

**SOLUTIONS FOR ASSIGNMENT 4, M6390 FALL 08**

3.4 This problem is an exercise in applying the examples in section 3.3 of Serre. The regular representation  $\beta_G$  of  $G$  is induced by the regular representation  $\beta_H$  of  $H$ . Decompose  $\beta_H$  into a direct sum of irreducible representations  $\rho_i$ . Then  $\beta_G \cong \text{Ind}(\beta_H) = \text{Ind}(\oplus_i \rho_i) \cong \oplus_i \text{Ind}(\rho_i)$ , where the last isomorphism is by example 3 in section 3.3. Thus  $\beta_G$  decomposes into a direct sum of representations induced by irreducible representations of  $H$ . Since  $\beta_G$  contains a copy of every irreducible representation of  $G$ , every irreducible representation of  $G$  is a subrepresentation of one of the  $\text{Ind}(\rho_i)$ , as desired.

It might not hurt to re-convince yourself that the examples in section 3.3 are actually true.

For the last (optional) part of problem 3.4, every irreducible representation  $\gamma$  of  $G$  appears as a subrepresentation of a representation induced by an irreducible representation  $\phi$  of  $A$ . Since  $A$  is abelian,  $\text{deg}(\phi) = 1$ . Then  $\text{Ind}(\phi)$  has degree  $g/a$ . Since  $\gamma \leq \text{Ind}(\phi)$ ,  $\text{deg}(\gamma) \leq g/a$ .

3.6 It suffices to show that the two representations have the same character. First of all,

$$\chi_{\theta \otimes r_K}(h, k) = \chi_\theta(h)\chi_{r_K}(k) = \begin{cases} |K|\chi_\theta(h) & \text{if } k = e \\ 0 & \text{otherwise.} \end{cases}$$

Next, note that for all  $(h', k') \in H \times K$ , we have that  $(h', k')(h, k)(h', k')^{-1} = (h'h(h')^{-1}, k'k(k')^{-1})$  is in  $H$  if and only if  $k = e$ . Then the character formula for the induced representation  $\rho$  is

$$\chi_\rho((h, k)) = \frac{1}{|H|} \sum_{\substack{g \in G \\ g(h, k)g^{-1} \in H}} \chi_\theta(g^{-1}(h, k)g) = \begin{cases} |K|\chi_\theta(h) & \text{if } k = e \\ 0 & \text{otherwise.} \end{cases}$$

Here we use the facts that  $\chi_\theta$  is a class function and  $\frac{|G|}{|H|} = |K|$ .

(2) By the universal property of  $A$ , there is a unique map  $h : A \rightarrow A'$  such that  $h \circ f = f'$ . By the universal property of  $A'$ , there is a unique map  $h' : A' \rightarrow A$  such that  $h' \circ f' = f$ . Thus  $f = h' \circ f' = h' \circ h \circ f$ . However, by the universal property of  $A$  again, the map  $h' \circ h$  is the unique map such that  $f = h' \circ h \circ f$ . Thus  $h' \circ h = \text{Id}_A$ . Similarly,  $h \circ h' = \text{Id}_{A'}$ . Thus  $f$  and  $f'$  are related by a unique isomorphism. One common way this situation is expressed is as follows: the abelianization  $A$  is unique up to unique isomorphism.