SOLUTIONS FOR ASSIGNMENT 2, M6390 FALL 08

(1) Serre's book, problems 2.1, 2,2, 2.4.

- Solution to 2.1:

  One way to do this problem is to use Proposition 3 in the book, which makes the problem an easy algebra exercise. However, it's not too hard to do it directly.

  For the symmetric square, let χ be the character of a representation V, and χ' the character of V'. Then $(V \oplus V') \otimes (V \oplus V')$ decomposes into a direct sum of invariant spaces $W_1 = V \otimes V$, $W_2 = V \otimes V' \oplus V' \otimes V$ and $W_3 = V' \otimes V'$. If B is a basis for V and B' a basis for V', then $B \cup B'$ forms a basis for $V \oplus V'$ and it is easy to see that each of the basis vectors for $(V \oplus V')^2_\sigma$ as defined in the text lies in either W_1, W_2 or W_3. Thus $(V \oplus V')^2_\sigma$ decomposes into three invariant subspaces. Intersecting $(V \oplus V')^2_\sigma$ with W_1 and W_3 gives the symmetric squares on W_1 and W_3 with characters χ^2_σ and ${\chi'}^2_\sigma$ respectively.

  Define $\phi : V \otimes V' \to W_2$ via $b_i \otimes b'_j \to b_i \otimes b'_j + b'_j \otimes b_i$. ϕ is easily seen to be an injective G-equivariant map. Thus $Im(\phi) \cong V \otimes V'$, with a basis given by $\{\phi(b_i \otimes b'_j)\}_{b_i \in B, b_j \in B'}$. But this is just the basis given above for the the intersection of $(V \oplus V')^2_\sigma$ with W_2, so $Im(\phi) \cong (V \oplus V')^2_\sigma \cap W_2$ and $(V \oplus V')^2_\sigma \cap W_2$ has character $\chi\chi'$. The character of the symmetric square $(V \oplus V')^2_\sigma$ is the sum of the characters of the representations corresponding to the three invariant subspaces, i.e. $\chi^2_\sigma + {\chi'}^2_\sigma + \chi\chi'$.

  The alternating square is similar.

- Solution to 2.2:

  $\chi_X(s)$ is equal to the sum of the diagonal elements of the permutation matrix ρ_s. The diagonal element corresponding to each $x \in X$ is 1 if $g \cdot x = x$, 0 otherwise. This gives the result.

- Solution to 2.4:

  The representation $\rho_1^* \otimes \rho_2$ is a homomorphism $G \to GL(V_1^* \otimes V_2)$. Since we can canonically identify $V_1^* \otimes V_2$ with $Hom(\mathbf{C}, V_1^* \otimes V_2)$ by sending $v \in V_1^* \otimes V_2 \to v'$ such that $v'(1) = v$, we can think of $\rho_1^* \otimes \rho_2$ as a homomorphism $G \to GL(Hom(\mathbf{C}, V_1^* \otimes V_2))$. There is a vector space isomorphism $\Phi : Hom(\mathbf{C}, V_1 * \otimes V_2) \to Hom(V_1, V_2)$ defined by $\Phi(f) = (ev_{V_1} \otimes Id_{V_2}) \circ (Id_{V_1} \otimes f)$.

  Now, (it might help to draw the pictures here to see what is going on)

$$\Phi(g \cdot f) = (ev_{V_1} \otimes Id_{V_2}) \circ (Id_{V_1} \otimes ((\rho_1^*(g) \otimes \rho_2(g)) \circ f))$$

$$= ((ev_{V_1} \circ (\rho_1(g^{-1}) \otimes \rho_1^*(g^{-1})) \otimes Id_{V_2}) \circ (Id_{V_1} \otimes ((\rho_1^*(g) \otimes \rho_2(g)) \circ f))$$

  by equivariance of ev_{V_1} ,

$$= (ev_{V_1} \otimes \rho_2(g)) \circ (\rho_1(g^{-1}) \otimes f)$$

  by pushing the ρ through composition and cancelling out the inverse pair,

$$= \rho_2(g) \circ \Phi(f) \circ \rho_1(g^{-1}).$$

For $f' : V_1 \to V_2$, we can define $g \cdot f' = \rho_2(g) \circ f' \circ \rho_1(g^{-1})$. From the above, we have an equivariance like property: $g \cdot \Phi(f) = \phi(g \cdot f)$. However, we do not yet have an honest G-equivariance because we don't yet know that the action on $Hom(V_1, V_2)$ is really a group action. However, we can use the equivariance-like property to prove that the action on $Hom(V_1, V_2)$ is a group action.

Each $f' = \Phi(f)$ for some $f \in Hom(\mathbf{C}, V_1^* \otimes V_2)$. We then have $e \cdot f' = e \cdot \Phi(f) = \Phi(e \cdot f) = \Phi(f) = f'$, and $g_1 \cdot (g_2 \cdot f') = g_1 \cdot (g_2 \cdot \Phi(f)) = g_1 \cdot \Phi(g_2 \cdot f) = \Phi(g_1 \cdot (g_2 \cdot f)) = \Phi((g_1 g_2) \cdot f) = (g_1 g_2) \cdot \Phi(f) = (g_1 g_2) \cdot f'$. Thus we get a (linear) action on $Hom(V_1, V_2)$, as desired, and the associated representation (call it ξ) is isomorphic to $\rho_1 * \otimes \rho_2$ since the equivariance-like property of Φ is now an honest G-equivariance. Thus ξ has the desired character $\chi_1^\dagger \otimes \chi_2$.

(2) Prove or disprove: the natural isomorphism between V and V^{**} is a G-equivariant map.

Solution: This is the map $\iota : V \to V^{**}$ that sends $v \in V$ to the map v^{**} such that for $f : V \to \mathbf{C}$, $v^{**}(f) = f(v)$. Recall that given a representation ϕ of G on $Aut(W)$, the action of ϕ_g^* on W^* is defined by precomposing maps $W \to \mathbf{C}$ with $\phi_{g^{-1}}$, i.e. $g \circ f(w) := f(g^{-1} \cdot w)$.

Then, with $W = V^*$ one gets $(g \cdot v^{**})(f) = v^{**}(g^{-1} \cdot f) = (g^{-1} \cdot f)(v)$, where the action of g^{-1} is the action on V^*. Using the same identity but with $W = V$, $(g^{-1} \cdot f)(v) = f(g \cdot v) = (g \cdot v)^{**}(f)$. Thus $g \cdot \iota = \iota \cdot g$, as desired.

(3) Give a picture proof that the definition of $coev_v$ is independent of choice of basis, to the same extent that ev_v is. To wit, once tensor products have been defined and a basis for V chosen, ev_v may be expressed as a matrix. A change of basis matrix m on V induces a change of dual basis (denoted m^*) on V^*, such that precomposing ev_v with $m \otimes m^*$ does not change the operator ev_v, or its matrix representation. Prove that postcomposition of $coev_v$ by $m^* \otimes m$ does not change the operator $coev_v$, or its matrix representation.

Solution: The proof is similar to the picture proof that $coev_V$ is a G-equivariant map from the notes, with m substituted for ϕ_g.