Category Theory and Representation Theory

Categories describe abstract objects and their composable relations.

Def: A category $\mathcal{C}$ is a collection of objects $Ob(\mathcal{C})$, a collection of sets $Mor(a,b)$ $\forall a,b \in Ob(\mathcal{C})$, and for each $a,b,c \in Ob(\mathcal{C})$ a composition function $\circ: Mor(a,b) \times Mor(b,c) \to Mor(a,c)$ satisfying

1) $(f \circ g) \circ h = f \circ (g \circ h)$ $\forall f \in Mor(a,b); g \in Mor(b,c); h \in Mor(c,d); a,b,c,d \in Ob(\mathcal{C})$
2) $\forall a \in Ob(\mathcal{C})$ $\exists\, Id \in Mor(a,a)$ s.t. $\forall b \in Ob(\mathcal{C})$ and $\forall f \in Mor(a,b), g \in Mor(b,a)$ $Id \circ f = f$ and $g \circ Id = g$.

(note that composition is 'backwards' compared to the convention for functions.)

Examples:

- A category with one object is a monoid.
- A category with one object and invertible morphisms is a group.
- A category with one object in which the morphisms have a commutative group structure is a ring.
- Grp: the category of groups and group homomorphisms.
- Top: the category of topological spaces and continuous functions
- Vect(F) Vector spaces over F and F-linear maps.
- Cob(o): Finite sequences of dots and Temperley-Lieb pictures up to isotopy.

Def: A functor $F: \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ is a collection of maps (all denoted F) such that

1) $F(c) \in Ob(\mathcal{D}) \;\; \forall c \in Ob(\mathcal{C})$
2) $F(f) \in Mor(F(a), F(b)) \;\; \forall f \in Mor(a,b)$
3) $F(Id_c) = Id_{F(c)}$
4) $F(f \circ g) = F(f) \circ F(g)$

Ex: 0) The identity functor.
1) Let G be a group. A representation is a functor $F: G \to Vect(\mathbb{C})$.
2) Abelianization is a functor from $Grp \to Ab$.
3) π_1 (fundamental group) is a functor $Top \to Grp$

In category theory, the internal structure of objects is ignored. One instead uses the structure of morphisms between objects.

For example, the zero dimensional vector space z in $Vect(\mathbb{C})$ is an initial object: for every object c there exists exactly one morphism $z \to c$. z is also a final object, which is the same definition but with the arrow reversed. Initial and final objects, if they exist, are unique up to isomorphism.

(One has a notion of isomorphism whenever there is an identity. Isomorphisms partition $Ob(\mathcal{C})$ into equivalence classes)

Since we ignore any internal structures of the objects, there is no way to tell isomorphic objects apart; special objects get defined up to isomorphism, often via universal properties they satisfy.

Example: Let $\mathcal{C}$ be a category with a zero (i.e. initial and final) object z. For objects $a, b \in Ob(\mathcal{C})$, the (unique) map $a \to z \to b$ is the zero map from a to b.
For any map $f: a \to b$, the kernel of f, if it exists, is an object c and a map $k: c \to a$ s.t. $k \circ f$ is a zero map, and for any $d \in Ob(\mathcal{C})$ and $g: d \to a$ s.t. $g \circ f$ is the zero map, there exists a unique map $h: d \to c$ s.t.

$$\begin{tikzcd} d \arrow[d, dashed, "h"'] \arrow[dr, "g"] & \\ c \arrow[r, "k"'] & a \end{tikzcd}$$ commutes.

Given a category $\mathcal{C}$ and $x \in Ob(\mathcal{C})$, one can construct a category $\mathcal{C}'$ by adding in isomorphic copies of x. However, $\mathcal{C}$ and $\mathcal{C}'$ would not differ in any interesting way. This leads to:

Def: A skeleton for a category $\mathcal{C}$ is a category consisting of one object from each isomorphism class of $Ob(\mathcal{C})$, with morphism sets and compositions the same as those in $\mathcal{C}$.

Skel$(\mathcal{C})$ is a <u>full subcategory</u> of $\mathcal{C}$.

Let $\overline{\mathcal{C}}$, $\hat{\mathcal{C}}$ be skeletons of $\mathcal{C}$, with $x \in Ob(\mathcal{C})$ represented by $\overline{x}$ and $\hat{x}$ respectively. For each $\overline{x} \in Ob(\overline{\mathcal{C}}) \subseteq Ob(\mathcal{C})$, fix an isomorphism $\varphi_x : \overline{x} \to \hat{x}$

Then there is a functor $F : \overline{\mathcal{C}} \to \hat{\mathcal{C}}$ that sends

$F(\overline{x}) = \hat{x}$ for each $\overline{x} \in Ob(\overline{\mathcal{C}})$

$F(f : \overline{x} \to \overline{y}) = \varphi_x^{-1} \circ f \circ \varphi_y$.

This functor has an inverse functor $G : \hat{\mathcal{C}} \to \overline{\mathcal{C}}$, i.e. $G \circ F = Id_{\hat{\mathcal{C}}}$, $F \circ G = Id_{\overline{\mathcal{C}}}$, given by

$F(\hat{x}) = \overline{x}$

$F(f : \hat{x} \to \hat{y}) = \varphi_x \circ f \circ \varphi_y^{-1}$

Def: Two categories $\mathcal{C}$, $\mathcal{D}$ are isomorphic if $\exists$ an invertible functor $F : \mathcal{C} \to \mathcal{D}$

Def: Two categories are naturally equivalent if their skeletons are isomorphic.

Many categories are defined to include

"all conceivable" objects satisfying certain axioms, and there is often no easy way to describe the skeleton.

Def: A *natural transformation* φ of functors $F, G : \mathcal{C} \to \mathcal{D}$ is a family of maps $\varphi_x : F(x) \to G(x)$ $\forall x \in Ob(\mathcal{C})$ s.t. $\forall x, y \in Ob(\mathcal{C})$ and $\forall f : x \to y$

$$\begin{tikzcd}
F(x) \arrow[r, "\varphi_x"] \arrow[d, "F(f)"'] & G(x) \arrow[d, "G(f)"] \\
F(y) \arrow[r, "\varphi_y"] & G(x)
\end{tikzcd}$$

commutes.

If the φ_x are isomorphisms, φ is called a *natural isomorphism*.

Ex: A G-equivariant map of representations is a natural transformation of functors.

Def: A natural equivalence of categories $\mathcal{C}$ and $\mathcal{D}$ is a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $FG \cong Id_{\mathcal{C}}$ and $GF \cong Id_{\mathcal{D}}$.

One can show that $\mathcal{C}$ and $\mathcal{D}$ admit a natural equivalence iff they are naturally equivalent.

Given categories $\mathcal{C}, \mathcal{D}$, functors $F : \mathcal{C} \to \mathcal{D}$ form a category, with natural transformations as the morphisms.

If $\mathcal{C}$ is a group, etc., $\mathcal{D} = \mathrm{Vect}_{fin}(\mathbb{C})$, one gets the category of complex representations of $\mathcal{C}$ with equivariant maps.

Additional structures:

In Cat, the category of categories, the objects are categories and the morphisms are functors. Natural transformations are then "morphisms between morphisms". Cat is an example of a 2-category.

A monoidal category is (roughly) a category which admits a tensor product. Monoidal categories can be thought of as bicategories with one object. For example, for Rep(G):

- There is one object (called a "0-morphism")
- Reps are "1-morphisms" from that object to itself, with tensor product as their composition
- G-equivariant maps are 2-morphisms between the 1-morphisms, with another composition
- The two compositions commute

Morphism spaces are themselves objects in the category Set. Set is a monoidal category (with cartesian product), and we need that to define composition. If, in the definition of a category, we replace Set with any other monoidal category $\mathcal{M}$, we say that $\mathcal{C}$ is <u>enriched</u> over $\mathcal{M}$.

<u>Examples</u>: $\mathrm{Rep}(G)$ is enriched over $\mathrm{Vect}_{fin}(\mathbb{C})$
$\mathrm{Vect}_{fin}(\mathbb{C})$ is enriched over itself.

These additional structures lead to new (more restrictive) notions of equivalence. For example, representation categories are typically studied as $\mathbb{C}$ (or $\mathbb{R}$)-linear monoidal categories.

<u>Def</u>: Two rings are said to be Morita equivalent if their representation categories are equivalent as $\mathbb{C}$-linear monoidal categories.