# Assignment 8

1) One first needs to relate the language in Lemma 1 with the problem to be solved. Here is a lexicon:

| Lemma 1 | Problem |
|---|---|
| $(V,\rho)$ | $\mathrm{Ind}(\rho)$ |
| $(W,\theta)$ | $\rho$ |
| $\rho'$ | $\sigma$ |
| $f(\theta_t w)=\rho'_t f(w)\ \forall t\in H, w\in W$ | $f$ satisfies $f(\rho_t w)=(\mathrm{Res}(\sigma))_t f(w)$ for all $t\in H$ |

Thus, in our language, Lemma one asserts that for each $f\in \mathrm{Hom}_H(\rho,\mathrm{Res}(\sigma))$ there exists a unique map $F\in \mathrm{Hom}_G(\mathrm{Ind}(\rho),\sigma)$ which extends $f$. The construction that sends $f\mapsto F$ is clearly a linear map.
It is also bijective, since any element of $\mathrm{Hom}_G(\mathrm{Ind}(\rho),\sigma)$ restricts to a unique element of $\mathrm{Hom}_H(\rho,\mathrm{Res}(\sigma))$.
Thus $\mathrm{Hom}_G(\mathrm{Ind}(\rho),\sigma)\cong \mathrm{Hom}_H(\rho,\mathrm{Res}(\sigma))$

2) Any element of $S_n$ which fixes both the rows and columns of the row-increasing $\lambda$-tableau $I_\lambda$ fixes every element, and thus is the identity.

3) Every summand of $b_\lambda a_\lambda T$ appears as $qpT$ for some $q \in Q_\lambda$, $p \in P_\lambda$.
If $T$, $T'$ are standard tableaux then $1$ appears in the first row and column of both.
If in addition $T' = qpT$, then both $q$ and $p$ fix $1$ at position $(1,1)$.

If, then, the shape of $T$ and $T'$ contains no box at $(2,2)$, clearly $T = T'$. Otherwise, the smallest partitions possible are $2,2$ and $2,2,1$.
There are two standard tableau of shape $2,2$:

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$$

It is easy to verify that no element $qp \neq e$ can fix both $1$ and $4$.
However, one has

$$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1} \end{array} \longrightarrow \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 5 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \longrightarrow \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1} \end{array}$$

It is easy to verify that any $qp$ which sends a standard tableau to a standard tableau and fixes the top row is the identity.
Thus in

$$\begin{array}{|c|c|}\hline 1 & a \\ \hline b & c \\ \hline d \\ \cline{1-1} \end{array}$$

$a$ must be moved down one square, by $q$. Since $a < c$, we must switch $b$ and $c$ first.

This would give $\begin{array}{|c|c|}\hline 1 & a \\ \hline b & c \\ \hline d \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|}\hline 1 & b \\ \hline c & a \\ \hline d \\ \cline{1-1}\end{array}$

since $c > a$, we must switch $c$ with $d$. Thus we must get

$$\begin{array}{|c|c|}\hline 1 & a \\ \hline b & c \\ \hline d \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|}\hline 1 & a \\ \hline c & b \\ \hline d \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|}\hline 1 & b \\ \hline d & a \\ \hline c \\ \cline{1-1}\end{array}$$

Thus $b < d < a < c$.

So the above is the <u>only</u> possibility for partition $2, 2, 1$ (other than $qp = e$)