SOLUTION FOR HOMEWORK 10, STAT 6331

1. Exerc.9.1. Given $L(X) \leq U(X)$, write *Solution A*:

$$P_\theta(L(X) \leq \theta \leq U(X)) = P_\theta((L(X) \leq \theta) \cap (U(X) \geq \theta)$$

(use De Morgan's law and then the total probability rule)

$$= 1 - P_\theta((L(X) > \theta) \cup (U(X) < \theta, L(X) \leq U(X))$$

(using the fact that the events $L(X) > \theta$ and $U(X) < \theta$ are mutually exclusive [sets are disjoint] whenever $L(X) \leq U(X)$, we continue)

$$= 1 - P_\theta(L(X) > \theta) - P_\theta(U(X) < \theta) = 1 - \alpha_1 - \alpha_2.$$

Solution B: Let $A := \{X : L(X) \leq \theta\}$ and $B := \{X : U(X) \geq \theta\}$. Then

$$A \cap B = \{X \;\; L(X) \leq \theta \leq U(X)\}$$

and

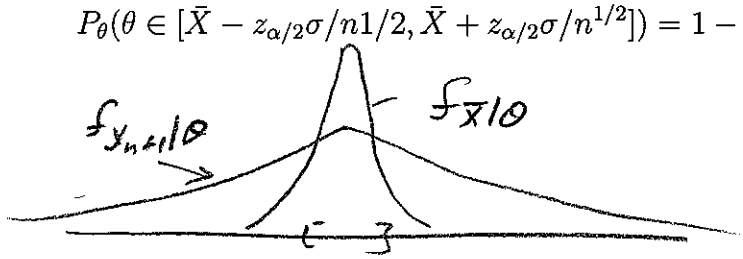
$$P_\theta(A \cup B) = P_\theta(L(X) \leq \theta \cup \theta \leq U(X)) = 1.$$

(please check the last equality!
if you have problems with understanding why note that $L(X) > \theta$ implies $U(X) > \theta$, and then continue to think in this direction). Then

$$P_\theta(A \cap B) = P_\theta(A) + P_\theta(B) - P_\theta(A \cup B) = (1 - \alpha_1) + (1 - \alpha_2) - 1 = 1 - \alpha_1 - \alpha_2.$$

2. Exerc. 9.2. Since $X_1, \ldots, X_n$ are iid according to $N(\theta, 1)$, we can write (I use $z_{\alpha/2}$ as notation for the $(1 - \alpha/2)$-quantile of the standard normal distribution)

$$P_\theta(\theta \in [\bar{X} - z_{\alpha/2}\sigma/n1/2, \bar{X} + z_{\alpha/2}\sigma/n^{1/2}]) = 1 - \alpha.$$



At the same time

$$P_\theta(X_{n+1} \in [\bar{X} - z_{\alpha/2}\sigma/n^{1/2}, \bar{X} + z_{\alpha/2}\sigma/n^{1/2}])$$

$$= P_\theta(\frac{X_{n+1} - \theta}{\sigma} \in [\frac{\bar{X} - \theta - z_{\alpha/2}\sigma/n^{1/2}}{\sigma}, \frac{\bar{X} - \theta + z_{\alpha/2}\sigma/n^{1/2}}{\sigma}])$$

(introduce two independent standard normal random variables Y and Y_1, and continue)

$$= P(Y \in [Y_1/n^{1/2} - z_{\alpha/2}/n^{1/2}, Y_1/n^{1/2} + z_{\alpha/2}/n^{1/2}]) = P(\frac{Y_1 - z_{\alpha/2}}{n^{1/2}} \leq Y \leq \frac{Y_1 + z_{\alpha/2}}{n^{1/2}})$$

(remember that Y and Y_1 are independent)

$$= \int_{-\infty}^{\infty} (2\pi)^{-1/2} e^{-y^2/2} [\int_{(y-z_{\alpha/2})/n^{1/2}}^{(y+z_{\alpha/2})/n^{1/2}} (2\pi)^{-1/2} e^{-x^2/2} dx] dy$$

(note that the inner integral in square brackets is less or equal to $2z_{\alpha/2}/[2\pi n]^{1/2}$, if you do not understand this inequality — plot a graphic and look at different possible ys and note that the integral takes on its maximum value when $y = 0$. We continue using this observation)

$$\leq \frac{2^{1/2} z_{\alpha/2}}{[\pi n]^{1/2}} = o_n(1).$$

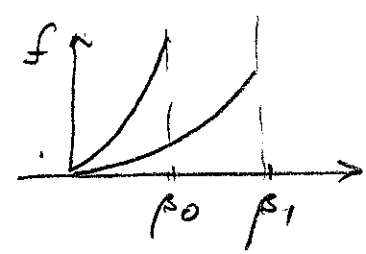
Remember that $o_n(1) \to 0$ as $n \to \infty$ is a standard notation for vanishing sequences in n.

3. Exers. 9.3(a) Given that $X_1, \ldots, X_n$ are iid as X according to the distribution

$$F_X(x|\beta, \alpha) = P_{\beta,\alpha}(X \leq x) = \begin{cases} 0 & \text{if } x \leq 0 \\ (x/\beta)^\alpha & \text{if } 0 < x < \beta \\ 1 & \text{if } x \geq \beta. \end{cases}$$

Since $\alpha = \alpha_0$ is given, from now on we will only use subscript for β to remind us that this is the unknown parameter of interest. Clearly $X_{(n)}$ is the MSS and note that β is the scale parameter, that is

$$Z := \frac{X}{\beta}$$

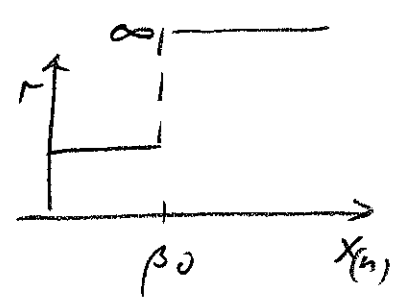


and Z, which often is referred to as the pivot, has the pdf defined as

$$f_Z(z) = \alpha_0 x^{\alpha_0 - 1} I(0 < x < 1).$$

Using this we can write,

$$.05 = P_\beta(\beta > U(\mathbf{X})) = P_\beta(\beta > U(X_{(n)})).$$



Now note that the likelihood ratio is

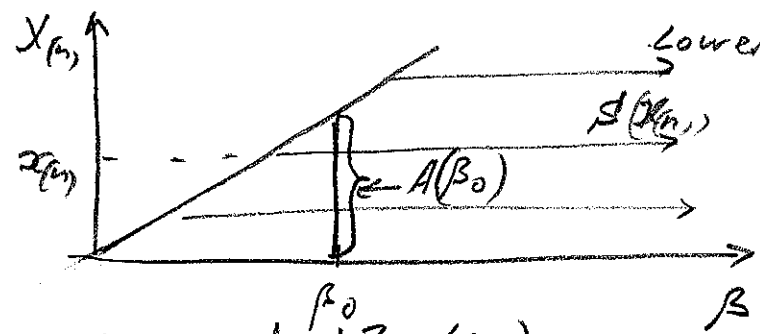
$$r(\mathbf{x}) = \frac{f_{\mathbf{X},\beta_1}(\mathbf{x})}{f_{\mathbf{X},\beta_0}(\mathbf{x})} = \frac{\beta_0^{n\alpha_0} (\prod_{l=1}^n x_l)^{\alpha_0 - 1} I(X_{(n)} < \beta_1)}{\beta_1^{n\alpha_0} (\prod_{l=1}^n x_l)^{\alpha_0 - 1} I(X_{(n)} < \beta_0)}$$

Graphics below can help you to with the analysis.

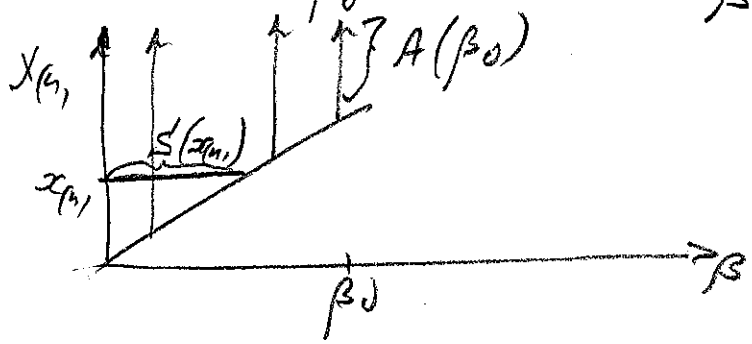
Based on the analysis of the likelihood ratio, we conclude that the rejection region is $X_{(n)} > c_\beta$. As a result, we reject if

$$\frac{X_{(n)}}{\beta} > k$$

where the constant k depends only on the level of significance. The two graphics below show the duality of the hypothesis testing and confidence interval estimation.



$$\begin{cases} H_0: \beta = \beta_0 \\ H_a: \beta > \beta_0 \end{cases}$$



$$\begin{cases} H_0: \beta = \beta_0 \\ H_a: \beta < \beta_0 \end{cases}$$

our case

Now we can write that

$$P_\beta(X_{(n)}/\beta < k) = \gamma = .05$$

and that $Z_{(n)} := X_{(n)}/\beta$ has the cdf which does not depend on β (remember that Z is the pivot)

$$\begin{aligned} F_{Z_{(n)}}(k) &= P(Z_{(n)} \le k) = P(X_1 \le k, \ldots, X_n \le k) \\ &= [k\beta/\beta]^{\alpha_0 n} = k^{\alpha_0 n} = .05. \end{aligned}$$

This implies $k = (.05)^{1/[\alpha_0 n]}$. We conclude that

$$P_\beta(X_{(n)}/\beta > k) = P_\beta(\beta < X_{(n)}/k) = P_\beta(\beta < X_{(n)}/(.05)^{1/[\alpha_0 n]}) = .95.$$

and, as a result,

$$U(\mathbf{X}) = \frac{X_{(n)}}{(.05)^{1/[\alpha_0 n]}}.$$

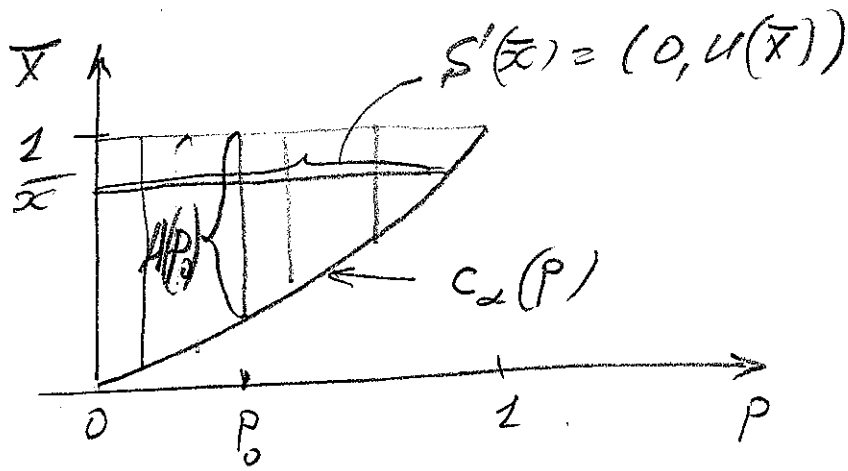
4. Exerc. 9.5 Here $X_1, \ldots, X_n$ is a sample from *Bernoulli*(p). We are asked to find $U(\mathbf{X})$.

(a) We invert the corresponding test

$$\begin{cases} H_0: & p = p_0 \\ H_1: & p < p_0. \end{cases}$$

which yields the familiar UMP test accepting the null hypothesis if $\bar{X} > C_\alpha(p_0)$ with a possible randomization at C_α.

Now look at the graphic which explains how the "inverting" works.



5. Exerc. 9.6(a) Here $X_1, \ldots, X_n$ is a sample from $Bernoulli(p)$, and denote $Y := \sum_{l=1}^n X_l$.

Consider the test $H_0: \ p = p_0$ versus $H_1: p \neq p_0$. The LRT statistic is

$$\lambda(Y) = \frac{\frac{n!}{Y!(n-Y)!} p_0^Y (1-p_0)^{n-Y}}{\frac{n!}{Y!(n-Y)!} \hat{p}^Y (1-\hat{p})^{n-Y}} = \Big[\frac{p_0}{(1-\hat{p})}\Big]^Y \Big[\frac{1-p_0}{1-\hat{p}}\Big]^n$$

where $\hat{p} = \bar{Y} := Y/n$ is the MLE of p.

Using this result we find that the Acceptance region is

$$A(p_0, \alpha) := \{y: \ \ [p_0/\hat{p}]^y [(1-p_0)/(1-\hat{p})]^{n-y} \geq k_\alpha\}$$

where $P(Y \in A(p_0, \alpha)) = 1 - \alpha$. The last equation allows us to calculate k_α in a usual way (with a possible randomization, if needed).

Inverting the acceptance region with respect to p_0 gives us a $1-\alpha$ level confidence set

$$S(Y, 1-\alpha) := \{p \ \ [p_0/\bar{Y}]^Y [(1-p_0)/(1-\bar{Y})^{n-Y} \geq k_\alpha\}.$$

6. Exerc. 9.16. Suppose that $X_1, \ldots, X_n$ are iid $Normal(\theta, \sigma^2)$, and σ^2 is known.

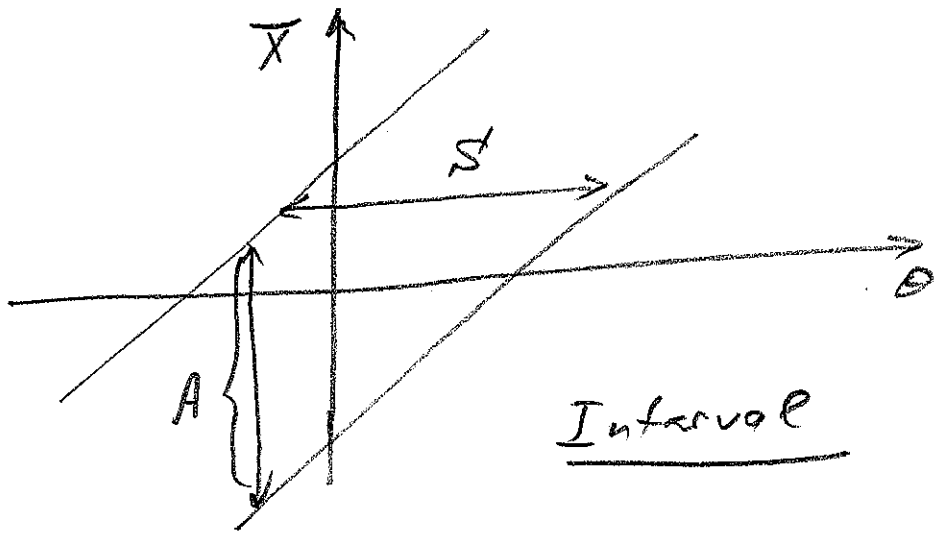
(a). Consider testing $H_0: \ \theta = \theta_0$ versus $H_1: \ \theta \neq \theta_0$. Then we have the following familiar Acceptance region

$$A(\theta_0, \alpha) := \{\bar{X}: \ \theta_0 - z_{\alpha/2}\sigma_{\bar{X}} < \bar{X} < \theta_0 + z_{\alpha/2}\sigma_{\bar{X}}\},$$

which yields the corresponding inverted confidence set

$$S(\bar{X}, 1-\alpha) := \{\theta : \ \bar{X} - z_{\alpha/2}\sigma_{\bar{X}} < \theta < \bar{X} + z_{\alpha/2}\sigma_{\bar{X}}\}.$$

Here I used a standard notation for $\sigma_{\bar{X}} = \sigma/n^{1/2}$. Please also look at the figure depicting the region and the set.



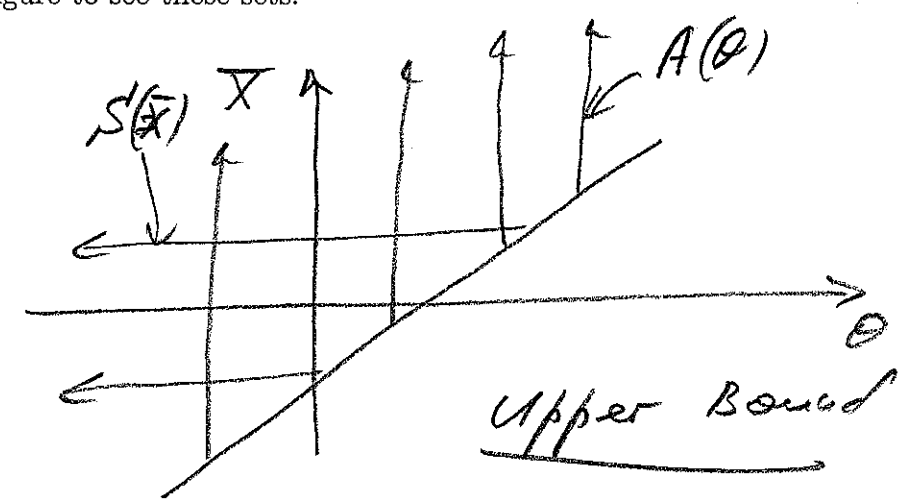
(b) Here we use the test $H_0 : \ \theta \geq \theta_0$ versus $H_1 : \ \theta < \theta_0$. The Acceptance and the inverted confidence set are

$$A(\theta_0, \alpha) := \{\bar{X} : \ \bar{X} > \theta_0 - z_\alpha \sigma_{\bar{X}}\}$$

and

$$S(\bar{X}, 1-\alpha) := \{\theta : \ \theta < \bar{X} + z_{\alpha/2}\sigma_{\bar{X}}\}.$$

Look at the figure to see these sets.



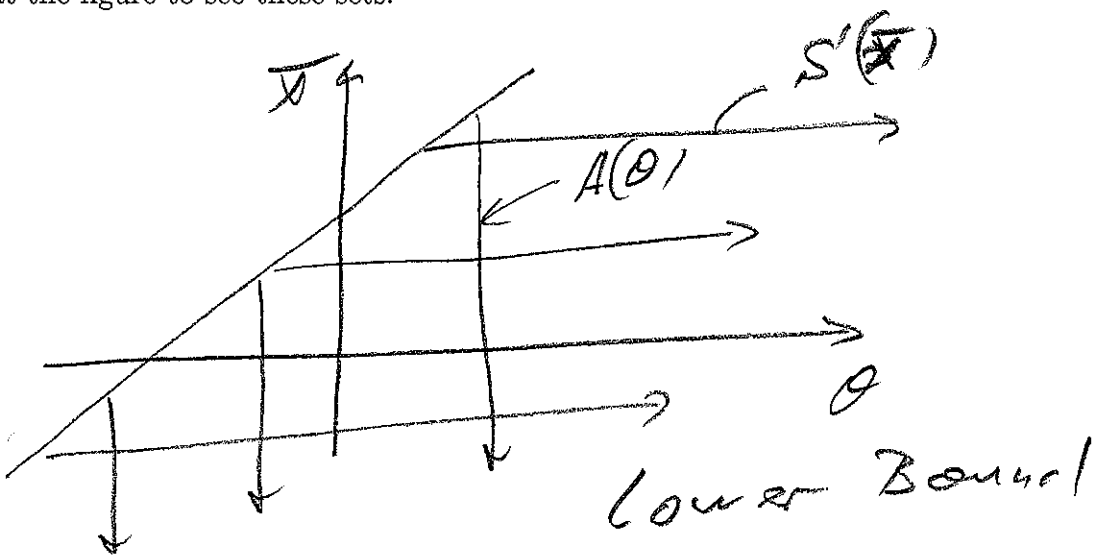
(c) Here we use the test $H_0: \ \theta \leq \theta_0$ versus $H_1: \ \theta > \theta_0$. The Acceptance and the inverted confidence set are

$$A(\theta_0, \alpha) := \{\bar{X}: \ \bar{X} < \theta_0 + z_\alpha \sigma_{\bar{X}}\}$$

and

$$S(\bar{X}, 1-\alpha) := \{\theta: \ \theta > \bar{X} - z_{\alpha/2}\sigma_{\bar{X}}\}.$$

Look at the figure to see these sets.



7. Exerc. 9.17. Consider problems in turn.

(a) Here $f(x|\theta) = I(\theta - 1/2, \theta + 1/2)$. Note that θ is a location parameter so $U := X - \theta$ is $Uniform(-1/2, 1/2)$. Thus any $-1/2 < a_{n,\alpha} < b_{n\alpha} < 1/2$ such that

$$P(a_{n,\alpha} < X_{(1)} - \theta < X_{(n)} - \theta < b_{n,\alpha}) = 1 - \alpha$$

yield a valid confidence set

$$-(b_{n,\alpha} - X_{(n)}) < \theta < X_{(1)} - a_{n,\alpha}.$$

Note that due to symmetry we can choose $a_{n,\alpha} = -b_{n,\alpha}$ and get

$$-(b_{n,\alpha} - X_{(n)}) < \theta < X_{(1)} + b_{n,\alpha}.$$

(b) Consider $f(x|\theta) = 2x/\theta^2 I(0 < x < \theta)$, $\theta > 0$. Note that θ is the scale parameter and $X_{(n)}$ is MSS. Then $Y = X/\theta$ is a random variable with $f_Y(y) = 2yI(o < y < 1)$. Furthermore, the cdf

$$F_{Y_{(n)}}(z) = P(Y_{(n)} \leq z) = (z^2)^n, \quad 0 < z < 1,$$

and the corresponding pdf

$$f_{Y_{(n)}}(z) = 2n(z)^{2n-1} = 2nz^{2n-1}I(0 < z < 1).$$

Thus we can choose any a and b such that

$$P_\theta(a < X_{(n)}/\theta < b) = 1 - \alpha$$

which is equivalent to choosing a and b such that

$$2n\int_a^b z^{2n-1}dz = b^{2n} - a^{2n} = 1 - \alpha.$$

The set is $S(X_{(n)}) = \{\theta : \frac{X_{(n)}}{b} < \theta < \frac{X_{(n)}}{a}\}$

8. Exerc. 9.33(a). Here X is $Normal(\mu, 1)$ and

$$S_\alpha(x) := \{\mu : \ \min(0, x - a) \leq \mu \leq \max(0, x + a)\}.$$

For $a = 1.645$ we know that $P_\mu(\mu \in S_\alpha(X)) = .95$. for all μ except of $\mu = 0$.
Solution: (A) Note that $0 \in S_\alpha(X)$ always so

$$P(0 \in S_\alpha(X)|\mu = 0) = 1.$$

(B) if $\mu > 0$ then

$$P_\mu(\mu \in S_\alpha(X)) = P_\mu(\mu \leq \max(0, X + a))$$

(because $0 \geq \min(0, x - a)$ we continue)

$$P_\mu(\mu < X + a) = P_\mu(Z > -a) = .95$$

where $Z = X - \mu$ is a standard normal random variable.
Absolutely similarly we consider $\mu < 0$.

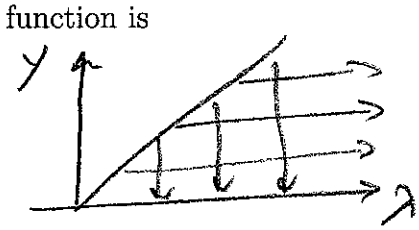
9. Exerc. 9.44 (a) Given $X_1, \ldots, X_n$ are iid $Poisson(\lambda)$. Find UMA interval.
Solution: Consider an appropriate hypothesis test $H_0 : \ \lambda = \lambda_0$ versus $H_a : \ \lambda > \lambda_0$. The UMP test accepts the null hypothesis if

$$Y := \sum_{l=1}^{n} X_l < c_\alpha$$

Note: $\sum_{l=1}^{n} X_l \sim Poisson(n\lambda)$

with a possible randomization at c_α. The corresponding critical function is

$$\phi(x) = \begin{cases} 1 & \text{if } Y > c_\alpha \\ \gamma & \text{if } Y \leq c_\alpha \\ 0 & \text{otherwise} \end{cases}$$



Then

$$A(\lambda_0) := \{Y : \ Y \leq c_\alpha \text{ and with probability } 1 - \gamma \text{ if } Y = c_\alpha\}.$$

Invert the Acceptence region and get a lower bound.

10. Exerc 9.45. Given: $X_1, \ldots, X_n$ is a sample from Exponential distribution with mean λ. Denote $Y := \sum_{l=1}^{n} X_l$.

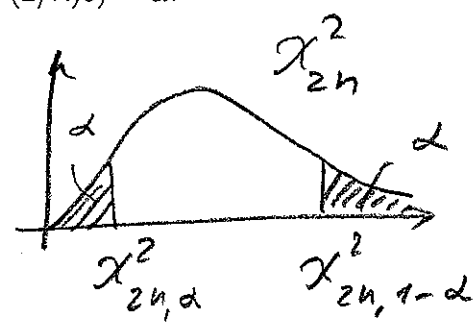
(a) Consider testing $H_0: \ \lambda = \lambda_0$ versus $H_1: \ \lambda < \lambda_0$. Then, as we know, the UMP test has the rejection region $A(\lambda_0) = \{Y: \ Y < c\}$.

We know that $(2/\lambda)Y$ has chi-squared distribution with $2n$ degrees of freedom. Using this we write,

$$P_\lambda(Y < c) = P((2/\lambda)Y < (2/\lambda)c) = P(\chi^2_{2n} < (2/\lambda)c) = \alpha.$$

This yields

$$(2/\lambda)c = \chi^2_{2n,\alpha}$$



and the critical function

$$\phi(Y) = I(Y < (\lambda/2)\chi^2_{2n,\alpha}).$$

(b) Since

$$A(\lambda) = \{Y \ \ Y > (\lambda/2)\chi^2_{2n,\alpha}\}$$

we get

$$S(Y) = \{\lambda: \ 0 < \lambda < (2Y)/\chi^2_{2n,\alpha}\}.$$

(c) Write

$$E_\lambda\{\frac{2Y}{\chi^2_{2n,1-\alpha}}\} = \frac{2}{\chi^2_{2n,\alpha}} E_\lambda(Y) = \frac{2}{\chi^2_{2n,\alpha}}(n\lambda) = \frac{2n\lambda}{\chi^2_{2n,\alpha}}.$$