Part II: Dimensions of irreps $\mathrm{Im}(c_\lambda)$

Def: Let T be a λ-tableau, and let $b_{i,j}$ be the contents of ~~each~~ the box at row i, column j.

- T has a <u>column inversion</u> at (i,j) if $b_{i+1,j}$ exists and $b_{i,j} > b_{i+1,j}$.
- T has a <u>row inversion</u> at (i,j) if $b_{i,j+1}$ exists and $b_{i,j+1} > b_{i,j}$.
- T is a <u>row-standard</u> tablean if it has no row-inversions.
- T is a <u>column-standard</u> tablean if it has no column inversions.
- T is a <u>standard tablean</u> if it has no row or column inversions.

<u>Example:</u> The standard tablean for $\lambda = 3, 2$ are:

$$\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5 \\ \cline{1-2}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 3&5 \\ \cline{1-2}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2&5\\ \hline 3&4 \\ \cline{1-2}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2&5 \\ \cline{1-2}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&3&5\\ \hline 2&4 \\ \cline{1-2}\end{array}$$

Also, I_λ is always standard.

Let $\Theta_\lambda = \{T \mid T \text{ is a } \lambda\text{-tablean}\}$
$\Theta_\lambda^s = \{T \mid T \text{ is a standard } \lambda\text{-tablean}\}$

<u>Claim:</u> $\{c_\lambda T \mid T \in \Theta_\lambda^s\}$ is a basis for $V_\lambda = \mathrm{Im}(c_\lambda)$
will show <u>lin. independence</u> and <u>spanning</u>

<u>Linear Independence</u> is given by the following:

Define a partial order on tableaux as follows:

Let T_λ, T_μ be tableaux. Then $T_\lambda < T_\mu$ if $\exists n$ s.t.

1) $\forall i < n$, i lies in the same row of T_λ and T_μ.
2) i lies in row r of T_λ and row $r+\delta$ ~~rows~~ of T_μ for some $r > 0$, $\delta > 0$.

<u>Prop:</u> Let $T \in \Theta_\lambda^S$. Suppose T' is a summand of $s_\lambda T$. Then $T' \leq T$.

<u>Pf:</u> Since T and $T' \in \Theta_\lambda^S$, 1 lies in the first row of both, and if $T' = qpT$, qp fixes $1 \ldots m-1$ for some $m > 1$. Since T is a standard tableau, the boxes containing $1 \ldots m-1$ form a standard tableau. The box containing m must lie directly to the right of and directly below boxes containing $1 \ldots m$, in both T and T'. Thus, q cannot move the box containing m down.

Thus, either every box is fixed or the least unfixed box is moved up by qp.

Thus $T' \leq T$.

The above partial order is a total order on standard tableaux,

Spanning:

Let T be a tableau, $p \in P_\lambda$ s.t. pT is row-standard.
Then $c_\lambda p T = c_\lambda T$.
The following shows that $\{c_\lambda T \mid T \in \Theta_\lambda^s\}$ spans V_λ:

Prop: Let T be row-standard. If T contains a column inversion, then
$c_\lambda T = \sum_{T' \in S} a_{T'} c_\lambda T'$ for some constants $a_{T'}$ and elements $T' \not\leq T$.
(The proof is by a "straightening argument")

Pf: Suppose there is a column inversion at i,j.
Then we have:

$$\begin{array}{l} \qquad\qquad\qquad\qquad b_{i,j} < b_{i,j+1} < \ldots < b_{i,\lambda_i} \\ \qquad\qquad\qquad\qquad\ \vee \\ b_{i+1,1} < b_{i+1,2} < \ldots < b_{i+1,j} \end{array}$$

Let H_i be the subgroup of S_{λ_i} which only permutes (i,j) through (i,λ_i).
Similarly, let $H_{i+1} \leq S_{\lambda_{i+1}}$ permute $(i+1,1)$ through $(i+1,j)$.
Let H be the group of permutations of elts
$H = \{(i,j), (i,j+1), \ldots, (i,\lambda_i), (i+1,1), (i+1,2), \ldots, (i+1,j)\}$.
Then $H_i \times H_{i+1} \leq H$.
Let S be a set of right coset representatives for $H_i \times H_{i+1}$ in H, one of which is e.
Then

$$\Big(\sum_{t \in H_i \times H_{i+1}} t\Big)\Big(\sum_{s \in S} s\Big) = \sum_{h \in H} h$$

Similarly, $H_i \times H_{i+1} \leq P_\lambda$.
Let L be a set of <u>left</u> coset representatives for $H_i \times H_{i+1}$ in P_λ.

Consider $c_\lambda \left(\sum_{s\in S} s\right)$.
So far we have

$$c_\lambda\left(\sum_{s\in S} s\right) = b_\lambda a_\lambda \sum_{s\in S} s = b_\lambda \sum_{p\in P_\lambda} p \sum_{s\in S} s$$

$$= b_\lambda \sum_{\ell\in L} \ell \sum_{h\in H_i\times H_{i+1}} h \sum_{s\in S} s = b_\lambda \sum_{\ell\in L} \ell \sum_{h\in H} h$$

Now, each $\ell \in L$ sends two elts of H, say e_1^ℓ and e_2^ℓ, into the same column.
Also, H has an index 2 subgroup H_{even} consisting of all even permutations in H, with $(e_1^\ell e_2^\ell)H_{even}$ as the nontrivial ~~*~~ left coset.

Then $c_\lambda\left(\sum_{s\in S} s\right) = b_\lambda \sum_{\ell\in L} \ell \sum_{h\in H} h = b_\lambda \sum_{\ell\in L} \ell \sum_{h\in H_{even}} \left(h + (e_1^\ell e_2^\ell)h\right)$

$$= b_\lambda \sum_{\ell\in L} (1 + (\ell(e_1^\ell)\, \ell(e_2^\ell)))\, \ell \sum_{h\in H_{even}} (h)$$

But $(\ell(e_1^\ell)\, \ell(e_2^\ell)) \in Q_\lambda$, so $b_\lambda(1 + (\ell(e_1^\ell)\,\ell(e_2^\ell)))$
$= b_\lambda - b_\lambda = 0$.

Thus $c_\lambda\left(\sum_{s\in S} s\right) = 0$, so $c_\lambda e = -\sum_{s\in S\setminus\{e\}} c_\lambda s$
and $c_\lambda T = -\sum_{s\in S\setminus\{e\}} c_\lambda s T$

Since all of the elements of H_{i+1} are smaller than all of the elements for H_i,
$sT \leq T \ \forall s\in S$.

Thus $\{c_\lambda T \mid T \in \Theta_\lambda^S\}$ is a spanning set.