

SOLUTIONS FOR ASSIGNMENT 3, M6390 FALL 08

(1) Solution to 3.1

Let $\rho : G \rightarrow GL(V)$ be a representation, with G an abelian group. Since G is abelian, for all $g, h \in G$, we have $\rho_g \rho_h = \rho_h \rho_g$. This makes each ρ_g a G -equivariant map from V to V . By Schur's lemma, each ρ_g is a homothety, i.e. $\rho_g = \lambda Id_V$ for some $\lambda \in \mathbf{C}$. Thus every vector in V is an eigenvector for ρ_g for every $g \in G$. Since ρ is irreducible, $\dim(V) = 1$.

(2) Solution to 3.2

(a) If $s \in C$, then for all $g \in G$, $sg = gs$. Thus $\rho_g \rho_s = \rho_s \rho_g$ and so ρ_s is a G -equivariant map. By Schur's lemma, $\rho_s = \lambda Id$. Since $s^m = e$ for some $m \neq 0$, $\lambda^m = 1$. Thus $|\chi(s)| = |Tr(\rho_s)| = |\lambda Tr(Id_n)| = n$.

(b) The formula $\sum_{s \in G} |\chi(s)|^2 = g$ is derived from the formula for the inner product of ρ with itself and the fact that ρ is irreducible. One then has the following:

$$g = \sum_{s \in G} |\chi(s)|^2 \geq \sum_{s \in C} |\chi(s)|^2 = cn^2.$$

Dividing both sides by c gives the result.

(c) The image of C under ρ is isomorphic to C (since ρ is faithful) and is a group of homotheties, which is isomorphic to a finite subgroup of \mathbf{C} . Every finite subgroup of \mathbf{C} is a cyclic group. Thus C is cyclic also.

The proof that every finite subgroup of \mathbf{C} is a cyclic group is as follows: Let $G \leq \mathbf{C}$ be finite. Then each element g is a primitive n_g -th root of unity. Thus G is a subgroup of the cyclic group generated by a primitive $\text{lcm}_{g \in G}(n_g)$ -th root of unity. Since every subgroup of a cyclic group is cyclic, the result holds.

(It is also possible to prove part (c) using a statement of the fundamental theorem of finitely generated abelian groups).