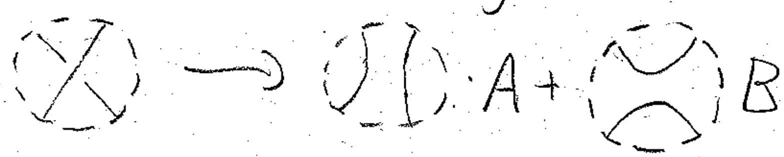
# The Jones Representation

(Historically Incorrect)

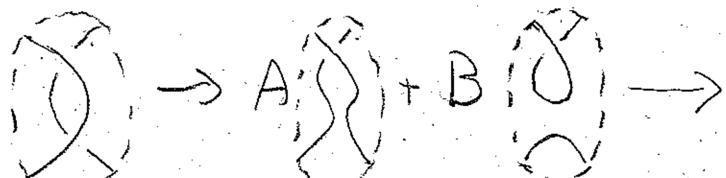
Let K be a knot. Fix a diagram for K.

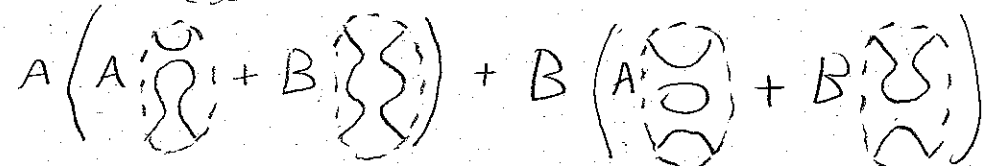
Lets replace K by a linear combination of pictures with smoothed crossings:

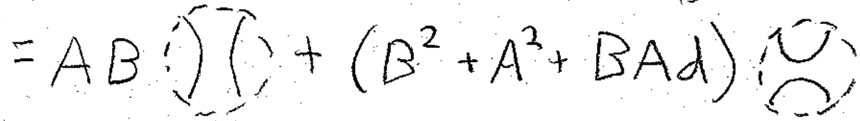



Replace each picture with $d^{(\text{\# of circles in the picture})}$

Then



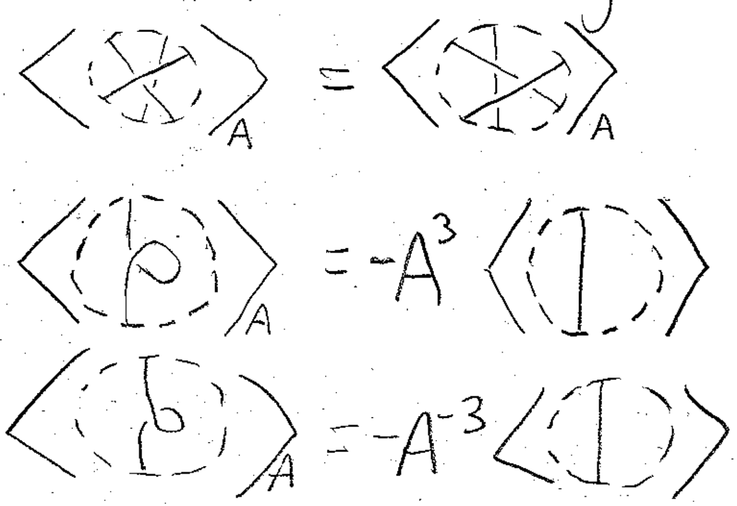




In order for this to be invariant under R-II, we need:

$$A = B^{-1} \quad d = -A^2 - A^{-2}$$

In this case, the resulting Laurent polynomial is the Kauffman Bracket $\langle K \rangle_A$ of K.

One can show the following:

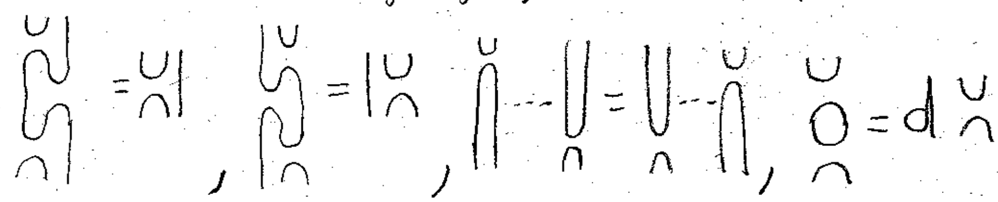



Performing the same operation on braids gives a homomorphism $\mathbb{C}[B_n] \to TL_n(d)$, where $TL_n(d)$ is the <u>Temperley-Lieb algebra</u>.

<u>Def</u>: $TL_n(d)$ is the $\mathbb{C}[A, A^{-1}]$-algebra

$$\left\langle \tau_1, \ldots, \tau_{n-1} \;\middle|\; \begin{array}{l} \forall_{i<n-1}\ \tau_i \tau_{i+1} \tau_i = \tau_i,\ \forall_{i<n-1}\ \tau_{i+1}\tau_i\tau_{i+1} = \tau_{i+1} \\ \forall_{i>j+1}\ \tau_i \tau_j = \tau_j \tau_i,\ \forall_i\ \tau_i^2 = d\,\tau_i \end{array} \right\rangle$$

i.e.:




Gives a braid rep with a Markov Trace

<u>Prop</u>: As a $\mathbb{C}[A, A^{-1}]$ vector space,

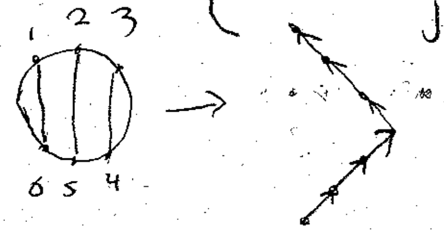
$$\dim(TL_n(d)) = \binom{2n}{n}\Big/ n+1.$$

<u>Pf</u>: $\dim(TL_n(d_0))$ is equal to the number of ways to connect $2n$ ordered points arranged in sequence around a circle using non-crossing straight lines.

This in turn is equal to the number of sequences $\{s_i\}_i$ of vectors such that $\forall i\ s_i \in \{(1,1),(-1,1)\}$, $\forall i \sum_{j<i} s_j$ has non-negative first coordinate, and $\sum_i s_i = (0,2n)$.

This can be seen as follows: for each marked point $i$, $s_i = \begin{cases} (1,1) & \text{if } s_i \text{ connects to a point } s_j \text{ s.t. } j>i \\ (-1,1) & \text{if } s_i \text{ connects to } s_j \text{ with } j<i \end{cases}$

e.g.




Let $p(i,j)$ = # of partial sequences summing to $(i,j)$.

Then $p(i,j) = \begin{cases} p(i-1,j-1)+p(i+1,j-1) & \text{if } i>0 \\ p(i+1,j-1) & \text{if } i=0, j>0 \\ 1 & \text{if } i=j=0 \\ 0 & \text{otherwise} \end{cases}$

By induction,

$p(i,j) = \begin{cases} \binom{j}{\frac{j-i}{2}} - \binom{j}{\frac{j-i}{2}-1} & \text{if } j \geq i \geq 0,\ j \text{ is even.} \\ 0 & \text{otherwise} \end{cases}$

Then $p(0,2n) = \binom{2n}{n} - \binom{2n}{n-1}$
$= \frac{1}{n+1}\binom{2n}{n}$. $\square$

If we set $A \in \mathbb{C}$, with $A$ not a root of unity then $TL_n(d)$ is semisimple. In addition, $\sigma_i - A^2 \sigma_i^{-1} \longrightarrow A\, Id + A^{-1}\tau_i - A^{1}\, id - A^3 \tau_i$
$= (A^{-1} - A^3)\tau_i$

so the Kauffman bracket map is surjective. Thus minimal idempotents for $TL_n(d)$ will give us irreps for $B_n$ for generic $d$.

For any idempotent $e_i \in TL_n(d)$, let $\rho_{e_i}$ denote the associated representation of $B_n$. We already know that the statement

$$\forall a \in TL_n(A) \quad e_i a e_i = k e_i,\ k \in \mathbb{C} \wedge \exists a \in TL_n(d)\ e_i a e_i \neq 0$$

holds iff $e_i$ is minimal.

In general, $\{e_i a e_i \mid a \in TL_n(d)\}$ is a vector space of dimension

$$\sum_{\text{irreps } \rho} \begin{pmatrix} \#\text{ of minimal idempotent summands } e_j \text{ of } e_i \\ \text{such that } \rho_{e_j} \cong \rho \end{pmatrix}^2$$

Recall also that in a semisimple algebra the identity can be written as a direct sum of minimal orthogonal idempotents, i.e. minimal idempotents such that $e_i e_j \neq 0$ when $i \neq j$.

With these tools we can find idempotents corresponding to some irreps of the Jones representation.

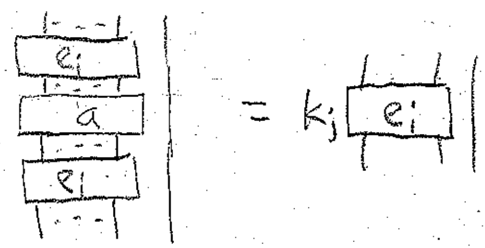
Induct on $n$. For the base case, $TL_1(d) \cong \mathbb{C}$, and Id is a minimal idempotent.

Suppose the set $E$ of minimal idempotents of $TL_{n-1}(d)$ is known.

Then for each $e_i \in E$, $\bar{e}_i :=$  is an idempotent and $\sum_{e_i \in E} \bar{e}_i = \mathrm{Id}$.

Splitting the $\bar{e}_i$ into minimal idempotents will give a complete set of idempotents for $TL_n(d)$.

We have




for all $a \in TL_{n-1}(d)$

In general, $\bar{e}_i$ will not be minimal and must be decomposed.

However, there exists a sequence of idempotents for which this decomposition is particularly easy.

<u>Prop</u>: $TL_n(d)$ admits a unique nonzero idempotent $\Pi_n$ such that $\Pi_n \tau_i = \tau_i \Pi_n$ for all $1 \le i \le n-1$.
(This idempotent is called the Jones-Wenzl projector.)

Pf: <u>Uniqueness:</u>

Suppose such a $\Pi_n$ exists. Since $\Pi_n \tau_i = 0 \ \forall 1 \le i \le n-1$, $\Pi_n = \Pi_n \cdot \Pi_n = \Pi_n \cdot k\, Id_n$, where $k$ is the coefficient of the summand $Id_n$ in $\Pi_n$. Thus $k = 1$.

Similarly, $\Pi_n \Pi_n = Id_n \Pi_n$.

Suppose $\Pi_n, \Pi_n'$ satisfy the conditions. Then

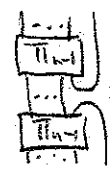
$\Pi_n = \Pi_n Id_n = \Pi_n \Pi_n' = Id_n \Pi_n' = \Pi_n'$

<u>Existence:</u>

Define $\Delta_n$ inductively via

$\Delta_{-1} = 0; \ \Delta_0 = 1, \ \Delta_{n+1} = d\Delta_n - \Delta_{n-1}$

Define $\Pi_n$ inductively via

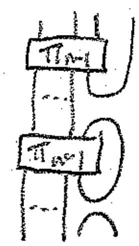
$\Pi_1 = Id_1$, $\Pi_n =$  $- \dfrac{\Delta_{n-2}}{\Delta_{n-1}}$ 

For $n=1$, $\Pi_n$ satisfies the required properties.

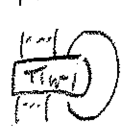
Suppose $\Pi_1, \dots, \Pi_{n-1}$ do.

Then $\tau_i \Pi_n = \Pi_n \tau_i = 0$ if $i < n-1$ by induction.

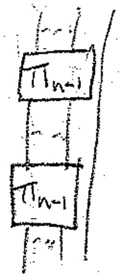
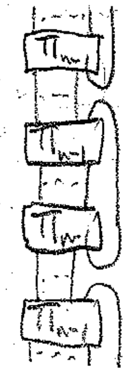
We have

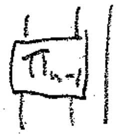
$\tau_{n-1} \Pi_n =$  $- \dfrac{\Delta_{n-2}}{\Delta_{n-1}}$ 

$= \left(1 - \dfrac{\Delta_{n-2}}{\Delta_{n-1}} k\right)$ , where $k$ is the coefficient

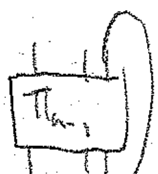
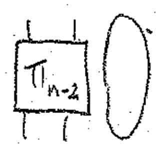
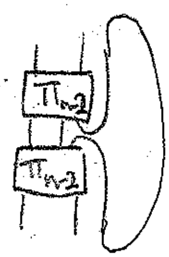
of $Id_{n-2}$ in 

Similarly, $\pi_n \tau_{n-1} = \left(1 - \frac{\Delta_{n-2}}{\Delta_{n-1}} k\right)$ 

and $\pi_n \pi_n =$  $- 2\frac{\Delta_{n-2}}{\Delta_{n-1}}$  $+ \left(\frac{\Delta_{n-2}}{\Delta_{n-1}}\right)^2$ 

$=$  $+ \left(\left(\frac{\Delta_{n-2}}{\Delta_{n-1}}\right)^2 k - 2\frac{\Delta_{n-2}}{\Delta_{n-1}}\right)$ 

so we must show $k = \frac{\Delta_{n-1}}{\Delta_{n-2}}$.

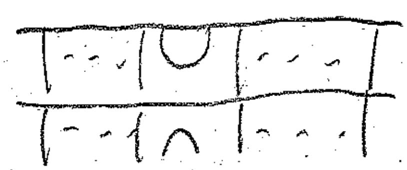
But  $=$  $- \frac{\Delta_{n-3}}{\Delta_{n-2}}$ 

by induction,

$$k = \frac{\Delta_{n-2}}{\Delta_{n-3}}\left(d - \frac{\Delta_{n-3}}{\Delta_{n-2}}\right) = \frac{d\Delta_{n-3} - \Delta_{n-2}}{\Delta_{n-2}} = \frac{\Delta_{n-1}}{\Delta_{n-2}}$$

$\pi_n$ is a minimal idempotent for $TL_n(d)$, since $\pi_n \pi_n = \pi_n$ and $\pi_n a \pi_n = k\pi_n$, where $k$ is the coefficient of $Id_n$ in $a$.

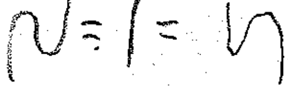
Using $\pi_n$ we can find other minimal idempotents. First, note that we can split each $\tau_i$ into a pair of "maps",

$\tau_i =$ $|\cdots|\overset{\cup}{\cap}|\cdots|$ $=$  $= \tau_i^{n-} \circ \tau_i^{n+}$

where $\tau_i^-$ is considered to be a map from "n dots" to "n-2 dots"

Then one has $\tau_i^{+}\tau_i^{-} = d\,Id_{n-2}$.

Define $TL_{m,n}(d)$ to be the vector space of linear combinations of compositions of the $\tau_i^-$, $\tau_i^+$, $Id$ such that the composition goes from m dots to n dots, subject to the relations

 $\left(\tau_i^{+}\tau_{i+1}^{-} = Id_{n-2} = \tau_{i+1}^{n+}\tau_i^{n-}\right)$

$|\cdots|O|\cdots| = d\,|\cdots||\cdots|$ $\left(\tau_i^{n+}\tau_i^{n-} = d\,Id_{n-2}\right)$

Then $TL_{n,n}(d) = TL_n(d)$, and the resulting structure for m,n is called the <u>Temperley Lieb algebroid</u>.

Then $\Pi_n$ is the projector onto the subspace of elements in $TL_n(d)$ which do not factor through a smaller $TL_{n-k}(d)$.

One can use $P_n$ to find other minimal idempotents:

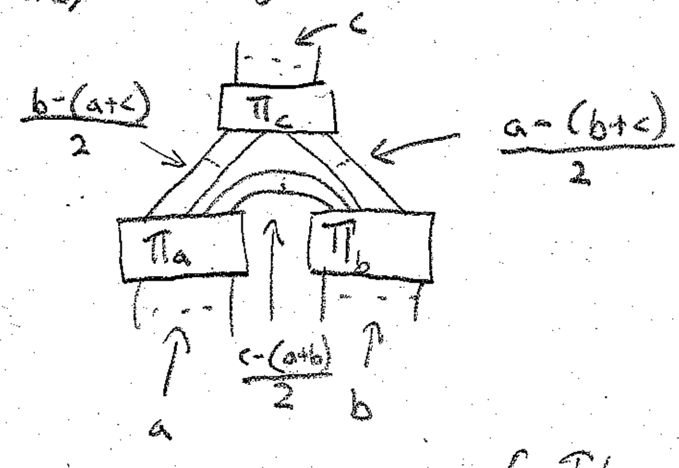
<u>Prop</u>: Let $p \in TL_{m,n}(d)$, $q \in TL_{n,m}(d)$.
Let $r = p\,\Pi_m\,q \in TL_n(d)$.
Then $\forall a \in TL_n(d)$, $\exists k_a \in \mathbb{C}$ s.t. $rar = k_a r$.

**Pf:** $rar = p\Pi_m q\, a\, p\Pi_m q$, with $qap \in TL_m(d)$. Then $\Pi_m$ kills every summand of $qap$ which is not $k\,\mathrm{Id}_m$ for some $k \in \mathbb{C}$. The result then follows with $k = k_a$.

**Def:** An $n$-acceptable binary tree is a rooted binary tree with labelled edges such that

1) The right branch of each non-terminal node is a terminal node
2) For ${}_a\lambda^c_b$ one has $a, b, c \geq 0$, $a+b \geq c$, $b+c \geq a$, $c+a \geq b$, $a+b+c$ even.
3) all non-root terminal nodes have label at least 1
4) the sum of the labels of the non-root terminal nodes is $n$.

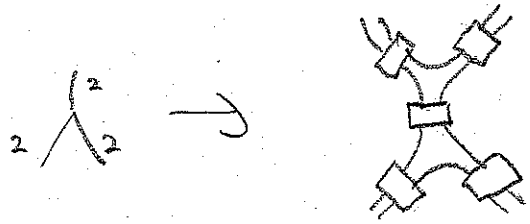
For each vertex ${}_a\lambda^c_b$ one gets an element $TL_{a+b,c}(d)$ given as




This gives an element of $TL_{n,r}(d)$, where $r$ is the labelling of the root node.

Composing this element with its vertical flip gives an element of $TL_n(d)$.

e.g.



where each box is a copy of $\Pi_2$

Prop: Any such element $i$ of $TL_n(d)$ is a minimal idempotent.

Pf Sketch: In light of the previous proposition, it suffices to show that $i \cdot i \neq 0$.
This computation is rather technical, but is worked out in detail in (Kauffman, Lins)

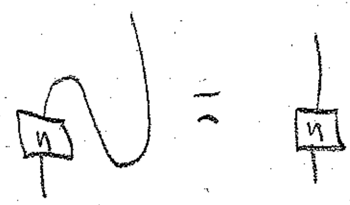
Let's rewrite  as .

Notice that the projectors $\Pi_n$ behave a lot like group representations.

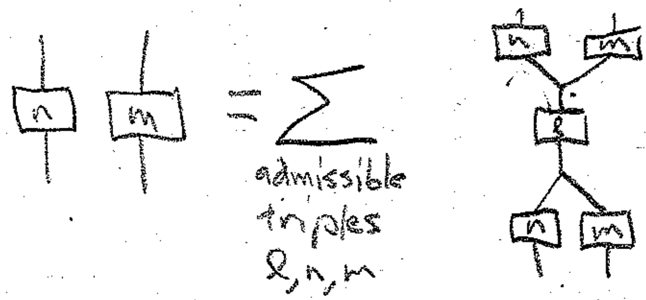
We have a version of Schur's lemma:

 $= \begin{cases} 0 & \text{if } n \neq m \text{ (because } f \text{ must contain a "turnback")} \\ \lambda \cdot \end{cases}$  if $m = n$

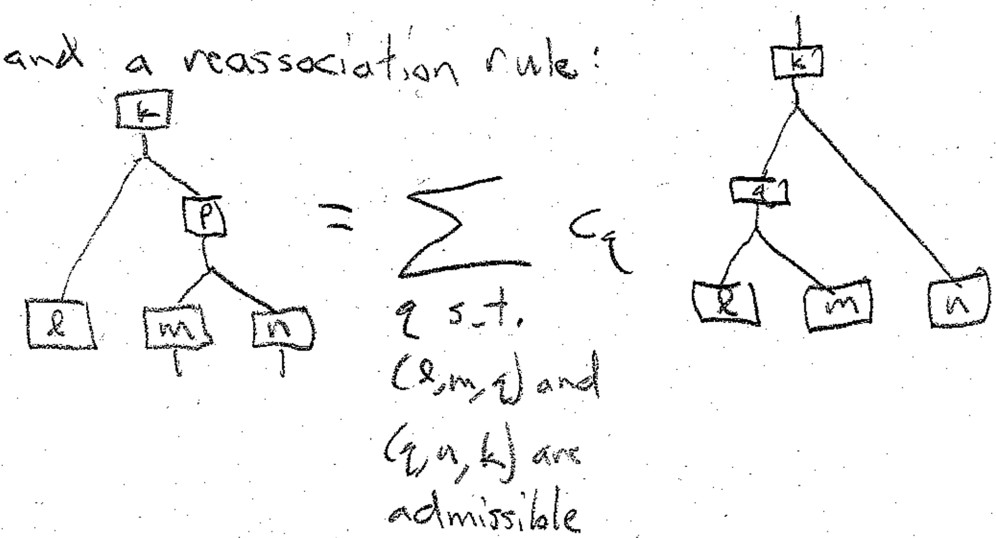
rigidity:




One can show a version of semisimplicity:




and a reassociation rule:




It seems like the projectors $\Pi_n$ might describe the representations of <u>something</u>. They do in fact, but it is not a group. It is the quantum group $U_q(SL_2)$. We'll describe what these are next time.