SOLUTION FOR HOMEWORK 1, STAT 4352

Welcome to your first homework. We begin with reviewing several classical distributions created by some functionals of iid standard normal random variables.

Try to find mistakes (and get extra points) in my solutions. Typically they are silly arithmetic mistakes (not methodological ones). They allow me to check that you did your HW on your own. Please do not e-mail me about your findings — just mention them on the first page of your solution and count extra points.

Now let us look at your problems.

1. Problem 8.1 First. let us recall that if $X_1, \ldots, X_n$ are independent, $Y_1 = \sum_{i=1}^n a_i X_i$ and $Y_2 = \sum_{i=1}^n b_i X_i$ then

$$Cov(Y_1, Y_2) = E\{(Y_1 - EY_1)(Y_2 - EY_2)\} = E\{(\sum_{i=1}^n a_i(X_i - EX_i)(\sum_{i=1}^n b_i(X_i - EX_i))\}$$

$$= \sum_{i=1}^n a_i b_i E(X_i - EX_i)^2 = \sum_{i=1}^n a_i b_i Var(X_i).$$

In the next to last equality we used $E(X_i - E(X_i))(X_j - E(X_j)) = 0$ whenever $i \neq j$ which holds due to independence of $X_1, \ldots, X_n$. This verifies Corollary 4.15.

Now we return to the problem at hand. Let us recall that a sample from an infinite population is a sequence of iid $X_1, X_2, \ldots$ Then

$$Cov(X_r - \bar{X}, \bar{X}) = Cov[(1 - n^{-1})X_r - n^{-1} \sum_{i \neq r} X_i, n^{-1} \sum_{i=1}^n X_i]$$

$$= [(1 - n^{-1})n^{-1} - (n-1)n^{-2}]Var(X) = 0.$$

Remark: You may check that if $X \sim \text{Normal}(\theta, \sigma^2)$ then $X_r - \bar{X}$ and $\bar{X}$ are independent. Recall that for normal RVs it is sufficient to check that they are uncorrelated.

2. Problem 8.2. (a) For the mean we always have $E(X + Y) = E(X) + E(Y)$. Then

$$E(\bar{X}_1 - \bar{X}_2) = E(\bar{X}_1) - E(\bar{X}_2) = \mu_1 - \mu_2.$$

(b) For computing the variance it is very important to look at dependence because in general

$$Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y).$$

As a result, in general variance is not additive (only if RVs are uncorrelated, and in particular independent).

If X and Y are independent then $Cov(X, Y) = 0$. Using the problem's assumption we get

$$Var(\bar{X}_1 - \bar{X}_2) = Var(\bar{X}_1) + Var(\bar{X}_2) = \sigma_1^2/n_1 + \sigma_2^2/n_2.$$

Please note that formulae (a) and (b) are absolutely classical; it is a good idea to memorize them.

3. Problem 8.7. We need to check: (a) Independence of X_i's — but here it is given; (b) Uniform boundness in probability — it is plain here because $|X_i| \le 1$ with probability 1; (c) For the variance we have

$$Var(X_i) = E(X_i^2) - E^2(X_i) = (1/2)[(1-(1/2)^i)^2 + ((1/2)^i - 1)^2] - 0$$
$$= (1-(1/2)^i)^2.$$

From this, using independence of X_i's we get

$$Var(Y_n) = Var(X_1 + \ldots + X_n) = \sum_{i=1}^{n} Var(X_i) = \sum_{i=1}^{n}(1-(1/2)^i)^2 \to \infty.$$

Thus the sufficient condition holds.

4. Problem 8.13 (a,b). Suppose that a random sample of size n is chosen from $\{1, 2, \ldots, N\}$, and $n < N$. Write:

(a)

$$E(\bar{X}) = n^{-1}\sum_{i=1}^{n} EX_i = n^{-1}(n\mu) = \mu.$$

Here $\mu = N^{-1}\sum_{i=1}^{N} i = N^{-1}(1+N)N/2 = (N+1)/2$. Here I used definition 8.4 and Theorem 8.6.

(b) Using Theorem 8.6 and Definition 8.4

$$Var(\bar{X}) = \frac{\sigma^2}{n}\frac{N-n}{N-1},$$

where

$$\sigma^2 = \sum_{i=1}^{N}(i-\mu)^2 N^{-1} = N^{-1}\sum_{i=1}^{N} i^2 - \mu^2.$$

Now using formula on page 554 we get,

$$\sum_{i=1}^{N} i^2 = (1/6)N(N+1)(2N+1),$$

and this yields

$$\sigma^2 = (1/6)(N+1)(2N+1) - [(N+1)/2]^2 = [(N+1)/2][(2N+1)/3 - (N+1)/2] = (N+1)(N-1)/12.$$

We conclude that

$$Var(\bar{X}) = \frac{(N+1)(N-1)(N-n)}{12n(N-1)} = \frac{(N+1)(N-n)}{12n}.$$

5. Problem 8.18. Let us prove Theorem 8.9. Here we use the moment generating function approach, namely, we know that χ^2_ν is a chi-squared RV with ν degrees of freedom iff (if anf only if) for all sufficiently small t

$$M_{\chi^2_\nu}(t) = (1-2t)^{-\nu/2}.$$

Let $Y = \sum_{i=1}^n X_i$ where X_i are independent chi-squared RVs with ν_i degrees of freedom. Then due to independence,

$$M_Y(t) = \prod_{i=1}^n M_{X_i}(t) = \prod_{i=1}^n (1-2t)^{-\nu_i/2} = (1-2t)^{-\nu/2}.$$

where $\nu = \sum_{i=1}^n \nu_i$. Theorem 8.9 is proved.

Another way to establish this result is to note that $X_i = \sum_{j=1}^{\nu_i} \xi_j^2$ where ξ_j are iid standard normal. Then we get the proof by a summation of all squared standard normal RVs.

6. Problem 8.19. Let us prove Theorem 8.10. Suppose that X_1 and X_2 are independent, $X_1 \sim ChiSq(\nu_1)$ and $X_1 + X_2 \sim ChiSq(\nu_1 + \nu_2)$. Under this assumption, let us show that $X_2 \sim ChiSq(\nu_2)$. Using independence of X_1 and X_2 we get the equality

$$M_{X_1+X_2}(t) = M_{X_1}(t) M_{X_2}(t)$$

which yields

$$M_{X_2}(t) = M_{X_1+X_2}(t)/M_{X_1}(t) = (1-2t)^{-(\nu_1+\nu_2)/2}/(1-2t)^{-\nu_1/2} = (1-2t)^{-\nu_2/2}.$$

This proves that $X_2 \sim ChiSq(\nu_2)$.

7. Let us prove Theorem 8.11 (this is a very useful one for many statistical issues). I'll show you probably the simplest direct method of proving part (1) [another proof, very simple one, is based on sufficiency/compliteness that will be discussed shortly in this course)].

(1). Write

$$S^2 = (n-1)^{-1} \sum_{i=1}^n (X_i - \bar{X})^2 = (n-1)^{-1}[(X_1 - \bar{X})^2 + \sum_{i=2}^n (X_i - \bar{X})^2]$$

$$= (n-1)^{-1}\{[\sum_{i=2}^n (X_i - \bar{X})]^2 + \sum_{i=2}^n (X_i - \bar{X})^2\}.$$

In the last equality I used $\sum_{i=1}^n (X_i - \bar{X}) = 0$. Please note that S^2 can be written as a function of only $(X_2 - \bar{X}, \dots, X_n - \bar{X})$.

Now, for the considered Theorem I can always assume that for the considered X mean is zero and variance is 1 (otherwise just apply z-scoring): the latter simplifies notation. Then the joint pdf of X_i is

$$f^{X_1,\dots,X_n}(x_1, \dots, x_n) = (2\pi)^{-n/2} e^{-\sum_{i=1}^n x_i^2/2}.$$

Now I am in a position to explain the main trick: make a one-to-one transformation $y_1 = \bar{x}, y_2 = x_2 - \bar{x}, \dots, y_n = x_n - \bar{x}$. Then we need to show that y_1 and $(y_2, \dots, y_n)$ are independent. To do this we calculate the joint pdf of $y_1, \dots, y_n$ and show that it factors.

Well, let us recall how to calculate the wished joint pdf. The transformation is one-to-one (actually it is linear) and the inverse one is $x_1 = y_1 - (y_2 + \dots + y_n), x_2 = y_2 + y_1, \dots, x_n = y_n + y_1$. Please accurately write down its Jacobian and see that it is equal to n. Then recalling the technique of Section 7.4 we get

$$f^{Y_1,\dots,Y_n}(y_1, \dots, y_n) = n(2\pi)^{-n/2} e^{-(y_1 - \sum_{i=2}^n y_i)^2/2} e^{-\sum_{i=2}^n (y_i + y_1)^2/2}$$

$$= \left[(n/2\pi)^{1/2} e^{-ny_1^2/2}\right] \left[n^{1/2}(2\pi)^{-(n-1)/2} e^{-[\sum_{i=2}^n y_i^2 + (\sum_{i=2}^n y_i)^2]}\right].$$

As we see, the joint pdf factors as we need, and this yields that Y_1 is independent of $(Y_2, \dots, Y_n)$. This proves part (1).

Part (2) is proved in the text – please read it and then repeat with closed book.

8. Problem 8.31. First of all, let us think about the following issue: why do the authors suggest a transformation approach (looking at distribution) and not a mgf approach that was used in other problems? The answer is that t-distribution has no mgf and no moments of larger order. Just by analyzing the pdf you can figure out that a t-distribution with ν degrees of freedom has only $\nu - 1$ moments. In particular, with 1 degree of freedom a t-distribution is the famous Cauchy distribution which has no moments at all, and when $\nu \to \infty$ the t-distribution approaches a normal one. As a result, as a family, t-distribution is extremely rich one. We conclude that variance exists only when $\nu \geq 3$.

Recall that t-distribution density is

$$f_\nu^T(t) = \frac{\Gamma((\nu+1)/2)}{\pi^{1/2}\nu^{1/2}\Gamma(\nu/2)}(1 + t^2/\nu)^{-(\nu+1)/2}. \tag{1}$$

Now we are making the recommended change of variable $1 + t^2/\nu = 1/u$. Then $t = (-1)^j \nu^{1/2}(u^{-1} - 1)^{1/2}$, $0 < u \leq 1$, $j = 0, 1$. Correspondingly, $|dt/du| = \nu^{1/2}(1/2)(1 - u)^{-1/2}u^{-3/2}$. Then

$$E(T^2) = \int_{-\infty}^{\infty} t^2 f_\nu^T(t)dt = 2\int_0^1 \frac{\Gamma((\nu+1)/2)}{\pi^{1/2}\nu^{1/2}\Gamma(\nu/2)} u^{(\nu+1)/2}\nu^{1/2}(1/2)u^{-3/2}(1-u)^{-1/2}\nu(u^{-1} - 1)du$$

$$= \frac{\nu\Gamma((\nu+1)/2)}{\pi^{1/2}\Gamma(\nu/2)} \int_0^1 u^{\nu/2-2}(1-u)^{1/2}du$$

$$= \frac{\nu\Gamma((\nu+1)/2)\Gamma(\nu/2-1)\Gamma(3/2)}{\pi^{1/2}\Gamma(\nu/2)\Gamma(\nu/2+1/2)} \int_0^1 \frac{\Gamma(\nu/2+1/2)u^{(\nu/2-1)-1}(1-u)^{3/2-1}}{\Gamma(\nu/2-1)\Gamma(3/2)}du.$$

The integral is equal to 1 because the integrable function is the pdf of a $Beta(\nu/2 - 1, 3/2)$ distribution. We can also use familiar $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ and $\Gamma(1/2) = \pi^{1/2}$. This yields that

$$E(T^2) = \frac{\nu\Gamma(\nu/2-1)(1/2)\Gamma(1/2)}{\pi^{1/2}(\nu/2-1)\Gamma(\nu/2-1)} = \nu/(\nu-2).$$

Finally, because t-distribution is symmetric about zero, its mean is zero, that is $ET = 0$, and thus

$$Var(T) = ET^2 - (ET)^2 = \nu/(\nu - 2).$$

9. Problem 8.34. Set $\nu = 1$ in (1), and using $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$, $\Gamma(1/2) = \pi^{1/2}$, $\Gamma(1) = 1$ we get that

$$f_1^T(t) = \frac{\Gamma(1)}{\pi^{1/2}\Gamma(1/2)}(1 + t^2)^{-1} = \frac{1}{\pi(1 + t^2)}.$$

This is the Cauchy pdf.

10. Problem 8.38. Suppose that T has Student distribution (another notation of t-distribution) with ν degrees of freedom. Then as we know

$$T = \frac{\xi_0}{[\sum_{i=1}^{\nu} \xi_i^2/\nu]^{1/2}}$$

where $\xi_0, \xi_1, \ldots, \xi_\nu$ are independent standard normal RVs. Then plainly

$$T^2 = \frac{\xi_0^2/1}{\sum_{i=1}^{\nu} \xi_i^2/\nu},$$

and, according to Theorem 8.14, $X = T^2$ has $F(1, \nu)$ distribution.

Of course, another way to establish this is via considering a pdf of T^2 via using our transformation technique.

11. Problem 8.41. This is a very useful property: if $Y \sim Beta(\alpha, \beta)$ with $\alpha = \nu_1/2$, $\beta = \nu_2/2$ then

$$X = \frac{\nu_2 Y}{\nu_1(1 - Y)}$$

has $F(\nu_1, \nu_2)$ distribution.

Note that the transformation is one-to-one with

$$Y = \frac{X}{\nu_2/\nu_1 + X},$$

so

$$|dy/dx| = \frac{(a + x) - x}{(a + x)^2} = \frac{a}{(a + x)^2}, \quad a := \nu_2/\nu_1.$$

Then

$$f^X(x) = f^Y(x/(x + a))\frac{a}{(a + x)^2}I(x > 0)$$

$$= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)}\Big(\frac{x}{a + x}\Big)^{\alpha - 1}\Big(\frac{a}{a + x}\Big)^{\beta - 1}\frac{a}{(a + x)^2}I(x > 0)$$

$$= \frac{\Gamma(\alpha + \beta)a^\beta}{\Gamma(\alpha)\Gamma(\beta)}\frac{x^{\alpha - 1}}{(a + x)^{\alpha + \beta}}I(x > 0)$$

$$= \frac{\Gamma((\nu_1 + \nu_2)/2)}{\Gamma(\nu_1/2)\Gamma(\nu_2/2)}(\nu_1/\nu_2)^{\nu_1/2}x^{\nu_1/2 - 1}(1 + (\nu_1/\nu_2)x)^{-(\nu_1 + \nu_2)/2}I(x > 0).$$

This is the pdf of $Gamma(\nu_1, \nu_2)$ distribution (see page 281).