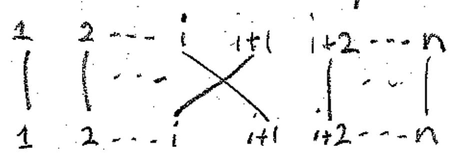
Braid Groups

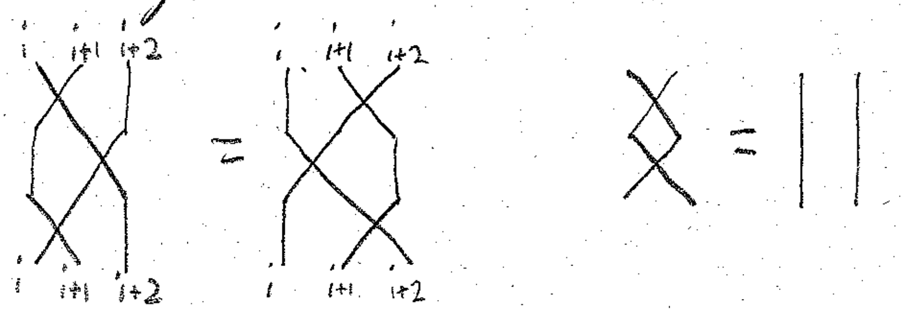
The symmetric group S_n is generated by pairwise transpositions $\sigma_i = (i \; i{+}1)$.
Here is a presentation:

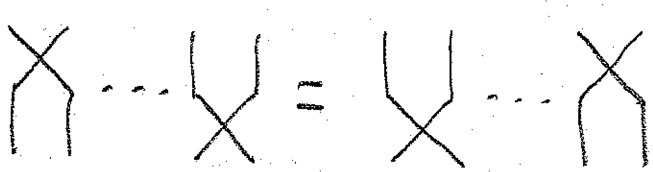
$$S_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \forall_i \; \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \; \forall_i \; \sigma_i^2 = e, \\ \forall_{i,j>i+1} \; \sigma_i \sigma_j = \sigma_j \sigma_i \end{array} \right\rangle$$

If we express each σ_i pictorially, like so:



then the above relations can be interpreted pictorially:

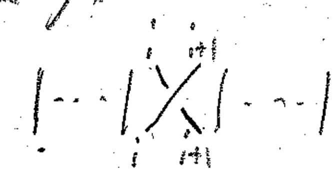




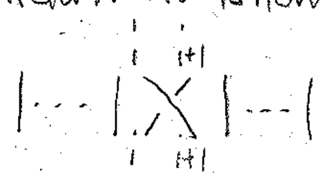
Removing the relation $\sigma_i^2 = e$ leads to the definition of the <u>n stranded braid group</u>

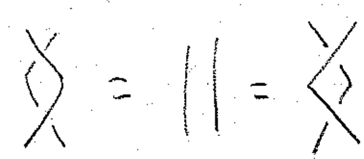
$$B_n := \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \forall i \;\; \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \forall i, j > i+1 \;\; \sigma_i \sigma_j = \sigma_j \sigma_i \end{array} \right\rangle$$

Pictorially:

$\sigma_i :=$ 

σ_i^{-1} is drawn as follows

$\sigma_i^{-1} :=$ 

giving: 

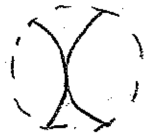
<u>Connection with knot theory:</u>

<u>Def:</u> A (tame) <u>knot</u> K is a smooth embedding $K: S^1 \to \mathbb{R}^3$. (Sometimes, $S^1 \to S^3$).

Knots K_1, K_2 are generally considered to be <u>equivalent</u> if there is an isotopy of $\mathbb{R}^3$ that takes K_1 to K_2. (WLOG, assume K has unit length tangent vectors.)

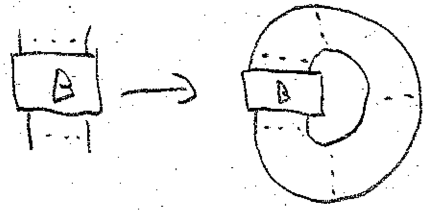
<u>Def:</u> Let p be a projection of $\mathbb{R}^3$ onto a 2-dim'l subspace. Then $p \circ K$ is in <u>general position</u> if

1) the tangent of $p \circ K$ is nonzero everywhere.
2) no triple intersections 

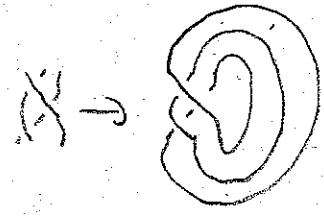
3) no self tangencies 

Additionally, if the double intersections of $Im(p \circ K)$ are labelled with decorations indicating the relative positions of the double pts, then $Im(p \circ K)$ is a <u>knot diagram</u>.

Every braid can be turned into a knot or link by taking the <u>trace closure</u>.



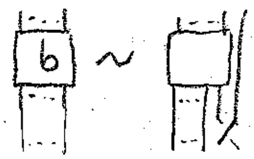
e.g.



(A <u>link</u> is just an embedding of one <u>or more</u> disjoint circles into $\mathbb{R}^3$ (or S^3).)

<u>Thm</u>: (Alexander's Thm): Every link can be represented as the trace closure of a braid.

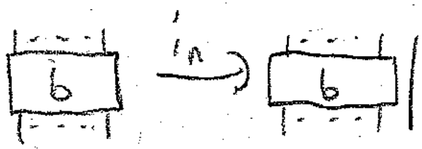
<u>Thm</u> (Markov): Let $b_1 \in B_m$, $b_2 \in B_n$. Then the trace closures of b_1, b_2 give the same knot iff b_1 and b_2 are related by a sequence of the following moves:

1) Braid relations
2) conjugations by the generators
3)  $\leftarrow$ markov moves

Note: For any family of representations $\{\rho_n : B_n \to GL(V_n)\}$, $tr(\rho_n(-))$ is invariant under (1) and (2).
If $tr(\rho_n(-))$ behaves nicely under Markov moves, one can construct a <u>link invariant</u>.

B_∞ and normalized Markov traces

There is a natural inclusion $B_n \to B_{n+1}$



Then B_∞ is the direct limit of
$B_1 \xrightarrow{i_1} B_2 \xrightarrow{i_2} B_3 \xrightarrow{i_3} \cdots$

<u>Def</u>: A trace function on a k-algebra A is a linear map $tr: A \to k$ s.t. $tr(ab) = tr(ba)\ \forall a, b \in A$.

<u>Def</u>: A <u>normalized Markov trace</u> on B_∞ is a tracefunction tr such that for all $b \in B_n \subseteq B_\infty$, $tr(b\sigma_n) = tr(\sigma_n b) = tr(b)$.

technically, $\mathbb{C}[B_\infty]$

A normalized Markov trace on B_∞ gives a link invariant. (Take a link L and compute the trace of any braid representing L).

(quantum) Yang-Baxter Equations

The symmetric group S_n acts on $V^{\otimes n}$ (where V is a vector space) by permutations, i.e.
$\sigma_i \to Id_V^{\otimes i-1} \otimes P \otimes Id_V^{\otimes (n-1-i)}$,
where $P: V \otimes V \to V \otimes V$ sends $v_i \otimes v_j \to v_j \otimes v_i$.

To get an action of the braid group on $V^{\otimes n}$, we want a map $R: V \otimes V \to V \otimes V$ that satisfies

$$(R \otimes Id_V)(Id_V \otimes R)(R \otimes Id_V) = (Id_V \otimes R)(R \otimes Id_V)(Id_V \otimes R)$$

This equation is called the quantum Yang-Baxter Equation (qYBE). (There is another equation called the classical Yang-Baxter Equation), qYBE first appeared in statistical mechanics and were used to study phase transitions in lattice models. Finding new solutions to qYBE is difficult, and an ongoing area of research.

qYBE solutions give representations of the braid group.

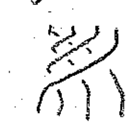
Positive ~~and~~ pure braids:

Def: S_n is a quotient of B_n via the map that sends generators $\sigma_i \in B_i$ to pairwise transpositions $(i, i+1) \in S_n$. The kernel of this homomorphism is P_n, the pure braid group. Pure braids always send the strands to themselves, e.g.

The generators $\sigma_1, \dots, \sigma_{n-1} \in B_n$ generate a monoid called the positive braid monoid B_n^+. A braid in B_n^+ is called a positive braid; its crossings are all "positive":

 positive  negative

The center $Z(B_n)$ maps to $Z(S_n) = \{(1)\}$, so $Z(B_n) \leq P_n$. In fact, $Z(B_n) \cong \mathbb{Z}$, with generator (Δ_n^2), where Δ_n is a "180° twist"

e.g.  $= \Delta_4$

Not that Δ_n satisfies the commutation relation $\sigma_i \Delta_n = \Delta_n \sigma_{n-i}$

180° rotation does not change the sign of a crossing!