CHAPTER 9

MAINTENANCE AND REPLACEMENT

The problem of determining the lifetime of an asset or an activity simultaneously with its management during that lifetime is an important problem in practice. The most typical example is the problem of optimal maintenance and replacement of a machine; see Rapp (1974) and Pierskalla and Voelker (1976). Other examples occur in forest management such as in Näslund (1969), Clark (1976), and Heaps (1984), and in advertising copy management as in Pekelman and Sethi (1978).

T = the sale date of the machine to be determined,

ρ = the constant discount rate,

$x(t)$ = the resale value of machine in dollars at time t; let $x(0) = x_0$,

$u(t)$ = the preventive maintenance rate at time t (maintenance here means money spend over and above the minimum required for necessary repairs),

$g(t)$ = the maintenance effectiveness function at time t (measured in dollars added to the resale value per dollar spent on preventive maintenance),

$d(t)$ = the obsolescence function at time t (measured in terms of dollars subtracted from x at time t),

π = the constant production rate in dollars per unit time per unit resale value; assume $\pi > \rho$ or else it does not pay to produce.

It is assumed that $g(t)$ is a nonincreasing function of time and $d(t)$ is a nondecreasing function of time, and that for all t

$$u(t) \in \Omega = [0, U], \tag{1}$$

where U is a positive constant.

The present value of the machine is the sum of two terms, the discounted income (production minus maintenance) stream during its life plus the discounted resale value at T:

$$J = \int_0^T [\pi x(t) - u(t)]e^{-\rho t}dt + x(T)e^{-\rho T}. \tag{2}$$

The state variable x is affected by the obsolescence factor, the amount of preventive maintenance, and the maintenance effectiveness function. Thus,

$$\dot{x}(t) = -d(t) + g(t)u(t), \;\; x(0) = x_0. \tag{3}$$

In the interests of realism we assume that

$$-d(t) + g(t)U \leq 0, \ t \geq 0. \tag{4}$$

The assumption implies that preventive maintenance is not so effective as to enhance the resale value of the machine over its previous values; rather it can at most slow down the decline of the resale value, even when preventive maintenance is performed at the maximum rate U.

The standard Hamiltonian as formulated in Section 2.2 is

$$H = (\pi x - u)e^{-\rho t} + \lambda(-d + gu), \tag{5}$$

where the adjoint variable λ satisfies

$$\dot{\lambda} = -\pi e^{-\rho t}, \quad \lambda(T) = e^{-\rho T}. \tag{6}$$

Since T is unspecified, the required additional terminal condition (3.14) for this problem is

$$-\rho e^{-\rho T} x(T) = -H, \tag{7}$$

which must hold on the optimal path at time T.

The adjoint variable λ can be easily obtained by integrating (6), i.e.,

$$\lambda(t) = e^{-\rho T} + \frac{\pi}{\rho}[e^{-\rho t} - e^{-\rho T}]. \tag{8}$$

The interpretation of $\lambda(t)$ is as follows. It gives in present value terms, the marginal profit per dollar of gain in resale value at time t. The first term represents the present value of one dollar of additional salvage value at T brought about by one dollar of additional resale value at the current time t. The second term represents the present value of incremental production from t to T brought about by the extra productivity of the machine due to the additional one dollar of resale value at time t.

The optimal control for a problem with any fixed T is bang-bang as in Model Type (a) in Table 3.3. Thus,

$$u^*(t) = \text{bang}\left[0, U; e^{-\rho T} g(t) + \frac{\pi}{\rho}(e^{-\rho t} - e^{-\rho T}) g(t) - e^{-\rho t}\right]. \tag{9}$$

Interpretation: The term $e^{-\rho T} g(t) + \frac{\pi}{\rho}(e^{-\rho t} - e^{-\rho T}) g(t)$ is the present value of the marginal return from increasing the preventive maintenance by one dollar at time t. The last term $e^{-\rho t}$ in the argument of the bang function is the present value of that one dollar spent for preventive maintenance at time t. If the former is larger than the latter, then perform the maximum possible preventive maintenance, otherwise do not carry out any at all.

To find how the optimal control switches, we rewrite the switching function in (9) as

$$e^{-\rho t}\left[\frac{\pi g}{\rho} - (\frac{\pi}{\rho} - 1)e^{\rho(t-T)}g - 1\right]. \tag{10}$$

- We can show that the expression inside the square brackets in (10) is monotonically decreasing with time t on account of the assumptions that $\pi/\rho > 1$ and that g is nonincreasing with t. It follows that there will not be any singular control for any finite interval of time.
- Since $e^{-\rho t} > 0$ for all t, the switching function can only go from positive to negative and not vice versa.

The switching time t^s is obtained as follows: equate (9.10) to zero and solve for t. If the solution is negative, let $t^s = 0$, and if the solution is greater than T, let $t^s = T$, otherwise set t^s equal to the solution.

It is clear that the optimal control in (9) can now be rewritten as

$$u^*(t) = \begin{cases} U & t \le t^s, \\ 0 & t > t^s. \end{cases} \tag{11}$$

Note that all the above calculations were made on the assumption that T was fixed, i.e., without imposing condition (7). On an optimal path, this condition with use of (5) and (8) can be restated as

$$\begin{aligned} & -\rho e^{-\rho T^*} x^*(T^*) \\ & = -\{\pi x^*(T^*) - u^*(T^*)\} e^{-\rho T^*} \\ & \quad -e^{-\rho T^*} \{-d(T^*) + g(T^*) u(T^*)\}. \end{aligned} \tag{12}$$

This means that when $u^*(T^*) = 0$ (i.e., $t^s < T^*$), we have

$$x^*(T^*) = \frac{d(T^*)}{\pi - \rho}, \tag{13}$$

and when $u^*(T^*) = U$ (i.e., $t^s = T^*$), we have

$$x^*(T^*) = \frac{d(T^*) - [g(T^*) - 1]U}{\pi - \rho}. \tag{14}$$

Since $d(t)$ is nondecreasing, $g(t)$ is nonincreasing, and $x(t)$ is nonincreasing, equation (13) or equation (14), whichever the case may be, has a solution for T^*.

Suppose $U = 1,\ x(0) = 100,\ d(t) = 2,\ \pi = 0.1,\ \rho = 0.05$, and $g(t) = 2/(1+t)^{1/2}$. Let the unit of time be one month.

First, we write the condition on t^s by equating (10) to 0, which gives

$$\pi - (\pi - \rho)e^{-\rho(T-t^s)} = \frac{\rho}{g}. \tag{15}$$

In doing so, we have assumed that the solution of (15) lies in the open interval $(0, T)$. As we shall indicate later, special care needs to be exercised if this is not the case.

Substituting the data in (15) we have

$$0.1 - 0.05e^{-0.05(T-t^s)} = 0.025(1+t^s)^{1/2},$$

which simplifies to

$$(1+t^s)^{1/2} = 4 - 2e^{-0.05(T-t^s)}. \tag{16}$$

Then integrating (3), we find

$$x(t) = -2t + 4(1+t)^{1/2} + 96, \;\; \text{if } t \le t^s,$$

and hence

$$\begin{aligned} x(t) &= -2t^s + 4(1+t^s)^{1/2} + 96 - 2(t - t^s) \\ &= 4(1+t^s)^{1/2} + 96 - 2t, \;\; \text{if } t > t^s. \end{aligned}$$

Since we have assumed $0 < t^s < T$, we substitute $x(T)$ into (13) and obtain

$$4(1+t^s)^{1/2} + 96 - 2T = 2/0.05 = 40,$$

which simplifies to

$$T = 2(1+t^s)^{1/2} + 28. \tag{17}$$

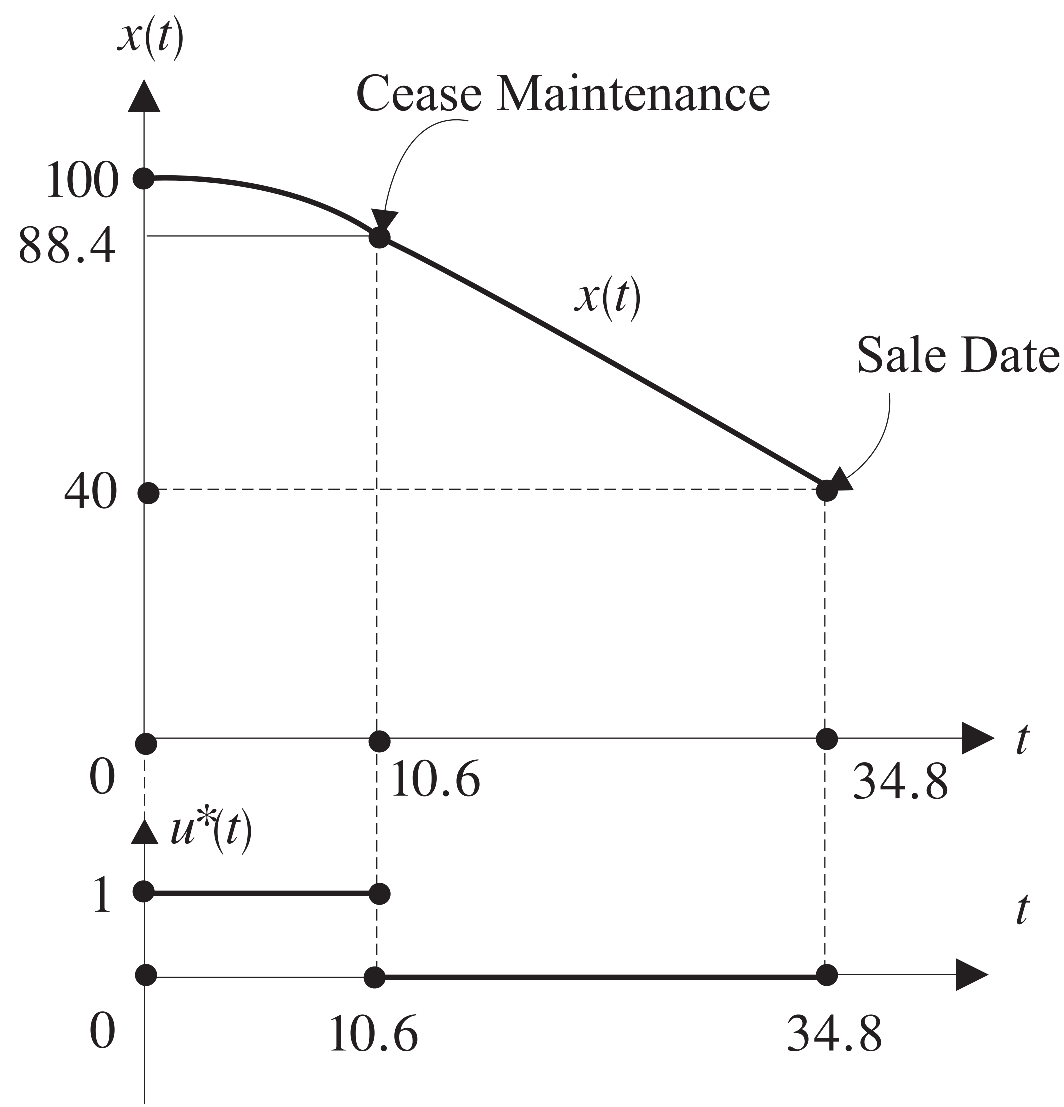
We must solve (16) and (17) simultaneously. Substituting (17) into (16), we find that t^s must be a zero of the function

$$h(t^s) = (1+t^s)^{1/2} - 4 + 2e^{-[2(1+t^s)^{1/2} - t^s + 28]/20}. \tag{18}$$

A simple binary search program was written to solve this equation, which obtained the value $t^s = 10.6$. Substitution of this into (17) yields $T = 34.8$.

Since this satisfies our supposition that $0 < t^s < T$, we can conclude our computations. Thus, the optimal solution is to perform preventive maintenance at the maximum rate during the first 10.6 months, and thereafter not at all. The sale date is at 34.8 months after purchase. Figure 9.1 gives the functions $x(t)$ and $u(t)$ for this optimal maintenance and sale date policy.

Figure 9.1: Optimal Maintenance and Machine Resale Value



If, on the other hand, the solution of (16) and (17) did not satisfy our supposition, we would need to follow the procedure outlined earlier in the section. This would result in $t^s = 0$ or $t^s = T$. If $t^s = 0$, we would obtain T from (17), and conclude $u^*(t) = 0,\ 0 \le t \le T$. Alternatively, if $t^s = T$, we would need to substitute $x(T)$ into (14) to obtain T. In this case the optimal control would be $u^*(t) = U,\ 0 \le t \le T$.

The pure bang-bang result in the model developed above is a result of the linearity in the problem. The result can be enriched as in Sethi (1973b) by generalizing the resale value equation (3) as follows:

$$\dot{x}(t) = -d(t) + g(u(t), t), \tag{19}$$

where g is nondecreasing and concave in u. For this section, we will assume the sale date T to be fixed for simplicity and g to be strictly concave in u, i.e., $g_u \geq 0$ and $g_{uu} < 0$ for all t. Also, $g_t \leq 0, \; g_{ut} \leq 0$, and $g(0, t) = 0$; see Exercise 9.5 for an example of the function $g(u, t)$.

The standard Hamiltonian is

$$H = (\pi x - u)e^{-\rho t} + \lambda[-d + g(u, t)], \tag{20}$$

where λ is given in (8). To maximize the Hamiltonian, we differentiate it with respect to u and equate the result to zero. Thus,

$$H_u = -e^{-\rho t} + \lambda g_u = 0. \tag{21}$$

If we let $u^0(t)$ denote the solution of (21), then $u^0(t)$ maximizes the Hamiltonian (20) because of the concavity of g in u. Thus, for a fixed T, the optimal control is

$$u^*(t) = \text{sat}[0, U; u^0(t)]. \tag{22}$$

To determine the direction of change in $u^*(t)$, we obtain $\dot{u}^0(t)$. For this, we use (21) and the value $\lambda(t)$ from (8) to obtain

$$g_u = \frac{e^{-\rho t}}{\lambda(t)} = \frac{1}{\frac{\pi}{\rho} - (\frac{\pi}{\rho} - 1)e^{\rho(t-T)}}. \qquad (23)$$

Since $\pi > \rho$, the denominator on the right-hand side of (23) is monotonically decreasing with time. Therefore, the right-hand side of 23) is increasing with time. Taking the time derivative of (23), we have

$$g_{ut} + g_{uu}\dot{u}^0 = \rho^2(\pi - \rho)e^{\rho(t-T)}/[\pi - (\pi - \rho)e^{\rho(t-T)}]^2 > 0.$$

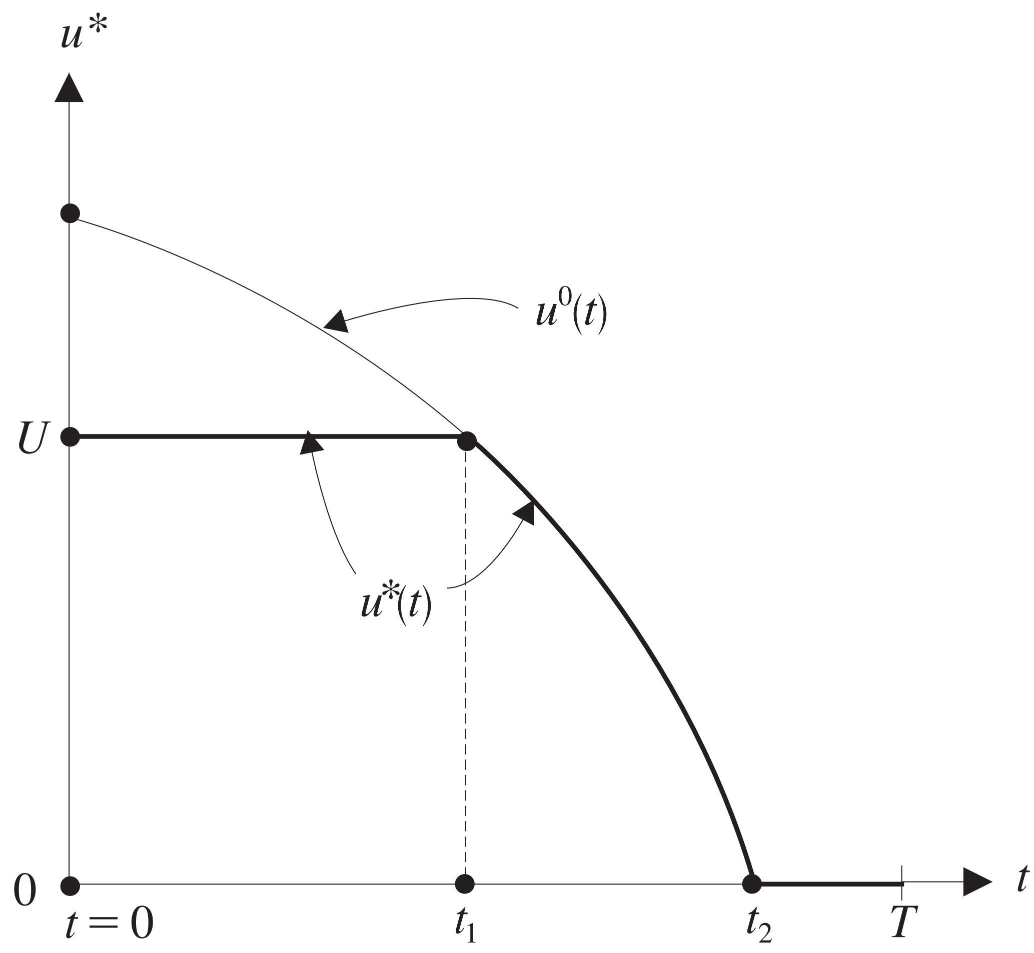
But $g_{ut} \leq 0$ and $g_{uu} < 0$, it is therefore obvious that $\dot{u}^0(t) < 0$.

In order to sketch the optimal control $u^*(t)$ specified in (22), let us define $0 \leq t_1 \leq t_2 \leq T$ such that $u^0(t) \geq U$ for $t \leq t_1$ and $u^0(t) \leq 0$ for $t \geq t_2$. Then, we can rewrite the sat function in (22) as

$$u^*(t) = \begin{cases} U & \text{for } t \in [0, t_1], \\ u^0(t) & \text{for } t \in (t_1, t_2), \\ 0 & \text{for } t \in [t_2, T]. \end{cases} \tag{24}$$

In (24) it is possible to have $t_1 = 0$ and/or $t_2 = T$. In Figure 9.2 we have sketched a case when $t_1 > 0$ and $t_2 < T$.

FIGURE 9.2: SAT FUNCTION OPTIMAL CONTROL



Note that while $u^0(t)$ in Figure 9.2 is decreasing over time, the way it will decrease will depend on the nature of the function g. Indeed, the shape of $u^0(t)$, while always decreasing, can be quite general. In particular, you will see in Exercise 9.5 that the shape of $u^0(t)$ is concave and, furthermore, $u^0(t) > 0,\ t \geq 0$, so that $t_2 = T$ in that case.

9.2.1 The Model

$$
\begin{aligned}
T &= \text{the sale date of a machine to be determined,} \\
u(t) &= \text{the preventive maintenance rate at time } t;\ 0 \le u(t) \le 1, \\
R &= \text{the constant positive rate of revenue produced by a functioning machine independent of its age at any time, net of all costs except preventive maintenance,} \\
\rho &= \text{the constant discount rate,} \\
L &= \text{the constant positive junk value of the failed machine independent of its age at failure,}
\end{aligned}
$$

$B(t)$ = the (exogenously specified) resale value of the machine at time t, if it is still functioning; $\dot{B}(t) \leq 0$,

$h(t)$ = the natural failure rate (also termed the natural hazard rate in the reliability theory); $h(t) \geq 0,\ \dot{h}(t) \geq 0$,

$F(t)$ = the cumulative probability that the machine has failed by time t,

$C(u, h)$ = the cost function depending on the preventive maintenance u when the natural failure rate is h.

To make economic sense, an operable machine must be worth at least as much as an inoperable machine and its resale value should not exceed the present value of the potential revenue generated by the machine if it were to function forever.

Thus,

$$0 \leq L \leq B(t) \leq R/\rho, \quad t \geq 0. \tag{25}$$

Also for all $t > 0$,

$$u(t) \in \Omega = [0, 1]. \tag{26}$$

The cost of reducing the natural failure rate is assumed to be proportional to the natural failure rate. Specifically, we assume that $C(u,h) = C(u)h$ denotes the cost of preventive maintenance u when the natural failure rate is h. In other words, when the natural failure rate is h and a controlled failure rate of $h(1-u)$ is sought, the action of achieving this reduction will cost $C(u)h$ dollars. It is assumed that

$$C(0) = 0, \; C_u > 0, \; C_{uu} > 0, \; \text{for } u \in [0,1]. \tag{27}$$

These conditions imply that a given absolute reduction becomes increasingly more costly as the machine gets older.

We note that $\dot{F}/(1-F)$) denotes the conditional probability density for the failure of machine at time t given that it has survived to time t. This is assumed to depend on two things, namely (i) the natural failure rate that governs the machine in the absence of preventive maintenance, and (ii) the current rate of preventive maintenance. Thus,

$$\frac{\dot{F}(t)}{1-F(t)} = h(t)[1-u(t)], \tag{28}$$

which gives the state equation

$$\dot{F} = h(1-u)(1-F),\; F(0) = 0. \tag{29}$$

Thus, the controlled failure rate at time t is $h(t)(1-u(t))$.

The expected present value of the machine is

$$J = \int_0^T e^{-\rho t}\left\{[R - C(u)h](1-F) + L\dot{F}\right\}dt + e^{-\rho T}B(T)[1 - F(T)].$$

Using (29), we can rewrite J as follows:

$$J = \int_0^T e^{-\rho t}\left[R - C(u)h + L(1-u)h\right](1-F)dt + e^{-\rho T}B(T)\left[1 - F(T)\right]. \quad (30)$$

The optimal control problem is to maximize J in (30) subject to (26) and (29).

The problem is similar to Model Type (f) in Table 3.3 subject to the free-end-point condition as in Row 1 of Table 3.1. Therefore, we follow the steps for solution by the maximum principle stated in Chapter 3. The standard Hamiltonian is

$$H = e^{-\rho t}[R - C(u)h + L(1-u)h](1-F) + \lambda[(1-u)h(1-F)], \tag{31}$$

and the adjoint variable satisfies

$$\begin{cases} \dot{\lambda} & = & e^{-\rho t}[R - C(u)h + L(1-u)h] + \lambda(1-u), \\ \lambda(T) & = & -e^{-\rho T}B(T). \end{cases} \tag{32}$$

Since T is unspecified, we apply the additional terminal condition (3.14) to obtain (see Exercise 9.6)

$$\begin{aligned} R - C[u^*(T^*)]h(T^*) + L[1 - u^*(T^*)]h[T^*] \\ -[\rho + \{1 - u^*(T^*)\}h(T^*)]B(T^*) = -B_T(T^*). \end{aligned} \quad (33)$$

Consider keeping the machine to time $T^* + \delta$. The first two terms in (33) when multiplied by δ give the incremental net cash inflow. The third term denotes the junk value L multiplied by the probability $[1 - u^*(T^*)]h(T^*)\delta$ that the machine fails during the short time δ. The fourth term represents the sum of loss of interest $\rho B(T^*)\delta$ on the resale value and the loss of the entire resale value, when the machine fails, with probability $[1 - u^*(T^*)]h(T^*)\delta$.

Thus, the LHS of (33) represents the marginal benefit of keeping the machine to $T^* + \delta$. On the other hand, the RHS term $-B_T(T^*)\delta$ is the decrease in the resale value from T^* to $T^* + \delta$. So it represents the marginal cost of keeping the machine to $T^* + \delta$. Hence, equation (33) determining the optimal sale date is the usual economic condition equating marginal benefit to marginal cost.

Next we analyze the problem to obtain the optimal maintenance policy for a fixed T. If the optimal solution is in the interior, i.e., $u^* \in (0,1)$, then the Hamiltonian maximizing condition gives

$$H_u = -e^{-\rho t} h(1-F)[C_u + L + e^{\rho t}\lambda] = 0. \tag{34}$$

In the trivial cases in which the natural failure rate $h(t)$ is zero or when the machine fails with certainty by time t (i.e., $F(t) = 1$), then $u^*(t) = 0$.

Assume therefore $h > 0$ and $F < 1$. Under these conditions, we can infer from (27) and (34) that

$$\left.\begin{array}{ll} \text{(i)} & C_u(0) + L + \lambda e^{\rho t} > 0 \Rightarrow u^*(t) = 0, \\ \text{(ii)} & C_u(1) + L + \lambda e^{\rho t} < 0 \Rightarrow u^*(t) = 1, \\ \text{(iii)} & \text{Otherwise, } C_u + L + \lambda e^{\rho t} = 0 \text{ determines } u^*(t). \end{array}\right\} \quad (35)$$

Using the terminal condition $\lambda(T) = -e^{-\rho T} B(T)$ from (32), we can derive $u^*(T)$ satisfying (35):

$$\left.\begin{array}{ll} \text{(i)} & C_u(0) > B(T) - L \text{ and } u^*(T) = 0, \\ \text{(ii)} & C_u(1) < B(T) - L \text{ and } u^*(T) = 1, \\ \text{(iii)} & \text{Otherwise, } C_u = B(T) - L \text{ determines } u^*(T). \end{array}\right\} \quad (36)$$

The next we determine how $u^*(t)$ changes over time. Kamien and Schwartz (1971a, 1998) have shown that $u^*(t)$ is non-increasing; see Exercise 9.7. Thus there exists $T \geq t_2 \geq t_1 \geq 0$ such that

$$u^*(t) = \begin{cases} 1 & \text{for } t \in [0, t_1], \\ u^0(t) & \text{for } t \in (t_1, t_2), \\ 0 & \text{for } t \in (t_2, T]. \end{cases} \tag{37}$$

Here $u^0(t)$ is the solution of (35)(iii), and it is easy to show that $\dot{u}^0(t) \leq 0$. Of course, $u^*(T)$ is immediately known from (36). If $u^*(T) \in (0, 1)$, it implies $t_2 = T$; and if $u^*(T) = 1$, it implies $t_1 = t_2 = T$.

For this model, the sufficiency of the maximum principle follows from Theorem 2.1; see Exercise 9.8.

We now extend the problem of maintenance and replacement to a chain of machines. By this we mean that given the time periods $0, 1, 2, \ldots, T-1$, we begin with a machine purchase at the beginning of period zero. Then, we find an optimal number of machines, say ℓ, and optimal times $0 < t_1 < t_2, ..., t_{\ell-1} < t_\ell < T$ of their replacements such that the existing machine will be replaced by a new machine at time $t_j,\ j = 1, 2, \ldots, \ell$. At the end of the horizon defined by the beginning of period T, the last machine purchased will be salvaged. Moreover, the optimal maintenance policy for each of the machines in the chain must be found.

Two approaches to this problem have been developed in the literature.

- The first attempts to solve for an infinite horizon $(T = \infty)$ with a simplifying assumption of identical machine lives, i.e.,

$$t_j - t_{j-1} = t_{j+1} - t_j \text{ for all } j \geq 1. \tag{38}$$

- The second approach relaxes the assumption (38) of identical machine lives, but then, it can only solve a finite horizon problem involving a finite chain of machines, i.e., ℓ is finite.

In This section, we deal with the latter problem as analyzed by Sethi and Morton (1972).

Consider buying a machine at the beginning of period s and salvaging it at the beginning of period $t > s$. Let J_{st} denote the present value of all net earnings associated with the machine. To calculate J_{st}, we need the following notation for $s \leq k \leq t - 1$.

x_s^k = the resale value of the machine at the beginning of period k,

P_s^k = the production quantity (in dollar value) during period k,

E_s^k = the necessary expense of the ordinary maintenance (in dollars) during period k,

$$\begin{aligned}
R_s^k &= P_s^k - E_s^k, \\
u^k &= \text{the rate of preventive maintenance (in dollars) during period } k, \\
C_s &= \text{the cost of purchasing machine at the beginning of period } s, \\
\rho &= \text{the periodic discount rate.}
\end{aligned}$$

It is required that

$$0 \le u^k \le U^{sk}, \; k \in [s, t-1]. \tag{39}$$

We can calculate J_{st} in terms of the variables and functions defined above:

$$J_{st} = \sum_{k=s}^{t-1} R_s^k (1+\rho)^{-k} - \sum_{k=s}^{t-1} u^k (1+\rho)^{-k} - C_s (1+\rho)^{-s} + x_s^t (1+\rho)^{-t}. \tag{40}$$

We must also have functions that will provide us with the ways in which states change due to the age of the machine and the amount of preventive maintenance.

Also, assuming that at time s, the only machines available are those that are up-to-date with respect to the technology prevailing at s, we can subscript these functions by s to reflect the effect of the machine's technology on its state at a later time k. Let $\Psi_s(u^k, k)$ and $\Phi_s(u^k, k)$ be such concave functions so that we can write the following state equations:

$$\Delta R_s^k = R_s^{k+1} - R_s^k = \Psi_s(u^k, k), \quad R_s^s \text{ given}, \tag{41}$$

$$\Delta x_s^k = \Phi_s(u^k, k), \; x_s^s = (1-\delta)C_s, \tag{42}$$

where δ is the fractional depreciation immediately after purchase of the machine at time s.

To convert the problem into the Mayer form, define

$$A_s^k = \sum_{i=s}^{k-1} R_s^i (1+\rho)^{-i}, \tag{43}$$

$$B_s^k = \sum_{i=s}^{k-1} u^i (1+\rho)^{-i}. \tag{44}$$

Using equations (43) and (44), we can write the optimal control problem as follows:

$$\max_{\{u_k\}} [J_{st} = A_s^t - B_s^t - C_s(1+\rho)^{-s} + x_s^t (1+\rho)^{-t}] \tag{45}$$

subject to

$$\Delta A_s^k = R_s^k (1+\rho)^{-k}, \quad A_s^s = 0, \tag{46}$$

$$\Delta B_s^k = u^k (1+\rho)^{-k}, \quad B_s^s = 0, \tag{47}$$

and the constraints (41), (42), and (39).

We associate the adjoint variables $\lambda_1^{k+1}, \lambda_2^{k+1}, \lambda_3^{k+1}$, and λ_4^{k+1}, respectively with the state equations (46), (47), (41), and (42). Therefore, the Hamiltonian becomes

$$
\begin{aligned}
H &= \lambda_1^{k+1} R_s^k (1+\rho)^{-k} + \lambda_2^{k+1} u^k (1+\rho)^{-k} \\
&\quad + \lambda_3^{k+1} \Psi_s + \lambda_4^{k+1} \Phi_s .
\end{aligned} \tag{48}
$$

The adjoint variables $\lambda_1, \lambda_2, \lambda_3,$ and λ_4 satisfy the following difference equations and terminal boundary conditions:

$$\Delta\lambda_1^k = -\frac{\partial H}{\partial A_s^k} = 0, \quad \lambda_1^t = 1, \tag{49}$$

$$\Delta\lambda_2^k = -\frac{\partial H}{\partial B_s^k} = 0, \quad \lambda_2^t = -1, \tag{50}$$

$$\Delta\lambda_3^k = -\frac{\partial H}{\partial R_s^k} = -\lambda_1^{k+1}(1+\rho)^{-k}; \quad \lambda_3^t = 0, \tag{51}$$

$$\Delta\lambda_4^k = -\frac{\partial H}{\partial x^k} = 0, \quad \lambda_4^t = (1+\rho)^{-t}. \tag{52}$$

The solutions of these equations are

$$\lambda_1^k = 1, \quad (53)$$

$$\lambda_2^k = -1, \quad (54)$$

$$\lambda_3^k = \sum_{i=k}^{t-1} (1+\rho)^{-i}, \quad (55)$$

$$\lambda_4^k = (1+\rho)^{-t}. \quad (56)$$

Note that λ_1^k, λ_2^k, and λ_4^k are constants for a fixed machine salvage time t.

To apply the maximum principle, we substitute (53)-(56) into the Hamiltonian (48), collect terms containing the control variable u^k, and rearrange and decompose H as

$$H = H_1 + H_2(u^k), \tag{57}$$

where H_1 is that part of H which is independent of u^k and

$$H_2(u^k) = -u^k(1+\rho)^{-k} + \sum_{i=k+1}^{t-1}(1+\rho)^{-i}\Psi_s + (1+\rho)^{-t}\Phi_s. \tag{58}$$

Next we apply the maximum principle to obtain the necessary condition for the optimal schedule of preventive maintenance expenditures in dollars. The condition of optimality is that H should be a maximum along the optimal path.

If u^k were unconstrained, this condition, given the concavity of Ψ_s and Φ_s, would be equivalent to setting the partial derivative of H with respect to u equal to zero, i.e.,

$$\begin{aligned} H_{u^k} = [H_2]_{u^k} &= -(1+\rho)^{-k} + (\Psi_s)_{u^k} \sum_{i=k+1}^{t-1} (1+\rho)^{-i} \\ &+ (\Phi_s)_{u^k}(1+\rho)^{-t} = 0. \end{aligned} \tag{59}$$

Equation (59) is an equation in u^k with the exception of the particular case when Ψ_s and Φ_s are linear in u^k (which will be treated later in this section). In general, (59) may or may not have a unique solution. For our case we will assume Ψ_s and Φ_s to be of the form such that they give a unique solution for u^k.

One such case occurs when Ψ_s and Φ_s are quadratic in u^k. In this case, (59) is linear in u^k and can be solved explicitly for a unique solution for u^k. Whenever a unique solution does exist, let this be

$$u^k = U^k_{st}. \tag{60}$$

The optimal control u^{k*} is given as

$$u^{k*} = \begin{cases} 0 & \text{if } U^k_{st} \leq 0, \\ U^k_{st} & \text{if } 0 \leq U^k_{st} \leq U^{sk}, \\ U^{sk} & \text{if } U^k_{st} \geq U^{sk}. \end{cases} \tag{61}$$

9.3.3 Special Case of Bang-Bang Control

We now treat the special case in which the problem, and therefore H, is linear in the control variable u^k. In this case, H can be maximized simply by having the control at its maximum when the coefficient of u^k in H is positive, and minimum when it is negative, i.e., the optimal control is of bang-bang type. In our problem, we obtain the special case if Ψ_s and Φ_s assume the form

$$\Psi_s(u^k, k) = u^k \psi_s^k \quad \text{and} \tag{62}$$

$$\Phi_s(u^k, k) = u^k \phi_s^k, \tag{63}$$

respectively, where ψ_s^k and ϕ_s^k are given constants.

Then the coefficient of u^k in H, denoted by $W_s(k,t)$, is

$$W_s(k,t) = -(1+\rho)^{-k} + \psi_s^k \sum_{i=k+1}^{t-1} (1+\rho)^{-i} + \phi_s^t (1+\rho)^{-t}, \quad (64)$$

and the optimal control u^{k*} is given by

$$u^{k*} = \text{bang}[0, U^{sk}; W_s(k,t)], \; k = s, s+1, \ldots, t-1. \quad (65)$$

Once u^{k*} has been obtained as in (61) or 65), we can substitute it into (41) and (42) to obtain R_s^{k*} and x_s^{k*}, which in turn can be used in (40) to obtain the optimal value of the objective function denoted by J_{st}^*. This can be done for each pair of machine purchase time s and sale time $t > s$. Let g_s denote the present value of the profit (discounted to period 0) of an optimal replacement and preventive maintenance policy for periods $s, s+1, ..., T-1$. Then,

$$g_s = \max_{t=s+1,...,T} [J_{st}^* + g_t], \; 0 \leq s \leq T-1 \quad (66)$$

with the boundary condition

$$g_T = 0. \quad (67)$$

The value of g_0 will give the required maximum.

- Machines may be bought at times $0, 1,$ and 2. The cost of a machine bought at time s is assumed to be

$$C_s = 1,000 + 500s^2.$$

- The discount rate, the fractional instantaneous depreciation at purchase, and the maximum preventive maintenance per period are assumed to be

$$\rho = 0.06,\ \delta = 0.25,\ \text{and } U = \$100,$$

respectively.

- Let R_s^s be the net return (net of necessary maintenance) of a machine purchased in period s and operated in period s. We assume

$$R_0^0 = \$600,\ R_1^1 = \$1,000,\ \text{ and } R_2^2 = \$1,100.$$

In a period k subsequent to the period s of machine purchase, the returns $R_s^k,\ k > s$, depends on the preventive maintenance performed on the machine in periods prior to period k. The incremental return function is given by $\Psi_s(u,k)$, which we assume to be linear. Specifically,

$$\Delta R_s^k = \Psi_s(u^k, k) = -d_s + a_s u^k,$$

where

$$d_0 = 200,\ d_1 = 50,\ d_2 = 100,\ \text{and}\ a_s = 0.5 + 0.1s^3.$$

This means that the return in period k on a machine purchased in period s goes down by an amount d_s every period between s and k, including s, in which there is no preventive maintenance. This decrease can be offset by an amount proportional to the amount of preventive maintenance. Note that the function Ψ_s is assumed to be stationary over time in order to simplify the example.

Let x_s^k be the salvage value at time k of a machine purchased at s. We assume

$$x_s^s = (1-\delta)C_s = 0.75[1,000 + 500s^2].$$

The incremental salvage value function is given by

$$\Delta x_s^k = -\ell_s C_s + b_s u^k,$$

where

$$\ell_s = \begin{cases} 0.1 & \text{when } s = 0, 1, \\ 0.2 & \text{when } s = 2, \end{cases}$$

and

$$b_s = (0.5 - 0.05s).$$

This means that the decrease in salvage value is a constant percentage of the purchase price if there is no preventive maintenance. With preventive maintenance, the salvage value can be enhanced by a proportional amount.

Let J^*_{st} be the optimal value of the objective function associated with a machine purchased at s and sold at $t \geq s+1$. We will now solve for $J^*_{st},\ s = 0, 1, 2,$ and $s < t \leq 3$, where t is an integer.

Before we proceed, we will as in (64) denote by $W_s(k,t)$, the coefficient of u^k in the Hamiltonian H, i.e.,

$$W_s(k,t) = -(1+\rho)^{-k} + a_s \sum_{i=k+1}^{t-1} (1+\rho)^{-i} + b_s(1+\rho)^{-t}.$$

The optimal control is given by (65).

It is noted in passing that

$$W_s(k+1,t) - W_s(k,t) = (1+\rho)^{-(k+1)}(\rho - a_s),$$

so that

$$\text{sgn}[W_s(k+1,t) - W_s(k,t)] = \text{sgn}[\rho - a_s]. \qquad (68)$$

This implies that

$$u^{(k+1)*} - u^{k*} \begin{cases} \geq 0 \ \text{ if } (\rho - a_s) > 0, \\ = 0 \ \text{ if } (\rho - a_s) = 0, \\ \leq 0 \ \text{ if } (\rho - a_s) < 0. \end{cases}$$

In this example $\rho - a_s < 0$, which means that if there is a switching in the preventive maintenance trajectory of a machine, the switch must be from $\$100$ to $\$0$.

We now solve the subproblems for various values of s and $t (s < t)$ by using the discrete maximum principle.

$\underline{s = 0,\ t = 1}$

$$W_0(0, 1) = -1 + 0.5(1.06)^{-1} < 0.$$

From (65) we have

$$u^{0*} = 0.$$

Now,

$$
\begin{aligned}
R_0^0 &= 600, \\
R_0^1 &= 600 - 200 = 400, \\
x_0^0 &= 0.75 \times 1,000 = 750, \\
x_0^1 &= 750 - 0.1 \times 1,000 = 650, \\
J_{01}^* &= 600 - 1,000 + 650 \times (1.06)^{-1} = \$213.2.
\end{aligned}
$$

Similar calculations can be carried out for other subproblems. We will list these results.

$s = 0,\ t = 2$

$$W_0(0,2) < 0, \quad W_0(1,2) < 0,$$
$$u^{0*} = 0, \qquad u^{1*} = 0,$$
$$J^*_{02} = 466.9.$$

$s = 0,\ t = 3$

$$W_0(0,3) > 0, \quad W_0(1,3) < 0, \quad W_0(2,3) < 0,$$
$$u^{0*} = 100, \qquad u^{1*} = 100, \qquad u^{2*} = 0,$$
$$J^*_{03} = 639.$$

$s = 1,\ t = 2$

$$W_1(1,2) < 0, \quad u^{1*} = 0, \quad J^*_{12} = 559.9.$$

$s = 1,\ t = 3$

$$W_1(1,3) > 0, \quad W_1(2,3) < 0,$$
$$u^{1*} = 100, \qquad u^{2*} = 0,$$
$$J^*_{13} = 1024.2.$$

$s = 2,\ t = 3$

$$W_2(2,3) < 0, \quad u^{2*} = 0, \quad J^*_{23} = 80.$$

With reference to the dynamic programming equation in (66) and (67), we have

$$
\begin{aligned}
g_3 &= 0, \\
g_2 &= J^*_{23} = \$80, \\
g_1 &= \max\,[J^*_{13}, J^*_{12} + g_2] \\
&= \max\,[1024.2, 559.9 + 80] \\
&= \$1024.2, \\
g_0 &= \max\,[J^*_{03}, J^*_{01} + g_1, J^*_{02} + g_2] \\
&= \max\,[639.0, 213.2 + 1024.2, 466.9 + 80] \\
&= \$1237.4.
\end{aligned}
$$

Now we can summarize the optimal solution. The optimal number of machines is 2, and their optimal policies are as follows:

First Machine Optimal Policy:

Purchase at $s = 0$; sell at $t = 1$; optimal preventive maintenance policy $u^{0*} = 0$.

Second Machine Optimal Policy:

Purchase at $s = 1$; sell at $t = 3$; optimal preventive maintenance policy $u^{1*} = 100, u^{2*} = 0$. The value of the objective function is $J^* =$ \$1237.4.