lambda(V)$ is irreducible

One can show that the $c_\lambda(V)$ give (essentially) all of the irreps for $GL(V)$ (or $SL(V)$ or $U(V)$).
This is known as __Schur-Weyl__ duality.