Chapter 8

The Maximum Principle: Discrete Time

We begin by stating a general form of a nonlinear programming problem.

y: be an n-component column vector;

a: be an r-component column vector;

b: an s-component column vector;

h: $E^n \to E^1$,

g: $E^n \to E^r$,

w: $E^n \to E^s$ be given functions.

We assume functions g and w to be column vectors with components r and s, respectively. We consider the nonlinear programming problem:

$$\max h(y) \tag{1}$$

subject to

$$\begin{aligned} g(y) &= a, & (2) \\ w(y) &\geq b. & (3) \end{aligned}$$

Suppose we want to solve (1) without imposing constraints (2) or (3). The problem is now the classical unconstrained maximization problem of calculus, and the first-order necessary conditions for its solution are

$$h_y = 0. \tag{4}$$

The points satisfying (4) are called *critical points*. With the equality constraints, the Lagrangian function

$$L(y, \lambda) = h(y) + \lambda[g(y) - a], \tag{5}$$

where λ is an $r-$component row vector.

The necessary condition for y^* to be a (maximum) solution to (1) and (2) is that there exists an r-component row vector λ such that

$$L_y = h_y + \lambda g_y = 0, \tag{6}$$

$$L_\lambda = g(y) - a = 0. \tag{7}$$

Suppose (y^*, λ) is in fact a solution to the equations (6) and (7). Note that y^* depends on a and we can show this dependence by writing $y^* = y^*(a)$. Now $h^* = h^*(a) = h(y^*(a))$ is the optimum value of the objective function.

The Lagrange multipliers satisfy the relation which has an important managerial interpretation

$$h_a^* = -\lambda, \tag{8}$$

namely, λ_i is the negative of the imputed value or *shadow price* of having one unit more of the resource a_i.

Consider the problem:

$$\begin{cases} \max \{h(x,y) = -x^2 - y^2\} \\ \text{subject to} \\ 2x + y = 10. \end{cases}$$

Solution. We form the Lagrangian

$$L(x, y, \lambda) = (-x^2 - y^2) + \lambda(2x + y - 10).$$

The necessary conditions found by partial differentiation are

$$\begin{aligned} L_x &= -2x + 2\lambda = 0, \\ L_y &= -2y + \lambda = 0, \\ L_\lambda &= 2x + y - 10 = 0. \end{aligned}$$

From the first two equations we get

$$\lambda = x = 2y.$$

Solving this with the last equation yields the quantities

$$x^* = 4,\ y^* = 2,\ \lambda = 4,\ h^* = -20,$$

$$L(y,\mu) = h(y) + \mu[w(y) - b]. \tag{9}$$

$$L_y = h_y + \mu w_y = 0, \tag{10}$$

$$w \geq b, \tag{11}$$

$$\mu \geq 0, \quad \mu(w - b) = 0. \tag{12}$$

Note that (10) is analogous to (6). Also (11) repeats the inequality constraint (3) in the same way that (7) repeated the equality constraint (2). However, the conditions in (12) are new and are particular to the inequality-constrained problem.

Solve the problem:

$$\begin{cases} \text{max } \{h(x) = 8x - x^2\} \\ \text{subject to} \\ x \geq 2. \end{cases}$$

Solution. We form the Lagrangian

$$L(x, \mu) = 8x - x^2 + \mu(x - 2).$$

The necessary conditions (10)-(12) become

$$L_x = 8 - 2x + \mu = 0, \tag{13}$$

$$x - 2 \geq 0, \tag{14}$$

$$\mu \geq 0, \quad \mu(x - 2) = 0. \tag{15}$$

Case 1: $\mu = 0$

From (13) we get $x = 4$, which also satisfies (14). Hence, this solution, which makes $h(4) = 16$, is a possible candidate for the maximum solution.

Case 2: $x = 2$

Here from (13) we get $\mu = -4$, which does not satisfy the inequality $\mu \geq 0$ in (15).

From these two cases we conclude that the optimum solution is $x^* = 4$ and $h^* = h(x^*) = 16$.

Solve the problem:

$$\begin{cases} \text{max } \{h(x) = 8x - x^2\} \\ \text{subject to} \\ x \geq 6. \end{cases}$$

Solution. The Lagrangian is

$$L(x, \mu) = 8x - x^2 + \mu(x - 6).$$

The necessary conditions are

$$\begin{aligned} L_x = 8 - 2x + \mu &= 0, && (16) \\ x - 6 &\geq 0, && (17) \\ \mu \geq 0,\ \mu(x - 6) &= 0. && (18) \end{aligned}$$

Solution of Example 8.3

Case 1: $\mu = 0$

From (16) we obtain $x = 4$, which does not satisfy (17), so that this case is infeasible.

Case 2: $x = 6$

Obviously (17) holds. From (16) we get $\mu = 4$, so that (18) holds as well. The optimal solution is then

$$x^* = 6, \; h^* = h(x^*) = 12,$$

since it is the only solution satisfying the necessary conditions.

Find the shortest distance between the point (2,2) and the upper half of the semicircle of radius one, whose center is at the origin. In order to simplify the calculation, we minimize h, the square of the distance. Hence, the problem can be stated as the following nonlinear programming problem:

$$\begin{cases} \text{max } \{-h(x,y) = -(x-2)^2 - (y-2)^2\} \\ \text{subject to} \\ x^2 + y^2 \leq 1, \\ y \geq 0. \end{cases}$$

The Lagrangian function for this problem is

$$L = -(x-2)^2 - (y-2)^2 + \mu(1 - x^2 - y^2) + \nu y. \tag{19}$$

The necessary conditions are

$$-2(x-2) - 2\mu x = 0, \tag{20}$$
$$-2(y-2) - 2\mu y + \nu = 0, \tag{21}$$
$$1 - x^2 - y^2 \geq 0, \tag{22}$$
$$y \geq 0, \tag{23}$$
$$\mu \geq 0, \quad \mu(1 - x^2 - y^2) = 0, \tag{24}$$
$$\nu \geq 0, \quad \nu y = 0. \tag{25}$$

From (24) we see that either $\mu = 0$ or $x^2 + y^2 = 1$, i.e., we are on the boundary of the semicircle. If $\mu = 0$, we see from (20) that $x = 2$. But $x = 2$ does not satisfy (22) for any y, and hence we conclude $\mu > 0$ and $x^2 + y^2 = 1$.

From (25) we conclude either $\nu = 0$ or $y = 0$. If $\nu = 0$, then from (20), (21), and $\mu > 0$, we get $x = y$. Solving the latter with $x^2 + y^2 = 1$ gives

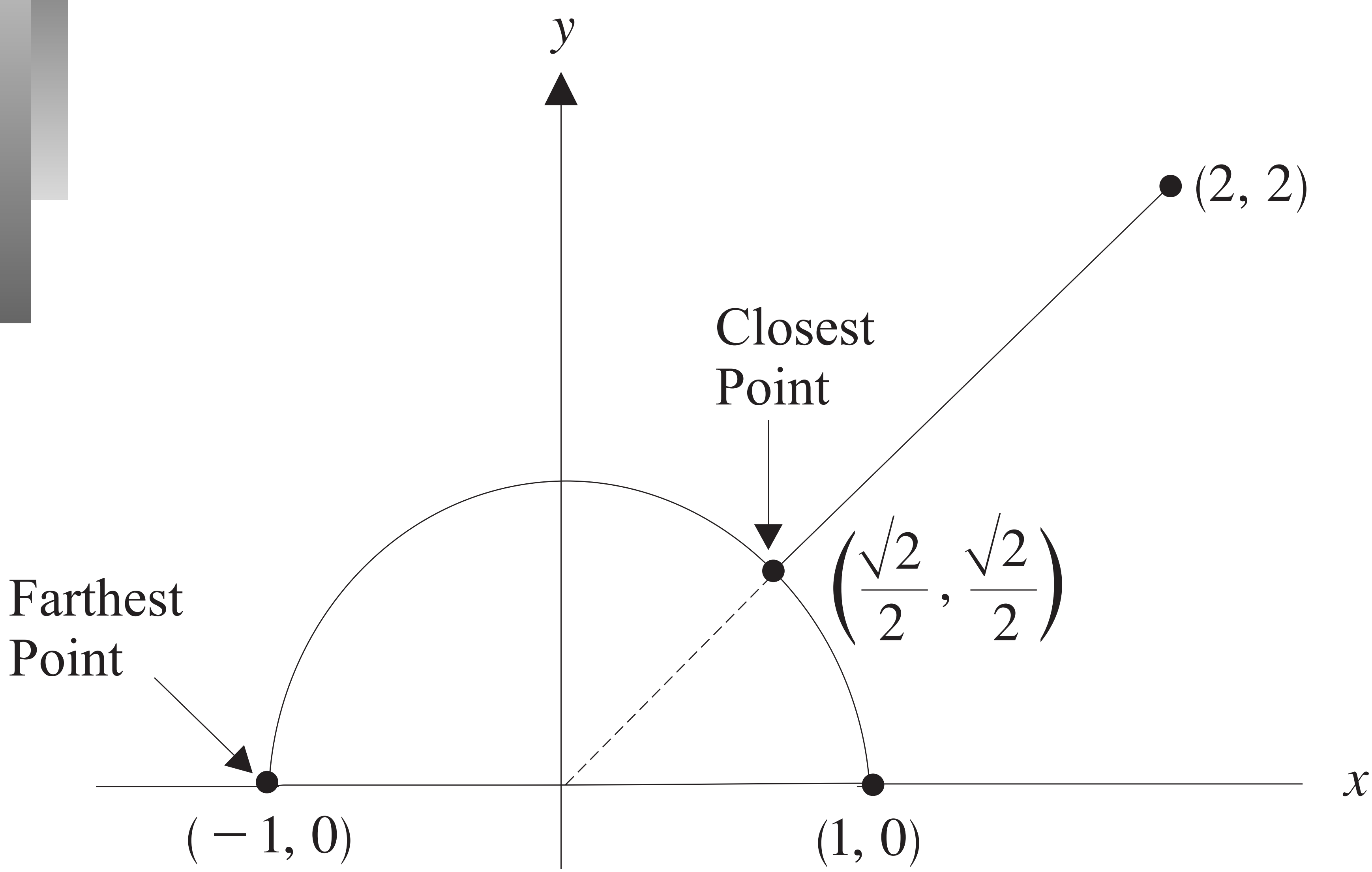
$$(a) \qquad (\sqrt{2}/2, \sqrt{2}/2) \text{ and } h = (9 - 4\sqrt{2})/4.$$

If $y = 0$, then solving with $x^2 + y^2 = 1$ gives

$$(b) \qquad (1, 0) \text{ and } h = 5,$$

$$(c) \qquad (-1, 0) \text{ and } h = 13.$$

Shortest Distance from a Point to a Semi-Circle



These three points are shown in Figure 8.1. Of the three points found that satisfy the necessary conditions, clearly the point $(\sqrt{2}/2, \sqrt{2}/2)$ found in (a) is the nearest point and solves the closest-point problem. The point $(-1, 0)$ in (c) is in fact the farthest point; and the point $(1, 0)$ in (b) is neither the closest nor the farthest point.

Consider the problem:

$$\max \{h(x,y) = y\} \tag{26}$$

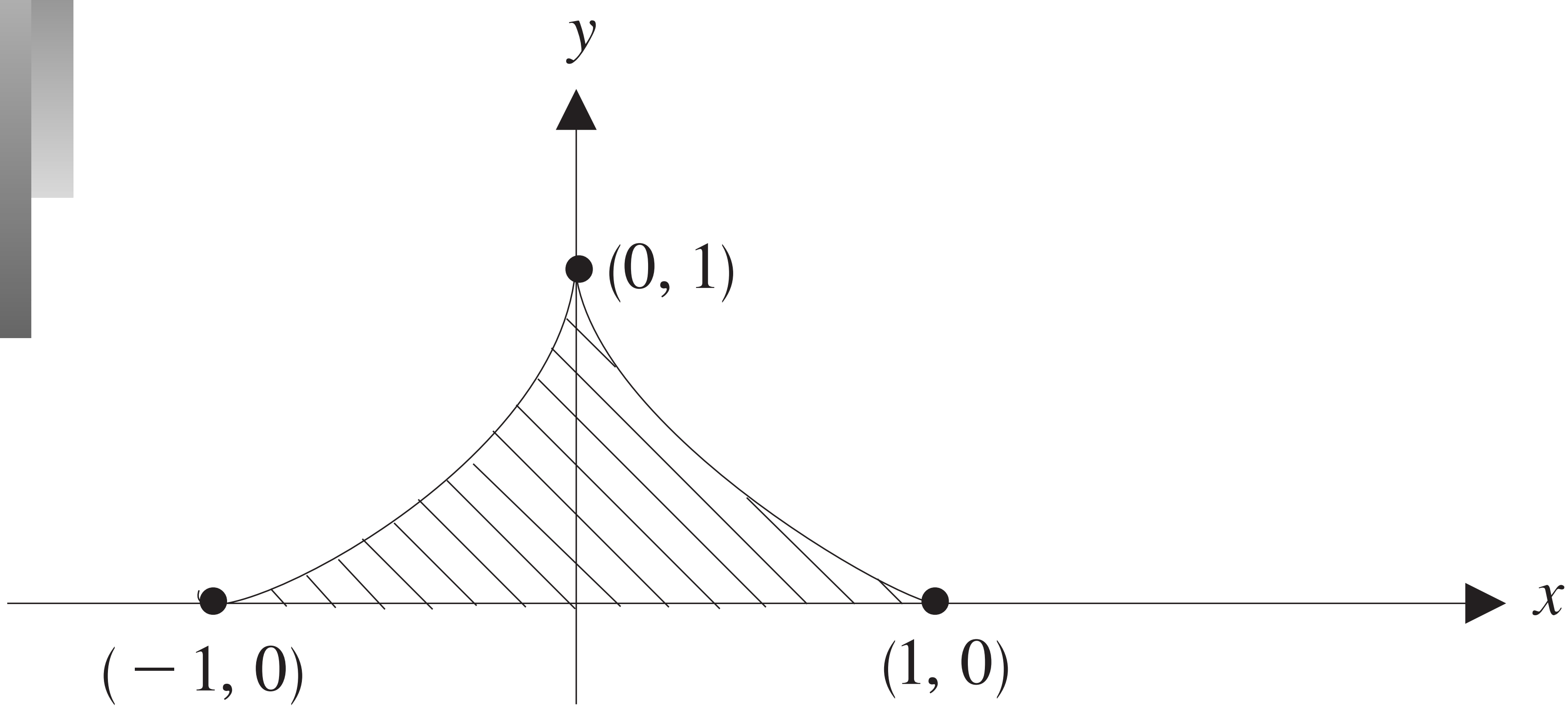
subject to

$$-x^{2/3} - y + 1 \geq 0, \tag{27}$$

$$y \geq 0. \tag{28}$$

The set of points satisfying the constraints is shown shaded in Figure 8.2. From the figure it is obvious that the solution point (0,1) maximizes the value of y. Let us see if we can find it using the above procedure.

Graph of Example 8.5



The Lagrangian is

$$L = y + \lambda(-x^{2/3} - y + 1) + \mu y. \tag{29}$$

The necessary conditions are

$$L_x = -\frac{2}{3}\lambda x^{-1/3} = 0, \tag{30}$$

$$L_y = 1 - \lambda + \mu = 0, \tag{31}$$

$$\lambda \geq 0,\ \lambda(-x^{2/3} - y + 1) = 0, \tag{32}$$

$$\mu \geq 0,\ \mu y = 0, \tag{33}$$

together with (27) and (28). From (30) we get $\lambda = 0$, since $x^{-1/3}$ is never 0 in the range $-1 \leq x \leq 1$. But substitution of $\lambda = 0$ into (31) gives $\mu = -1 < 0$, which fails to solve (33).

Consider the problem:

$$\max \{h(x,y) = y\} \tag{34}$$

subject to

$$(1-y)^3 - x^2 \geq 0, \tag{35}$$

$$y \geq 0. \tag{36}$$

The constraints are now differentiable and the set of feasible solutions is exactly the same as for Example 8.5, and is shown in Figure 8.2. Hence, the optimum solution is $(x^*, y^*) = (0, 1)$ and $h^* = 1$. But once again the Kuhn-Tucker method fails, as we will see.

The Lagrangian is

$$L = y + \lambda[(1-y)^3 - x^2] + \mu y, \tag{37}$$

so that the necessary conditions are

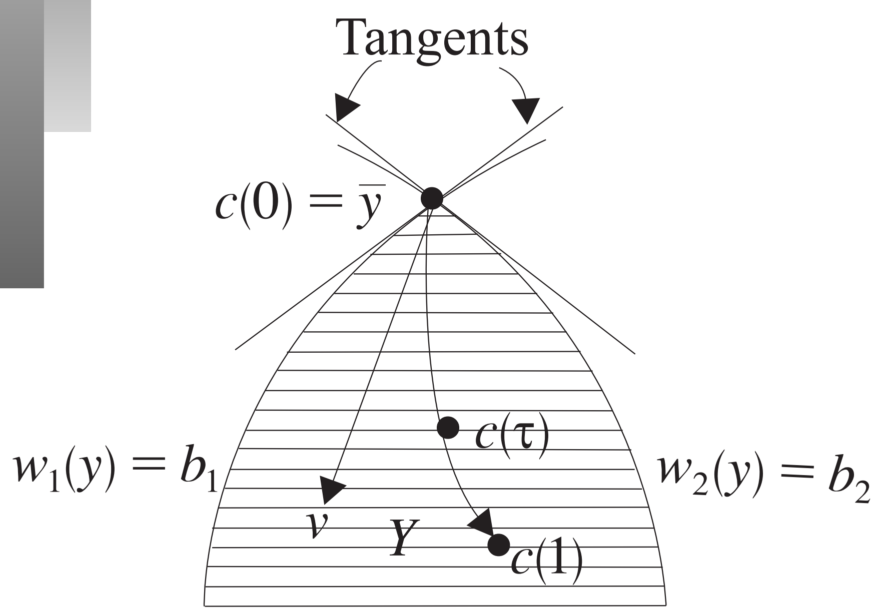
$$\begin{aligned} L_x &= -2x\lambda = 0, && (38)\\ L_y &= 1 - 3\lambda(1-y)^2 + \mu = 0, && (39)\\ \lambda &\geq 0, \;\; \lambda[(1-y)^3 - x^2] = 0, && (40)\\ \mu &\geq 0, \;\; \mu y = 0, && (41) \end{aligned}$$

together with (35) and (36).

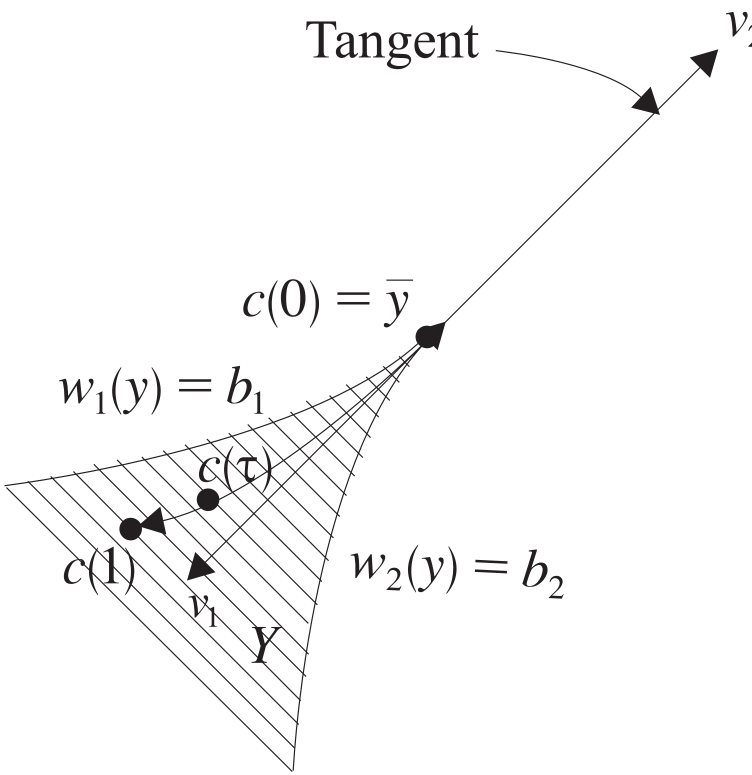
From (41) we get, either $y = 0$ or $\mu = 0$. Since $y = 0$ minimizes the objective function, we choose $\mu = 0$. From (38) we get either $\lambda = 0$ or $x = 0$. Since substitution of $\lambda = \mu = 0$ into (39) shows that it is not satisfied, we choose $x = 0, \lambda \neq 0$. But then (40) gives $(1 - y)^3 = 0$ or $y = 1$. However, $\mu = 0$ and $y = 1$ means that once more that (39) is not satisfied.

The reason for failure of the method in Example 8.6 is that the constraints do not satisfy what is called the *constraint qualification*, which will be discussed in the next section.

In order to motivate the definition we illustrate two different situations in Figure 8.3. In Figure 8.3(a) we show two boundary curves $w_1(y) = b_1$ and $w_2(y) = b_2$ intersecting the boundary point $\bar{y}$. The two tangents to these curves are shown, and v is a vector lying between the two tangents. Starting at $\bar{y}$, there is a differentiable curve $c(\tau), 0 \leq \tau \leq 1,$ drawn so that it lies entirely within the feasible set Y, such that its initial slope is equal to v. Whenever such a curve can be drawn from every boundary point $\bar{y}$ in Y and every v contained between the tangent lines, we say that the constraints defining Y satisfy the *Kuhn-Tucker constraint qualification* at $\bar{y}$.



a) Constraint Qualifications Satisfied



b) Constraint Qualifications Not Satisfied

Figure 8.3(b) illustrates a case of a cusp at which the constraint qualification does not hold. Here the two tangents to the graphs $w_1(y) = b_1$ and $w_2(y) = b_2$ coincide, so that v_1 and v_2 are vectors lying between these two tangents. Notice that for vector v_1 it is possible to find the differentiable curve $c(\tau)$ satisfying the above conditions, but for vector v_2 no such curve exists. Hence, the constraint qualification does not hold for the example in Figure 8.3(b).

We first state the constraint qualification symbolically. For the problem defined by (1), (2), and (3), let Y be the set of all feasible vectors satisfying (2) and (3), i.e.,

$$Y = \{y | g(y) = a,\ w(y) \geq b\}\,.$$

Let $\bar{y}$ be any point of Y and let $z(\bar{y}) = d$ denote the vector of tight constraints at point $\bar{y}$, i.e., z includes all the g constraints in (2) and those constraints in (3) which are satisfied as equalities.

Define the set

$$V = \left\{ v \,\middle|\, \frac{\partial z(\bar{y})}{\partial y} v \leq 0 \right\}. \tag{42}$$

Then, we shall say that the constraint set Y satisfies the *Kuhn-Tucker constraint qualification* at $\bar{y} \in Y$ if z is differentiable at $\bar{y}$ and if, for every $v \in V$, there exists a differentiable curve $c(\tau)$ defined for $0 \leq \tau \leq 1$ such that

(i) $c(0) = \bar{y}$,

(ii) $c(\tau) \in Y$ for all t satisfying $0 \leq \tau \leq 1$,

(iii) $\left.\dfrac{dc(\tau)}{d\tau}\right|_{\tau=0} = kv$ for some constant $k > 0$.

The Lagrangian function for this problem is

$$L(y, \lambda, \mu) = h + \lambda(g(y) - a) + \mu(w(y) - b). \tag{43}$$

The Kuhn-Tucker conditions at $\bar{y} \in Y$ for this problem are

$$\begin{aligned} L_y(\bar{y}, \lambda, \mu) = h_y(\bar{y}) + \lambda g_y(\bar{y}) + \mu w_y(\bar{y}) &= 0, && (44) \\ g(\bar{y}) &= a, && (45) \\ w(\bar{y}) &\geq b, && (46) \\ \mu \geq 0, \quad \mu w(\bar{y}) &= 0, && (47) \end{aligned}$$

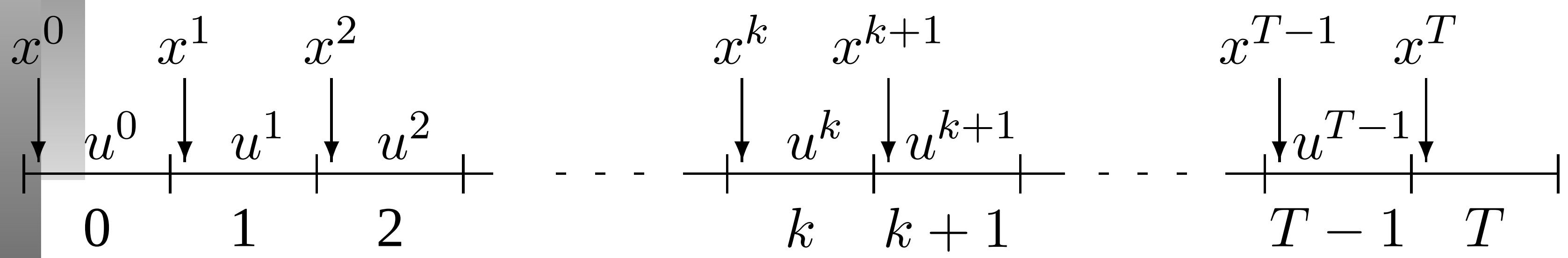
where λ and μ are row vectors of multipliers to be determined.

Theorem 8.1 *(Sufficient Conditions). If $h, g,$ and w are differentiable, h is concave, g is affine, w is concave, and $(\bar{y}, \lambda, \mu)$ solve the conditions* (44)-(47), *then $\bar{y}$ is a solution to the maximization problem* (1)-(3).

Theorem 8.2 *(Necessary Conditions). If $h, g,$ and w are differentiable, $\bar{y}$ solves the maximization problem, and the constraint qualification holds at $\bar{y}$, then there exist multipliers λ and μ such that $(\bar{y}, \lambda, \mu)$ satisfy conditions* (44)-(47).

$$
\begin{aligned}
\Theta &= \text{the set } \{0, 1, ..., T-1\}, \\
x^k &= \text{an } n\text{-component column state vector; } k = 0, 1, ..., T, \\
u^k &= \text{an } m\text{-component column control vector;} \\
&\qquad k = 0, 1, ..., T-1, \\
b^k &= \text{an } s\text{-component column vector of constants;} \\
&\qquad k = 0, 1, ..., T-1.
\end{aligned}
$$

Here, the state x^k is assumed to be measured at the beginning of period k and control u^k is implemented during period k. This convention is depicted in Figure 8.4.



We also define continuously differentiable functions

$$f : E^n \times E^m \times \Theta \to E^n,$$

$$F : E^n \times E^m \times \Theta \to E^1,$$

$$g : E^m \times \Theta \to E^s,$$

$$S : E^m \times \Theta \cup \{T\} \to E^1.$$

Then, a discrete-time optimal control problem in the Bolza form (see Section 2.1.4) is:

$$\max \left\{ J = \sum_{k=0}^{T-1} F(x^k, u^k, k) + S(x^T, T) \right\} \tag{48}$$

subject to the difference equation

$$\begin{aligned} \triangle x^k &= x^{k+1} - x^k = f(x^k, u^k, k),\ k = 0, ..., T-1, \\ & \qquad x^0 \text{ given}, \end{aligned} \tag{49}$$

$$g(u^k, k) \geq b^k,\ k = 0, ..., T-1. \tag{50}$$

In (49) the term $\triangle x^k = x^{k+1} - x^k$ is known as the *difference operator*.

The Lagrangian function of the problem is

$$\begin{aligned} L &= \sum_{k=0}^{T-1} F(x^k, u^k, k) + S(x^T, T) \\ &+ \sum_{k=0}^{T-1} \lambda^{k+1}[f(x^k, u^k, k) - x^{k+1} + x^k] \\ &+ \sum_{k=0}^{T-1} \mu^k [g(u^k, k) - b^k]. \end{aligned} \tag{51}$$

We now define the Hamiltonian function H^k to be

$$H^k = H(x^k, u^k, k) = F(x^k, u^k, k) + \lambda^{k+1} f(x^k, u^k, k). \tag{52}$$

Using (52) we can rewrite (51) as

$$\begin{aligned} L &= S(x^T, T) + \sum_{k=0}^{T-1} [H^k - \lambda^{k+1}(x^{k+1} - x^k)] \\ &\quad + \sum_{k=0}^{T-1} \mu^k [g(u^k, k) - b^k]. \end{aligned} \tag{53}$$

If we differentiate (53) with respect to x^k for $k = 1, 2, ..., T-1$, we obtain

$$\frac{\partial L}{\partial x^k} = \frac{\partial H^k}{\partial x^k} - \lambda^k + \lambda^{k+1} = 0,$$

which upon rearranging terms becomes

$$\triangle\lambda^k = \lambda^{k+1} - \lambda^k = -\frac{\partial H^k}{\partial x^k}, \; k = 0, 1, ..., T-1. \tag{54}$$

If we differentiate (53) with respect to x^T, we get

$$\frac{\partial L}{\partial x^T} = \frac{\partial S}{\partial x^T} - \lambda^T = 0, \text{ or } \lambda^T = \frac{\partial S}{\partial x^T}. \tag{55}$$

The difference equations (54) with terminal boundary conditions (55) are called the *adjoint equations*.

If we differentiate L with respect to u^k and state the corresponding Kuhn-Tucker conditions for the multiplier μ^k and constraint (50), we have

$$\frac{\partial L}{\partial u^k} = \frac{\partial H^k}{\partial u^k} + \mu^k \frac{\partial g}{\partial u^k} = 0$$

or

$$\frac{\partial H^k}{\partial u^k} = -\mu^k \frac{\partial g}{\partial u^k}, \tag{56}$$

and

$$\mu^k \geq 0, \quad \mu^k[g(u^k, k) - b^k] = 0. \tag{57}$$

We note that, provided H^k is concave in u^k, $g(u^k, k)$ is concave in u^k, and the constraint qualification holds, then conditions (56) and (57) are precisely the necessary and sufficient conditions for solving the following Hamiltonian maximization problem:

$$\left\{\begin{array}{l} \max\limits_{u^k} H^k \\ \text{subject to} \\ g(u^k, k) \geq b^k. \end{array}\right. \tag{58}$$

Theorem 8.3

If for every k, H^k *in* (52) *and* $g(u^k, k)$ *are concave in* u^k, *and the constraint qualification holds, then the necessary conditions for* $u^{k*}, k = 0, 1, ..., T-1$, *to be an optimal control for the problem* (48)-(50) *are*

$$\begin{cases} \triangle x^{k*} = f(x^{k*}, u^{k*}, k), \; x^0 \text{given}, \\ \triangle \lambda^k = -\frac{\partial H^k}{\partial x^k}[x^{k*}, u^{k*}, \lambda^{k+1}, k], \; \lambda^T = \frac{\partial S(x^{T*}, T)}{\partial x^T}, \\ H^k(x^{k*}, u^{k*}, \lambda^{k+1}, k) \geq H^k(x^{k*}, u^k, \lambda^{(k+1)}, k), \\ \text{for all } \; u^k \; \text{ such that } \; g(u^k, k) \geq b^k, \quad k = 0, 1, ..., T-1. \end{cases} \tag{59}$$

Consider the discrete-time optimal control problem:

$$\max \left\{ J = \sum_{k=1}^{T-1} -\frac{1}{2}(x^k)^2 \right\} \tag{60}$$

subject to

$$\triangle x^k = u^k, x^0 = 5, \tag{61}$$

$$u^k \in \Omega = [-1, 1]. \tag{62}$$

We shall solve this problem for $T = 6$ and $T \geq 7$.

The Hamiltonian is

$$H^k = -\frac{1}{2}(x^k)^2 + \lambda^{k+1}u^k, \tag{63}$$

from which it is obvious that the optimal policy is bang-bang. Its form is

$$u^k = \text{bang}[-1, 1; \lambda^{k+1}] = \begin{cases} 1 & \text{if } \lambda^{k+1} > 0, \\ \text{singular} & \text{if } \lambda^{k+1} = 0, \\ -1 & \text{if } \lambda^{k+1} < 0. \end{cases} \tag{64}$$

Let us assume, as we did in Example 2.3, that $\lambda^k < 0$ as long as x^k is positive so that $u^k = -1$.

Given this assumption, (61) becomes $\triangle x^k = -1$, whose solution is

$$x^k = -k + 5 \text{ for } k = 1, 2, ..., T - 1. \tag{65}$$

By differentiating (63), we obtain the adjoint equation

$$\triangle \lambda^k = -\frac{\partial H^k}{\partial x^k} = x^k, \; \lambda^T = 0. \tag{66}$$

Let us assume $T = 6$. Substitute (65) into (66) to obtain

$$\triangle \lambda^k = -k + 5, \; \lambda^6 = 0.$$

From Appendix A.11, we find the solution to be

$$\lambda^k = -\frac{1}{2}k^2 + \frac{11}{2}k + c,$$

where c is a constant. Since $\lambda^6 = 0$, we can obtain the value of c by setting $k = 6$ in the above equation. Thus,

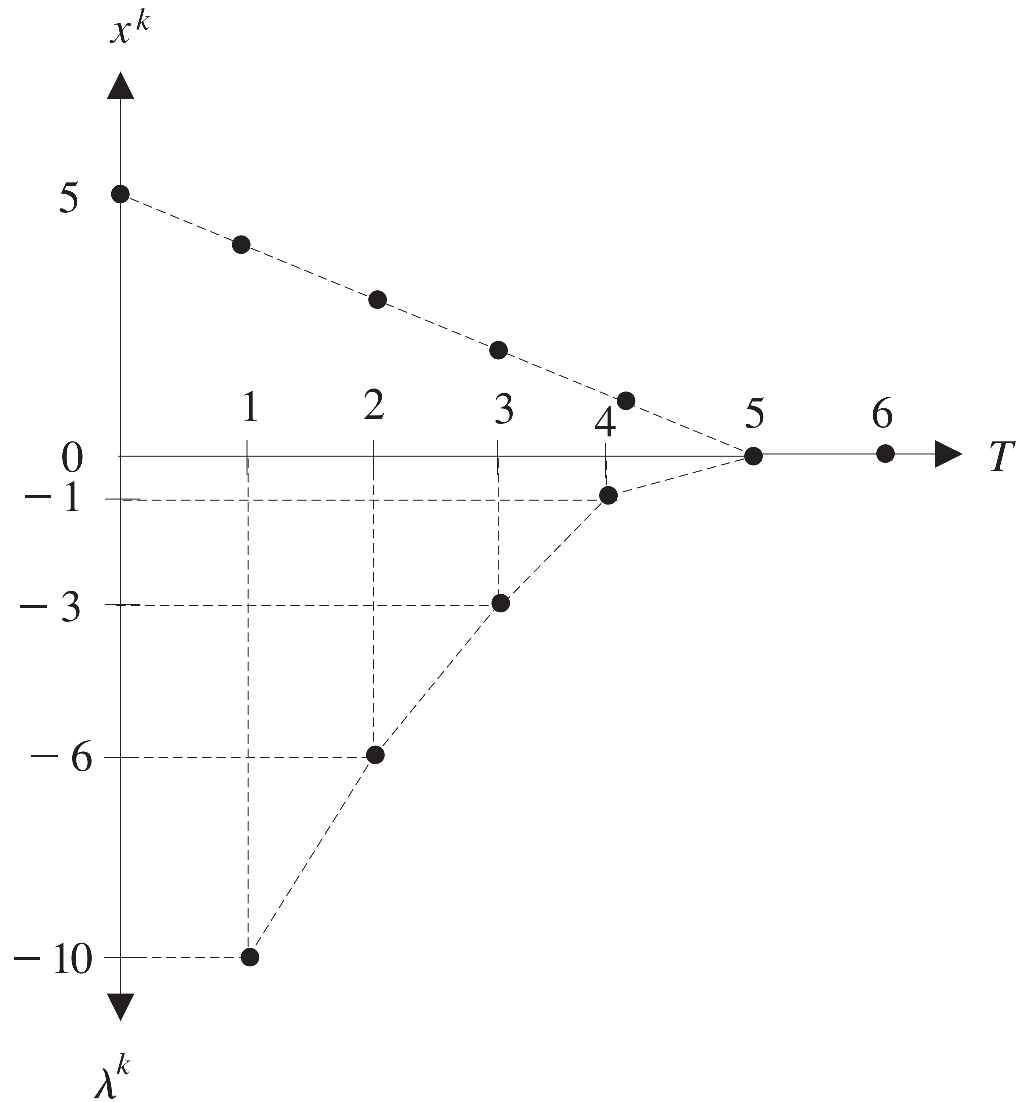
$$\lambda^6 = -\frac{1}{2}(36) + \frac{11}{2}(6) + c = 0 \Rightarrow c = -15,$$

so that

$$\lambda^k = -\frac{1}{2}k^2 + \frac{11}{2}k - 15. \tag{67}$$

A sketch of the values for λ^k and x^k appears in Figure 8.5. Note that $\lambda^5 = 0$, so that the control u^4 is singular. However, since $x^4 = 1$ we choose $u^4 = -1$ in order to bring x^5 down to 0.

Sketch of x^k and λ^k



The solution of the problem for $T \geq 7$ is carried out in the same way that we solved Example 2.3. Namely, observe that $x^5 = 0$ and $\lambda^5 = \lambda^6 = 0$, so that the control is singular. We simply make $\lambda^k = 0$ for $k \geq 7$ so that $u^k = 0$ for all $k \geq 7$. It is clear without a formal proof that this maximizes (60).

Let us consider a discrete version of the production-inventory example of Section 6.1; see Kleindorfer, Kriebel, Thompson, and Kleindorfer (1975). Let I^k, P^k, and S^k be the inventory, production, and demand at time k, respectively. Let I^0 be the initial inventory, let $\hat{I}$ and $\hat{P}$ be the goal levels of inventory and production, and let h and c be inventory and production cost coefficients. The problem is:

$$\max_{P^k \geq 0} \left\{ J = \sum_{k=0}^{T-1} -\frac{1}{2}[h(I^k - \hat{I})^2 + c(P^k - \hat{P})^2] \right\} \qquad (68)$$

subject to

$$\triangle I^k = P^k - S^k, I^0 \text{ given.} \qquad (69)$$

Form the Hamiltonian

$$H^k = -\frac{1}{2}[h(I^k - \hat{\mathrm{I}})^2 + c(P^k - \hat{P})^2] + \lambda^{k+1}(P^k - S^k), \quad (70)$$

where the adjoint variable satisfies

$$\triangle\lambda^k = -\frac{\partial H^k}{\partial I^k} = h(I^k - \hat{\mathrm{I}}),\ \lambda^T = 0. \quad (71)$$

To maximize the Hamiltonian, let us differentiate (70) to obtain

$$\frac{\partial H^k}{\partial P^k} = -c(P^k - \hat{P}) + \lambda^{k+1} = 0.$$

Since production must be nonnegative, we obtain the optimal production as

$$P^{k*} = \max\,[0, \hat{P} + \lambda^{k+1}/c]. \tag{72}$$

Expressions (69), (71), and (72) determine a two-point boundary value problem. For a given set of data, it can be solved numerically by using a spreadsheet software like EXCEL; see Section 2.5. If the constraint $P^k \geq 0$ is dropped it can be solved analytically by the method of Section 6.1, with difference equations replacing the differential equations used there.

The problem to be solved is:

$$\max\left\{J=\sum_{k=0}^{T-1}F(x^k,u^k,k)\right\} \tag{73}$$

subject to

$$\triangle x^k = f(x^k,u^k,k), \quad x^0 \text{ given}$$

$$u^k \in \Omega_k,\ k=0,1,...,(T-1). \tag{74}$$

Assumptions required are:
(i) $F(x^k, u^k, k)$ and $f(x^k, u^k, k)$ are continuously differentiable in x^k for every u^k and k.
(ii) The sets $\{-F(x, \Omega^k, k), f(x, \Omega^k, k)\}$ are *b-directionally convex* for every x and k, where $b = (-1, 0, ..., 0)$. That is, given v and w in Ω^k and $0 \leq \lambda \leq 1$, there exists $u(\lambda) \in \Omega^k$ such that

$$F(x, u(\lambda), k) \geq \lambda F(x, v, k) + (1 - \lambda) F(x, w, k)$$

and

$$f(x, u(\lambda), k) = \lambda f(x, v, k) + (1 - \lambda) f(x, w, k)$$

for every x and k. It should be noted that convexity implies *b*-directional convexity, but not the converse.
(iii) Ω^k satisfies the Kuhn-Tucker constraint qualification.