Assignment 6:

1) Changes needed

Lemma 2: switch "p" with "q", "rows" with "columns"

Lemma 3: parts 1, 2, 4 - no change

part 3: $q c_\lambda p \, sgn(q) \longrightarrow p c_\lambda q \, sgn(q)$

Claim 1: $sgn(q) q c_\lambda a c_\lambda p \longrightarrow sgn(q) p c_\lambda a c_\lambda q$

claim 2: swap order in which a_λ, b_λ are analyzed

claim 3: switch "row" with "column"

switch end to:

"Thus $c_\lambda a c_\mu = c_\lambda q q a c_\mu = c_\lambda q a p c_\mu$

$= sgn(q) c_\lambda a c_\mu = -c_\lambda a c_\mu$,

so $c_\lambda a c_\mu = 0$."

2) Let $c_\lambda = b_\lambda a_\lambda$, $\bar{c}_\lambda = a_\lambda b_\lambda$

These are both minimal idempotents by the above, so by the Lemma on p.g. 3 of symmetric group notes I, we need to find $a \in \mathbb{C}[G]$ such that

$c_\lambda a \bar{c}_\lambda \neq 0$.

Set $a = 1$

Then $c_\lambda a \bar{c}_\lambda = c_\lambda \bar{c}_\lambda = b_\lambda a_\lambda a_\lambda b_\lambda = b_\lambda a_\lambda b_\lambda$.

Lets expand this to $\sum_{q_1, p, q_2} \underset{sgn(q_1) sgn(q_2)}{q_1 p q_2}$ with $q_1, q_2 \in Q_\lambda$, $p \in P_\lambda$

The summands of $b_\lambda a_\lambda b_\lambda$ which preserve rows are exactly those such that $p = e$

Thus $b_\lambda a_\lambda b_\lambda = b_\lambda b_\lambda + \left\{\begin{array}{c}\text{summands not}\\ \text{preserving rows}\end{array}\right\}$

Every summand $\pm q_1 q_2$ s.t. $q_1 q_2 = e$ appears with positive sign, so the coefficient of e in $b_\lambda b_\lambda$ is strictly positive.

Thus $b_\lambda a_\lambda b_\lambda \neq 0$

3) In the definition of Q_λ we used the row increasing tableau. Other choices of fixed tableau give elements that differ from Q_λ by conjugation.
In particular, $\exists\, g \in S_n$ s.t.
$P_\lambda = g Q_{\lambda^t} g^{-1}$ and $Q_\lambda = g P_{\lambda^t} g^{-1}$

Then $c_\lambda = b_\lambda a_\lambda = \sum_{\substack{p \in P_\lambda \\ q \in Q_\lambda}} \operatorname{sgn}(q) q p = \sum_{\substack{q \in Q_\lambda^t \\ p \in P_\lambda^t}} \operatorname{sgn}(p)\, g p q g^{-1}$

$= g \left(\sum_{\substack{q \in Q_\lambda^t \\ p \in P_\lambda^t}} \operatorname{sgn}(pq) \operatorname{sgn}(q)\, pq \right) g^{-1}$

$= g\,(f(a_{\lambda^t} b_{\lambda^t}))\,g^{-1}$, where $f: \mathbb{C}[G] \to \mathbb{C}[G]$ sends $g \to \operatorname{sgn}(g)\, g \;\; \forall g \in G$.

Let $c_t = a_{\lambda^t} b_{\lambda^t}$.

Now, this gives a right representation of S_n on $gf(c_t)g^{-1}\mathbb{C}[S_n]$

via

$$(gf(c_t)g^{-1}r)\cdot h = gf(c_t)g^{-1}rh \quad \forall h \in S_n$$

Since $\mathbb{C}[S_n] = g^{-1}\mathbb{C}[S_n]$, there is an isomorphism $f(c_t)\mathbb{C}[S_n] \to gf(c_t)g^{-1}\mathbb{C}[S_n]$ given by $f(c_t)r \to gf(c_t)r$.
The inverse isomorphism is left multiplication by g^{-1}.

Then the actions of S_n on $gf(c_t)g^{-1}\mathbb{C}[S_n]$ and $f(c_t)\mathbb{C}[S_n]$ are conjugate, and the two representations are isomorphic.

By the lemma presented in class, and since c_t gives a representation isomorphic to that given by c_λ (see part 2) $f(c_t)$ generates a representation isomorphic to $\rho \otimes \alpha$, where

α = alternating representation
ρ = representation given by c_λ.