We give the simplest example in each case.

a) the empty set on \{a\} (vacuously symmetric and antisymmetric)

b) \{(a, b), (b, a), (a, c)\} on \{a, b, c\}

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a) The union of two relations is the union of these sets. Thus \( R_1 \cup R_3 \) holds between two real numbers if \( R_1 \) holds or \( R_3 \) holds (or both, it goes without saying). Here this means that the first number is greater than the second or vice versa—in other words, that the two numbers are not equal. This is just relation \( R_6 \).

b) For \((a, b)\) to be in \( R_3 \cup R_6 \), we must have \( a > b \) or \( a = b \). Since this happens precisely when \( a \geq b \), we see that the answer is \( R_3 \).

c) The intersection of two relations is the intersection of these sets. Thus \( R_2 \cap R_4 \) holds between two real numbers if \( R_2 \) holds and \( R_4 \) holds as well. Thus for \((a, b)\) to be in \( R_2 \cap R_4 \), we must have \( a \geq b \) and \( a \leq b \). Since this happens precisely when \( a = b \), we see that the answer is \( R_3 \).

d) For \((a, b)\) to be in \( R_3 \cap R_3 \), we must have \( a < b \) and \( a = b \). It is impossible for \( a < b \) and \( a = b \) to hold at the same time, so the answer is \( \emptyset \), i.e., the relation that never holds.

e) Recall that \( R_1 - R_2 = R_1 \setminus R_2 \). But \( R_2 = R_3 \), so we are asked for \( R_1 \cap R_3 \). It is impossible for \( a > b \) and \( a < b \) to hold at the same time, so the answer is \( \emptyset \), i.e., the relation that never holds.

f) Reasoning as in part (f), we want \( R_2 \cap R_3 = R_2 \cap R_4 \), which is \( R_6 \) (this was part (c)).

g) Recall that \( R_1 \oplus R_3 = (R_1 \cap R_3) \cup (R_3 \cap R_1) \). We see that \( R_1 \cap R_3 \) holds if \( R_1 \cap R_3 = R_1 \), and \( R_3 \cap R_1 = R_3 \cap R_4 = R_3 \). Thus our answer is \( R_1 \cup R_3 = R_6 \) (as in part (a)).

h) Recall that \( R_3 \oplus R_4 = (R_3 \cap R_4) \cup (R_4 \cap R_3) \). We see that \( R_3 \cap R_4 = R_2 \cap R_1 = R_1 \), and \( R_4 \cap R_3 = R_4 \cap R_3 = R_3 \). Thus our answer is \( R_1 \cup R_3 = R_6 \) (as in part (a)).

2. In each case we use a \( 4 \times 4 \) matrix, putting a 1 in position \((i, j)\) if the pair \((i, j)\) is in the relation and a 0 in position \((i, j)\) if the pair \((i, j)\) is not in the relation.

\[
\begin{array}{cccc}
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

4. a) Since the \((1, 1)^{th}\) entry is a 1, \((1, 1)\) is in the relation. Since \((1, 3)^{th}\) entry is a 0, \((1, 3)\) is not in the relation. Continuing in this manner, we see that the relation contains \((1, 1), (1, 2), (1, 4), (1, 2), (2, 1), (2, 3), (3, 2), (3, 3), (3, 4), (4, 1), (4, 3), \) and \((4, 4)\).

b) \((1, 1), (1, 2), (1, 3), (2, 2), (3, 3), (3, 4), (4, 1),\) and \((1, 4)\)

c) \((1, 2), (1, 4), (2, 1), (2, 3), (3, 2), (3, 4), (4, 1),\) and \((4, 3)\)
25. Algorithm 1 finds the transitive closure by computing the successive powers and taking their join. We exhibit our answers in matrix form as $M_R \lor M_R^2 \lor \ldots \lor M_R^n = M_R^n$.

\[\begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix} \lor 
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix} \lor 
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0
\end{bmatrix} \lor 
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix}
\]

12. This follows from Exercise 9, where $f$ is the function that takes a bit string of length $n \geq 3$ to its last $n - 3$ bits.

36. In each case, the equivalence class of 4 is the set of all integers congruent to 4, modulo $n$.

\[\begin{align*}
a) \{4 + 2n \mid n \in \mathbb{Z}\} &= \{\ldots, -2, 0, 2, 4, \ldots\} \\
b) \{4 + 3n \mid n \in \mathbb{Z}\} &= \{\ldots, -2, 1, 4, 7, \ldots\} \\
c) \{4 + 6n \mid n \in \mathbb{Z}\} &= \{\ldots, -2, 4, 10, 16, \ldots\} \\
d) \{4 + 8n \mid n \in \mathbb{Z}\} &= \{\ldots, -4, 4, 12, 20, \ldots\}
\end{align*}\]

6. The question in each case is whether the relation is reflexive, antisymmetric, and transitive.

\[\quad\]

a) The equality relation on any set satisfies all three conditions and is therefore a partial order. (It is the smallest partial order; reflexivity insures that every partial order contains at least all the pairs $(a, a)$.)

b) This is not a poset, since the relation is not reflexive, although it is antisymmetric and transitive. Any relation of this sort can be turned into a partial ordering by adding in all the pairs $(a, a)$.

c) This is a poset, very similar to Example 1.

d) This is not a poset, since the relation is not reflexive, not antisymmetric, and not transitive (the absence of one of these properties would have been enough to give a negative answer).

14. a) These are comparable, since $5 \mid 15$.

b) These are not comparable since neither divides the other.

c) These are comparable, since $8 \mid 16$.

d) These are comparable, since $7 \mid 7$. 

22. In each case we put a above b and draw a line between them if b|a but there is no element c other than a and b such that b|c and c|a.
   a) Note that 1 divides all numbers, so the numbers on the second level from the bottom are the primes.

   ![Diagram](image1)

   b) In this case these numbers are pairwise relatively prime, so there are no lines in the Hasse diagram.

   3 5 7 11 13 16 17

34. The reader should draw the Hasse diagram to aid in answering these questions.
   a) Clearly the numbers 27, 48, 60, and 72 are maximal, since each divides no number in the list other than itself. All of the other numbers divide 72, however, so they are not maximal.
   b) Only 2 and 9 are minimal. Every other element is divisible by either 2 or 9.
   c) There is no greatest element, since, for example, there is no number in the set that both 60 and 72 divide.
   d) There is no least element, since there is no number in the set that divides both 2 and 9.
   e) We need to find numbers in the list that are multiples of both 2 and 9. Clearly 18, 36, and 72 are the numbers we are looking for.
   f) Of the numbers we found in the previous part, 18 satisfies the definition of the least upper bound, since it divides the other two upper bounds.
   g) We need to find numbers in the list that are divisors of both 60 and 72. Clearly 2, 4, 6, and 12 are the numbers we are looking for.
   h) Of the numbers we found in the previous part, 12 satisfies the definition of the greatest lower bound, since the other three lower bounds divide it.

44. In each case, we need to decide whether every pair of elements has a least upper bound and a greatest lower bound.
   a) This is not a lattice, since the elements 6 and 9 have no upper bound (no element in our set is a multiple of both of them).
   b) This is a lattice; in fact it is a linear order, since each element in the list divides the next one. The least upper bound of two numbers in the list is the larger, and the greatest lower bound is the smaller.
   c) Again, this is a lattice because it is a linear order. The least upper bound of two numbers in the list is the smaller number (since here “greater” really means “less”!), and the greatest lower bound is the larger of the two numbers.
   d) This is similar to Example 24, with the roles of subset and superset reversed. Here the g.l.b. of two subsets A and B is \( A \cup B \), and their l.u.b. is \( A \cap B \).