So far we have seen representations of the following types of objects

1) finite groups
2) compact groups
3) $GL_n(\mathbb{C})$ (hinted at)
4) associative algebras.

These were defined axiomatically, with an exception. The diagonal map, used to define tensor products of representations, was not specified axiomatically. What axioms does it satisfy?

Let's recall the definition of an F-algebra (reworded slightly):

Def: An F-algebra is a triple (A, μ, η) with
  A a vector space over F
  $\mu: A \otimes A \to A$ a linear map (called <u>multiplication</u>)
  $\mu: F \to A$ a linear map (called the <u>unit</u>)

s.t.

1)
$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mu \otimes Id_A"] \arrow[d, "Id_A \otimes \mu"'] & A \otimes A \arrow[d, "\mu"] \\
A \otimes A \arrow[r, "\mu"] & A
\end{tikzcd}$$

2)
$$\begin{tikzcd}
F \otimes A \arrow[r, "\eta \otimes Id_A"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "\mu"] & A \otimes F \arrow[l, "Id_A \otimes \eta"'] \arrow[dl, "\cong"] \\
& A &
\end{tikzcd}$$

Let $A = \mathbb{C}[G]$. Then the diagonal map on groups extends to a linear map $\Delta : A \to A \otimes A$ satisfying these axioms:

1)
$$\begin{tikzcd} A\otimes A\otimes A & A\otimes A \arrow[l, "\Delta\otimes Id_A"'] \\ A\otimes A \arrow[u, "Id_A\otimes\Delta"] & A \arrow[l, "\Delta"'] \arrow[u, "\Delta"'] \end{tikzcd}$$
commutes.

2) There exists a linear map $\varepsilon : A \to F$, (given by $k \cdot g \to k$) such that

$$\begin{tikzcd} F\otimes A & A\otimes A \arrow[l, "\varepsilon\otimes Id_A"'] \arrow[r, "Id_A\otimes\varepsilon"] & A\otimes F \\ & A \arrow[ul, "\cong"] \arrow[u, "\Delta"'] \arrow[ur, "\cong"'] & \end{tikzcd}$$
commutes.

The triple (A, Δ, ε) is called a coalgebra for A. Δ is the *comultiplication*, ε is the *counit*

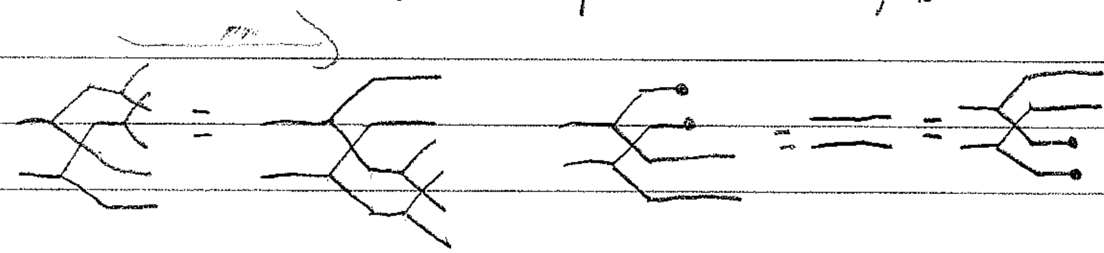
Let $\tau : A \otimes A \to A \otimes A$ be the linear map which exchanges tensor factors.
Then Δ satisfies

$$\begin{tikzcd} A\otimes A \arrow[rr, "\tau"] & & A\otimes A \\ & A \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] & \end{tikzcd}$$

This makes (A, Δ, ε) a cocommutative coalgebra.

If (A,μ,η) is an algebra then $(A\otimes A, (\mu\otimes\mu)\circ(Id_A\otimes\tau\otimes Id_A), \eta\otimes\eta)$ is an algebra also.

For the case $A=\mathbb{C}[G]$ the diagonal map and its associated counit become homomorphisms of algebras.



<u>Prop</u> Let (A,μ,η) an algebra, (A,Δ,ε) a coalgebra. TFAE

1) Δ, ε are morphisms of algebras
2) μ, η are " " coalgebras

<u>Def</u>: If either of the above holds, $(A,\mu,\eta,\Delta,\varepsilon)$ is called a bialgebra.

If $A=\mathbb{C}[G]$, then the map of group elements that sends $g\to g^{-1}$ extends to a linear map $S:A\to A$. S satisfies the following

$$\Delta\circ(Id_A\otimes S)\circ\mu = Id = \Delta\circ(S\otimes Id_A)\circ\mu$$

Def: A map $S:A\to A$ satisfying the above is an <u>antipode</u>. A bialgebra with an antipode is called a <u>Hopf algebra</u>.

A representation of a Hopf algebra is just a representation of the underlying algebra. The "extra bits" are used as follows.

1) Δ allows us to define a tensor product of representations.
2) The existence of ε is equivalent to the existence of a trivial representation ν s.t. $\nu \otimes \rho \cong \rho \otimes \nu \cong \rho \;\; \forall \rho$.
3) S allows one to construct dual representations for each representation.

It is relatively difficult to come up with examples of Hopf algebras which are not cocommutative. The most well known way (due to Drinfeld) is via universal enveloping algebras of Lie algebras.

Def: A Lie algebra L is a vector space with a bilinear map $[\,,\,]: L \times L \to L$ satisfying $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$ and $[x,x] = 0$.

One could easily spend a semester on Lie algebras; we'll just give the barest outline.

Def: A Lie group is a topological group which is a smooth manifold s.t. mult. and inverse are smooth.

Finite dimensional Lie algebras describe the behavior of "infinitesimal vectors" of the tangent space of a Lie group. Given a Lie group, one can construct a (unique) Lie algebra, and from a Lie algebra one can construct a (locally unique) Lie group.

Ex: $GL_n(\mathbb{C})$ with the commutator bracket $[x,y] = xy - yx$ is a Lie algebra, called $gl_n(\mathbb{C})$

Ex: More generally, for any associative algebra A, one has a Lie algebra L_A given by A with commutator bracket.

Def: A (finite dimensional) representation of a Lie algebra L is a Lie algebra homomorphism $L \to gl_n(\mathbb{C})$.

It turns out (Ado's Theorem) that every finite dimensional Lie algebra over $\mathbb{C}$ (actually, any field of characteristic 0) has a faithful representation.

There exists a Hopf algebra $U(L)$ such that $L_{U(L)}$ contains L.

$U(L)$ is called the universal enveloping algebra for L. It has the property that

For any associative algebra A, any map $L \to L_A$ of Lie algebras extends uniquely to a map $U(L) \longrightarrow A$ of algebras.

For the case $A = GL_n$, $L_A = gl_n$, this implies that $U(L)$ and L have the same representation theory (one can define tensor products and duals for representations of Lie Algebras).

It has been known for a long time that Lie algebras are rigid; one cannot slightly deform the structures and get a Lie algebra.

However, Drinfeld and Jimbo showed that for a semisimple Lie algebra g, $U(g)$ admits a one-parameter family of deformations $U_q(g)$ with $q \notin \{0,1\}$. These deformations are non-cocommutative Hopf algebras which limit to $U(g)$ in a certain sense as $q \longrightarrow 1$.

The deformation makes $U_q(g)$ a <u>quasi-triangular</u> or <u>braided</u> Hopf algebra. This means two things:

1) $U_q(g) \otimes U_q(g)$ contains a <u>universal R-matrix</u> R which satisfies the quantum Yang Baxter equations (giving reps of the braid group)
2) Representations of $U_q(g)$ are braided in a certain sense.