SOLUTIONS FOR ASSIGNMENT 5, M6390 FALL 08

Q_8 has five conjugacy classes. Thus there are five irreps. The sum of the squares of their dimensions must add up to eight. The only possibility then is that four of the representations have degree one and one has degree two. For the degree one representations, the matrices are 1×1 with each single entry being equal to the trace of that matrix. Thus the character table determines the matrix representation, and vice versa.

The degree one characters may be determined by experimentation, or by computing the abelianization of Q_8. One could do this by working with the presentation explicitly, or as follows. Writing Z_2 as $\{1, -1\}$ with multiplication as the group operation, let's take the homomorphism h that sends $i \to (-1, 1)$ and $j \to (1, -1)$. One can verify that this is a homomorphism. The image is clearly an abelian group, so the kernel must contain the commutator subgroup. However, the kernel has only two elements, and Q_8 is not abelian, so the kernel must be the commutator subgroup. Thus h is the abelianization map.

As discussed in class and in the notes, the degree one representations of any abelian group are given by sending generators of order n to n-th roots of unity. The degree one representations of Q_8 are given by composing the degree one representations of its abelianization with the abelianization map. (Recall that the universal property of the abelianization map guarantees that this gives us all degree one representations). This means that the degree one irreps of Q_8 are given by sending 1 and -1 to 1 and i and j to ± 1. This gives the characters in the table below.

The degree two representation ρ can be found easily if one knows how to express the field of quaternions as complex $2x2$ matrices and realizes that Q_8 is a subgroup under multiplication of the nonzero quaternions. Otherwise, one can find it by experimentation. For a hint, one could use orthonormality to compute the character for ρ since it is the only character missing from the character table it this point. However, here is another way:

By a previous homework exercise, ρ_{-1} is a homothety since -1 is in the center of Q_8. If ρ_{-1} is the identity then ρ is not faithful and factors through some quotient of Q_8. This is impossible since all quotients are abelian and then ρ would be reducible. Thus we must have ρ faithful, $\rho_{-1} = -Id$ and $\chi_\rho(-1) = -2$.

By an appropriate choice of basis we can make one of the other elements, say i, be given by a diagonal matrix. Since i has order four the eigenvalues of ρ_i must be fourth roots of unity. Since ρ is faithful, ρ_i will not compute with (say) ρ_j, so ρ_i cannot be a homothety. Since $i^2 = -1$, the square of ρ_i must be $\rho_{-1} = -Id$. Thus, up to choice of basis, $\rho_i \begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}$.

Solving the matrix linear equation $\rho_i \rho_j = -\rho_j \rho_i$, one finds that ρ_j must be zero along the diagonal. By an argument similar to the previous paragraph, the eigenvalues of ρ_j must be $\pm i$. The obvious possibilities for j are then $\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}$ and $\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}$, or -1 times either of these. Any choice gives a representation of Q_8 (computation of the rest of the matrices is left to the reader). The representation ρ must be irreducible because it is degree two and no two degree one characters add to give χ_ρ (this can be seen from just the first two columns of the table).

The tensor product structure may be found as follows. The character of a tensor product of representations is the elementwise product of the characters. Since the first four representations are degree one, the tensor product of any (not necessarily distinct) pair is a degree $1 \times 1 = 1$ representation, and thus is irreducible. All we have to do for tensor products of these representations is multiply the characters together and look up the result in our character table. This is left as an exercise. Recall that we showed in class that the set C degree one characters of a group G has a group structure with tensor product as group multiplication, and that group is isomorphic to the abelianization of G; one could use this fact as well if desired.

It is easy to see by inspection of the character table that the character of the tensor product of any degree one representation with ρ is just χ_ρ. Thus ρ times any degree one representation is isomorphic to ρ. The character of $\rho \otimes \rho$ would appear in the table as $(4, 4, 0, 0, 0)$. Since the entry in the second column is equal to the entry in the first column, this character must be a sum of characters from the first four rows. The decomposition of this sum is unique, and it is clear from looking at the table that the sum of the first four rows is $(4, 4, 0, 0, 0)$. Thus $\rho \otimes \rho$ is isomorphic to a direct sum of one copy of each of the four degree one representations.

Conj. class	$\{1\}$	$\{-1\}$	$\{\pm i\}$	$\{\pm j\}$	$\{\pm k\}$
υ	1	1	1	1	1
ϕ_1	1	1	-1	1	-1
ϕ_2	1	1	1	-1	-1
ϕ_3	1	1	-1	-1	1
ρ	2	-2	0	0	0

Note that this is the same character table that we computed for D_4, so the charecters have the same structure under multiplication in Q_8 and D_4. Taking the tensor product of representations modulo isomorphisms of representations gives a ring structure, called the *Grothendeick ring* for Q_8 and D_4. Q_8 and D_4 have the same Grothendeick rings, though they are not Morita equivalent (this notion was defined on the last day of class).