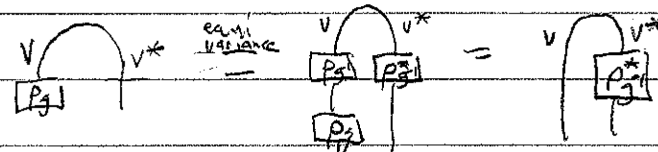
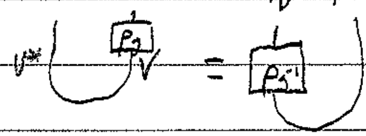
1) Let $\varphi: G \to \mathrm{End}(V)$, $\psi: G \to \mathrm{End}(W)$ be representations and $f: V \to W$ a G-equivariant map.

Recall that

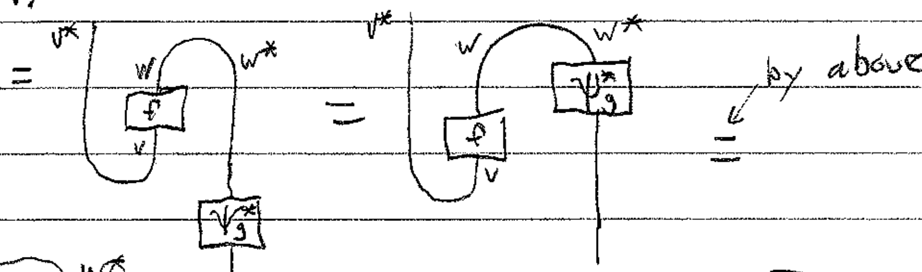


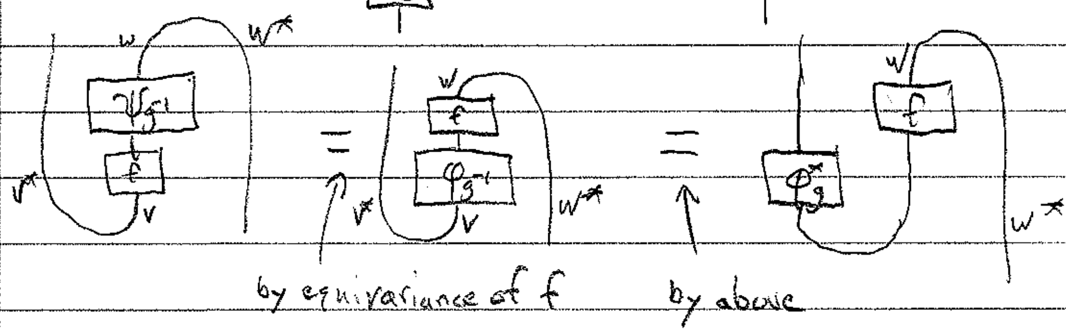
and similarly

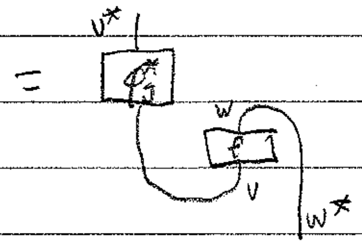


for any vector space V and any representation ρ on V.

Then $f^* \circ \psi^*_g =$







$= \varphi^*_g \circ f^*$, as desired.

2 & 3) There are a number of ways to get at the character table for problem 2. Here is a solution that works for all n.

No relation changes the number of f's modulo 2, so no group element expressed with an odd number of f's may be conjugate to an element with an even number.

Let $g = (1\ n)(2\ n-1)\ldots(\lfloor\frac{n-1}{2}\rfloor\ \lceil\frac{n-1}{2}\rceil)$
Then for any cycle $(a_1, \ldots, a_k)$,
g satisfies $(a_1, \ldots, a_k)g = g\,(n+1-a_1, \ldots, n+1-a_k)$
Then $gf = fg = (gf)^{-1} \in Z(T_n)$

Every $g \in T_n - S_n$ has a unique expression $s\,gf$ for some $s \in S_n$.
Then $\forall t \in S_n, s \in S_n$
$t^{-1}sgft = t^{-1}st\,gf$,
$(gf)^{-1}t^{-1}s\,gf\,tgf = t^{-1}stgf$ and
$(gf)^{-1}t^{-1}s\,tgf = t^{-1}st$
Thus there are exactly twice as many conjugacy classes in T_n as in S_n.

Define a map $\varphi: T_n \to S_n$ s.t. $\forall s \in S_n$,
$\varphi(s) = s$, $\varphi(f) = g$.
This map is a surjective homomorphism.

Then for every irrep ρ of S_n, $\rho \circ \varphi$ is an irrep of T_n. This gives non-isomorphic irreps such that the sum of the squares of the irreps is $n!$ and gives the character table $\begin{array}{|c|c|} \hline T & T \\ \hline \end{array}$ where T is the character table for S_n.

There is also a representation $\bar{\alpha}$ of degree 1 given by sending all $s \in S_n$ to 1, $f \to -1$.

For any irrep ρ, $\rho \otimes \bar{\alpha}$ is irreducible, since $\rho \otimes \bar{\alpha} \otimes \bar{\alpha} \cong \rho$. This gives a table

$$\begin{array}{|c|c|} \hline T & T \\ \hline T & -T \\ \hline \end{array}$$

where the second row of blocks is given by irreps in the first row tensored with $\bar{\alpha}$. This gives a set of linearly independent irreducible characters of total (summed) squared dimension $2n!$, so these are all of the irreps.

4) Let $\lambda = (k+1, 1, \ldots, 1)$ be a partition of n.
Then by the hook length formula,
$$\dim(V_\lambda) = \frac{n!}{n \cdot k!(n-1-k)!} = \binom{n-1}{k}.$$

5) The easiest way to prove this is to invoke the formula for computing induced representations from S_{n-1} into S_n mentioned in the notes. Then the trivial representation $V_{(n-1)}$ in S_{n-1} induces $V_{(n-1,1)} \oplus V_{(n)}$ in S_n, which must be the permutation representation on n letters. Since $V_{(n)}$ is trivial, $V_{(n-1,1)}$ is the standard irrep on S_n.

Another way: $c_\lambda\left(\begin{array}{|c|c|c|}\hline a_1 & \cdots & a_{n-1} \\ \hline a_n & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}\right)$ only depends on the value of a_n, since $c_\lambda = b_\lambda a_\lambda$ and a_λ symmetrizes rows.
Recall that $\overset{=V_\lambda}{c_\lambda(\mathbb{C}[S_n])}$ has a basis B given by the $n-1$ standard $(n-1,1)$ tableaux and S_n acts on $c_\lambda(\mathbb{C}[S_n])$ on the right.
Then S_n permutes an n element set $\overline{B}$ consisting of the basis elements and $c_\lambda\left(\begin{array}{|c|c|c|}\hline 2 & \cdots & n \\ \hline 1 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}\right)$.

Let $T_k = \{T \in \Theta_{(n-1,1)} \mid k \text{ is in position } (2,1)\}$
$\overline{T}_k = \{T \in \Theta_{(n-1,1)} \mid k \text{ is in position } (1,1)\}$
$a_{k,1}, \ldots, a_{k,n-1}$ = the sequence $1, \ldots, n$ excluding k

Then $c_\lambda\left(\begin{array}{|c|c|c|}\hline a_{k,1} & \cdots & a_{k,n-1} \\ \hline k \\ \cline{1-1}\end{array}\right) = \sum_{T\in T_k} T - \sum_{T\in \bar{T}_k} T$

and $\sum_{k=1}^{n} c_\lambda\left(\begin{array}{|c|c|c|}\hline a_{k,1} & \cdots & a_{k,n-1} \\ \hline k \\ \cline{1-1}\end{array}\right) = \sum_{k=1}^{n}\left(\sum_{T\in T_k} T - \sum_{T\in \bar{T}_k} T\right)$

$= \sum_{T\in\Theta_\lambda} T - \sum_{T\in\Theta_\lambda} T = 0.$

Thus $c_\lambda\left(\begin{array}{|c|c|c|}\hline 2 & \cdots & n \\ \hline 1 \\ \cline{1-1}\end{array}\right) = \sum_{b\in B} -b.$

We can then compute the character of $s\in S_n$.
For each basis element $b\in B$, b contributes
- 1 if b is sent to b.
- -1 if b is sent to the element of $\bar{B}$ not in B.
- 0 otherwise.

Then either the element in $\bar{B}-B$ is fixed or not.
In either case,

$$\chi(s) = (\#\text{ of elements in } \bar{B} \text{ fixed by } s) - 1.$$

Thus V_λ is the standard irrep of S_n

6) If b_1^+ and b_2^+ are positive, one can convert them to GNF using only R-III moves as described in the notes.
Composing one move sequence with the inverse of the other gives a sequence of R-III moves from b_1^+ to b_2^+.

7) This group is huge, but it is possible to find quotient groups and use irreps of those.

The simplest way to proceed is to quotient by the normal subgroup that fixes the center squares on each of the faces.
The convex hull of the center points of the faces is a regular octahedron, which is acted upon by rigid motions by twists that move the center faces. Any such action is represented by a "twist" which consists of rotating the whole cube as if it were rigid.
Thus cube group / center preserving moves is isomorphic to the group of rigid motions of a cube, which is isomorphic to $A_4 \times \mathbb{Z}_2$ as discussed in the book and class.
Take the 3-dim'l irrep of A_4 tensored with the trivial rep on $\mathbb{Z}_2$, and compose

with the quotient map. This gives a 3-dimensional irrep of the cube group.

Typically, Rubik's cube enthusiasts don't count moves that move the center squares as being in the Rubik's cube group, since one can get from the group of center-fixing moves to the full group of moves by adding rigid motions (i.e. turning the cube in your hand) In this case, it's a bit harder to proceed.

One could quotient by the (normal) subgroup consisting of moves that fix the locations (but not necessarily the orientations) of the corners. The quotient is obviously a subgroup of S_8. It would be nice to show that it is in fact S_8. Then one can use the standard representation (say) to get a 7-dimensional irrep of the reduced cube group.

It turns out that the quotient is in fact S_8, but we can proceed even without this knowledge.

Let $Q \leq S_8$ be the quotient in question, and $\rho : S_8 \longrightarrow \mathbb{C}^8$ be the permutation representation.
Let ρ_Q be the restriction of ρ to Q.
If ρ_Q is not irreducible it decomposes into a sum $\oplus \rho_i$ of irreducibles.
If we restrict further to ρ_S for some $S \leq Q$
These ρ_i will decompose further into irreps $\rho_{i,j}$ which are (possibly non-proper) subrepresentations of the ρ_i considered as representations of S.

Let ρ act by permuting the corners of the cube labeled as follows:

c_1 c_5

c_2 c_6

c_4 c_8

c_3 c_7

Then Q has a subgroup $S \cong \mathbb{Z}_4 \oplus \mathbb{Z}_4$ generated by $(c_1\ c_2\ c_3\ c_4)$ and $(c_5\ c_6\ c_7\ c_8)$.
Then ρ_S decomposes into 8 1 dim'l representations with invariant spaces spanned by vectors

$v_1 = (1,1,1,1,1,1,1,1)$ $v_5 = (1,-i,-1,i,0,0,0,0)$
$v_2 = (1,1,1,1,-1,-1,-1,-1)$ $v_6 = (0,0,0,0,1,i,-1,-i)$
$v_3 = (1,i,-1,-i,0,0,0,0)$ $v_7 = (0,0,0,0,1,-1,1,-1)$
$v_4 = (1,-1,1,-1,0,0,0,0)$ $v_8 = (0,0,0,0,1,-i,-1,i)$

These vectors span mutually non-isomorphic representations except for the first two, which both give the trivial representation. Thus, except for the first two subspaces, the decomposition is unique.

Thus, each of $v_3, v_4, \dots, v_8$ lie within same irreducible component ρ_i of ρ_Q.

$\langle v_1 \rangle$ is invariant under the action of Q. We will show that the irreducible invariant space containing v_4 also contains v_2, v_3, v_5, v_6, v_7 and v_8 and thus is spanned by them, giving the solution.

Now, $((c_1\ c_2\ c_5\ c_6)(c_4\ c_3\ c_7\ c_8))^2$ sends

$v_3 \longrightarrow i v_8$

(a) $v_4 \longrightarrow v_7$

$v_5 \longrightarrow -i v_6$

while $((c_1\ c_2\ c_5\ c_6)(c_4\ c_3\ c_7\ c_8))$ sends

(b) $v_4 \longrightarrow (0, 1, -1, 0, -1, 0, 1, 0)$

$= \frac{1}{4} - \frac{i}{4} v_3 - \frac{1}{2} v_4 + (\frac{1}{4} + \frac{i}{4}) v_5 - \frac{1}{2} v_6 - \frac{1}{2} v_8$

and

(c) $v_2 \longrightarrow \frac{-1-i}{2} v_3 + \frac{1-i}{2} v_5 + v_7$

(a) and (b) imply that the span of $v_3, v_4, v_5, v_6, v_7, v_8$ is a subspace of a single irrep since each vector lies in an irreducible invariant space. Using the inverse map, (c) implies that v_2 is contained in the same invariant space.