1. Assignment 1

Due Monday, 9/8/08

(1) Group homomorphisms were defined in the notes as structure-preserving maps. However, the definition contains a redundancy. Prove that the multiplication-preserving property of a group homomorphism implies the following:
   (a) $f(e_G) = e_H$
   (b) for all $g \in G$, $f(g^{-1}) = (f(g))^{-1}$

(2) Prove that for any finite group G and any $g \in G$, there exists $n \in \mathbf{N}^+$ such that $g^n = 1$. (Note: the smallest such n is called the *order* of g, and is denoted $|g|$).

(3) Prove the following:

   Proposition (The Subgroup Criterion)**.** *: A subset H of a group G is a subgroup if and only if*
   (a) *H is non-empty.*
   (b) *For all $h_1, h_2 \in H$, $h_1 h_2^{-1} \in H$.*
   Furthermore, if H is finite then it suffices to show that H is non-empty and closed under multiplication.

(4) Prove that the orbits of a group action partition a group G into equivalence classes as described in Proposition 1.11 in the notes.

(5) The *center* $Z(G)$ of a group G is the set of elements that commute with every element in G. Prove the following:
   (a) $Z(G)$ is a normal subgroup of G.
   (b) Action of G on itself by conjugation is a faithful action iff $Z(G) = \{e\}$.

(6) Let $\phi : G \to H$ be a surjective homomorphism. Prove the following:
   (a) For each $h \in H$, $\phi^{-1}(h)$ is a coset of $ker(\phi)$.
   (b) The quotient group of G by $ker\phi$ is isomorphic to H.

(7) We required both $e * s = s$ and $s * e = e$ in the definition of a monoid. Give an example of a semigroup which is not a monoid, but has a *right identity*, i.e. $s * e = s$ for all $s \in S$.

(8) The groups D_4 and Q_8 have eight elements. Prove that they are not isomorphic.

   Q_8 is defined as follows. As a set, $Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$. 1 is the identity element. $-1 * -1 = 1$ and for $x \in \{i, j, k\}$, $-x = -1 * x = x * -1$. Finally, we have the following table (which should be read such that $i * j = k$):

	i	j	k
i	-1	k	-j
j	-k	-1	i
k	j	-i	-1

2. Assignment 2

Due Monday, 9/29/08

(1) Serre's book, problems 2.1, 2,2, 2.4.
(2) Prove or disprove: the natural isomorphism between V and V^{**} is a G-equivariant map.
(3) Give a picture proof that the definition of $coev_v$ is independent of choice of basis, to the same extent that ev_v is. To wit, once tensor products have been defined and a basis for V chosen, ev_v may be expressed as a matrix. A change of basis matrix m on V induces a change of dual basis (denoted m^*) on V^*, such that precomposing ev_v with $m \otimes m^*$ does not change the operator ev_v, or its matrix representation. Prove that postcomposition of $coev_v$ by $m^* \otimes m$ does not change the operator $coev_v$, or its matrix representation.

3. Assignment 3

Due Monday, 10/13/08

(1) Exercises 3.1 and 3.2 from Serre.

4. Assignment 4

Due Monday, 10/20/08

(1) Exercises 3.4 and 3.6 from Serre. Exercise 3.4 was stated earlier in the notes as Corollary 4.13; you should do a careful proof of that corollary.
(2) Let (A, f) and (A', f') be abelianizations of G. Show that A and A' are isomorphic and state the relation between f and f'.

5. Assignment 5

Due Monday, 11/02/08

Classify the irreducible representations of $\mathbf{Q}_8$, the order 8 group described in the first homework. Your solution should include:

(1) A character table,
(2) Matrices for each of the irreducible representations,
(3) For each triplet of irreducibles ϕ, ψ, and χ, the number of copies of χ in $\phi \otimes \psi$.

6. Assignment 6

Due Monday, 11/17/08

Let ρ be an irreducible representation of S_n constructed from the partition λ. Let α be the alternating representation, i.e. the 1-dimensional representation of S_n which is not trivial. Prove that $\rho \otimes \alpha$ is isomorphic to the irreducible representation constructed from the partition $\lambda^\dagger$ (the conjugate partition to λ). The following outline is suggested (and if you don't get the whole problem I'll grade these parts separately):

(1) "Prove" that $\alpha_\lambda \beta_\lambda$ generates a minimal ideal corresponding to an irreducible representation by indicating what changes would need to be made to the proof in the text (for $c_\lambda = \beta_\lambda \alpha_\lambda$). You do not need to duplicate the argument in the notes; just indicate the changes.

(2) Prove that the minimal scaled idempotents $\alpha_\lambda\beta_\lambda$ and $c_\lambda = \beta_\lambda\alpha_\lambda$ give isomorphic representations.
(3) Determine how to change c_λ to give a representation isomorphic to $\rho \otimes \alpha$ and prove the result.

7. Assignment 7

(1) Give a proof of Example 3 in Section 3.3 of Serre's book.
(2) The picture version of the relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ is not the only way to pass a strand through a crossing. Give an algebraic proof that $\sigma_i\sigma_{i+1}\sigma_i^{-1} = \sigma_{i+1}^{-1}\sigma_i\sigma_{i+1}$.
(3) Give an algebraic proof that $\Delta_n\sigma_i = \sigma_{n-i}\Delta_n$ using the recursive formula from the notes.

8. Assignment 8

(1) Use Lemma 1 in section 3.3 of Serre's book to show the following:

    Let $H \leq G$ be finite groups, $Ind(\rho)$ a representation of G induced by a representation ρ of H. Let σ be a representation of G and $Res(\sigma)$ the restriction of σ to a representation of H. Use a result from Serre to prove that $Hom_H(\rho, Res(\sigma)) \cong Hom_G(Ind(\rho), \sigma)$.

    This property, when extended to arbitrary class functions (and defining $Ind(f)$ in a sensible way), is known as *Frobenius reciprocity*.
(2) Prove that $P_\lambda \cap Q_\lambda = \{e\}$.
(3) We showed that the image of a standard tableau T under $b_\lambda a_\lambda$ contains T as a summand and contains no lexicographically larger standard tableaux. Prove that partition $(2, 2, 1)$ is the lexicographically smallest partition admitting a standard tableau T such that $b_\lambda a_\lambda T$ contains a summand $T' \neq T$ such that T' is a standard tableau. In other words, there exists $p \in P_\lambda$ and $q \in Q_\lambda$ such that $T' = qpT \neq T$ is standard. What is the lexicographically smallest T which has this property? (Note that there are two orderings here, one for partitions and one for tableaux).