CHAPTER 7

APPLICATIONS TO MARKETING

- State Equation:
  Rate of sales expressed in terms of advertising, which is a control variable
- Objective:
  Profit maximization
- Constraints:
  Advertising rate to be nonnegative

Let $G(t) \geq 0$ denote the stock of goodwill at time t, with

$$\dot{G} = u - \delta G, \qquad G(0) = G_0, \tag{1}$$

where $u = u(t) \geq 0$ is the advertising effort at time t measured in dollars per unit time. Sales rate $S(t)$ is given by

$$S = S(p, G, Z). \tag{2}$$

Assuming the rate of total production costs is $c(S)$, we can write the total revenue net of production costs as

$$R(p, G, Z) = pS(p, G, Z) - c(S). \tag{3}$$

The revenue net of advertising expenditure is therefore $R(p, G, Z) - u$.

The firm wants to maximize the present value of net revenue streams discounted at a fixed rate ρ, i.e.,

$$\max_{u \geq 0, p \geq 0} \left\{ J = \int_0^\infty e^{-\rho t} [R(p, G, Z) - u] \, dt \right\} \tag{4}$$

subject to (1). Since the only place that p occurs is in the integrand, we can maximize J by first maximizing R with respect to price p holding G fixed, and then maximize the result with respect to u. Thus,

$$\frac{\partial R(p, G, Z)}{\partial p} = S + p \frac{\partial S}{\partial p} - c_S \frac{\partial S}{\partial p} = 0, \tag{5}$$

which implicitly gives the optimal price

$$p^*(t) = p(G(t), Z(t)).$$

Defining $\eta = -(p/S)(\partial S/\partial p)$ as the elasticity of demand with respect to price, we can rewrite condition (5) as

$$p^* = \frac{\eta c_S(S)}{\eta - 1}, \tag{6}$$

which is the usual price formula for a monopolist, known sometimes as the Amoroso-Robinson relation. In words, the formula means that the marginal revenue $(\eta - 1)p/\eta$ must equal the marginal cost $c_S(S)$. See, e.g., Cohen and Cyert (1965, p.189).

Defining $\pi(G,Z) = R(p^*,G,Z)$, the objective function in (4) can be rewritten as

$$\max_{u\geq 0}\left\{ J = \int_0^\infty e^{-\rho t}[\pi(G,Z) - u]\,dt \right\}.$$

For convenience, we assume Z to be a given constant and restate the optimal control problem which we have just formulated:

$$\begin{cases} \max_{u\geq 0}\left\{J = \int_0^\infty e^{-\rho t}[\pi(G) - u]\,dt\right\} \\ \text{subject to} \\ \dot{G} = u - \delta G, \quad G(0) = G_0. \end{cases} \tag{7}$$

$$H = \pi(G) - u + \lambda[u - \delta G], \tag{8}$$

$$\dot{\lambda} = \rho\lambda - \frac{\partial H}{\partial G} = (\rho + \delta)\lambda - \frac{\partial \pi}{\partial G}, \tag{9}$$

$$\lim_{t \to +\infty} e^{-\rho t}\lambda(t) = 0. \tag{10}$$

The adjoint variable $\lambda(t)$ is the shadow price associated with the goodwill at time t. Thus, the Hamiltonian in (8) can be interpreted as the dynamic profit rate which consists of two terms: (i) the current net profit rate $(\pi(G) - u)$ and (ii) the value $\lambda\dot{G} = \lambda[u - \delta G]$ of the new goodwill created by advertising at rate u.

Equation (9) corresponds to the usual equilibrium relation for investment in capital goods; see Arrow and Kurz (1970) and Jacquemin (1973). It states that the marginal opportunity cost $\lambda(\rho + \delta)dt$ of investment in goodwill should equal the sum of the marginal profit $(\partial\pi/\partial G)dt$ from increased goodwill and the capital gain $d\lambda := \dot{\lambda}dt$.

Defining $\beta = (G/S)(\partial S/\partial G)$ as the elasticity of demand with respect to goodwill and using (3), (5), and (9), we can derive (see Exercise 7.3)

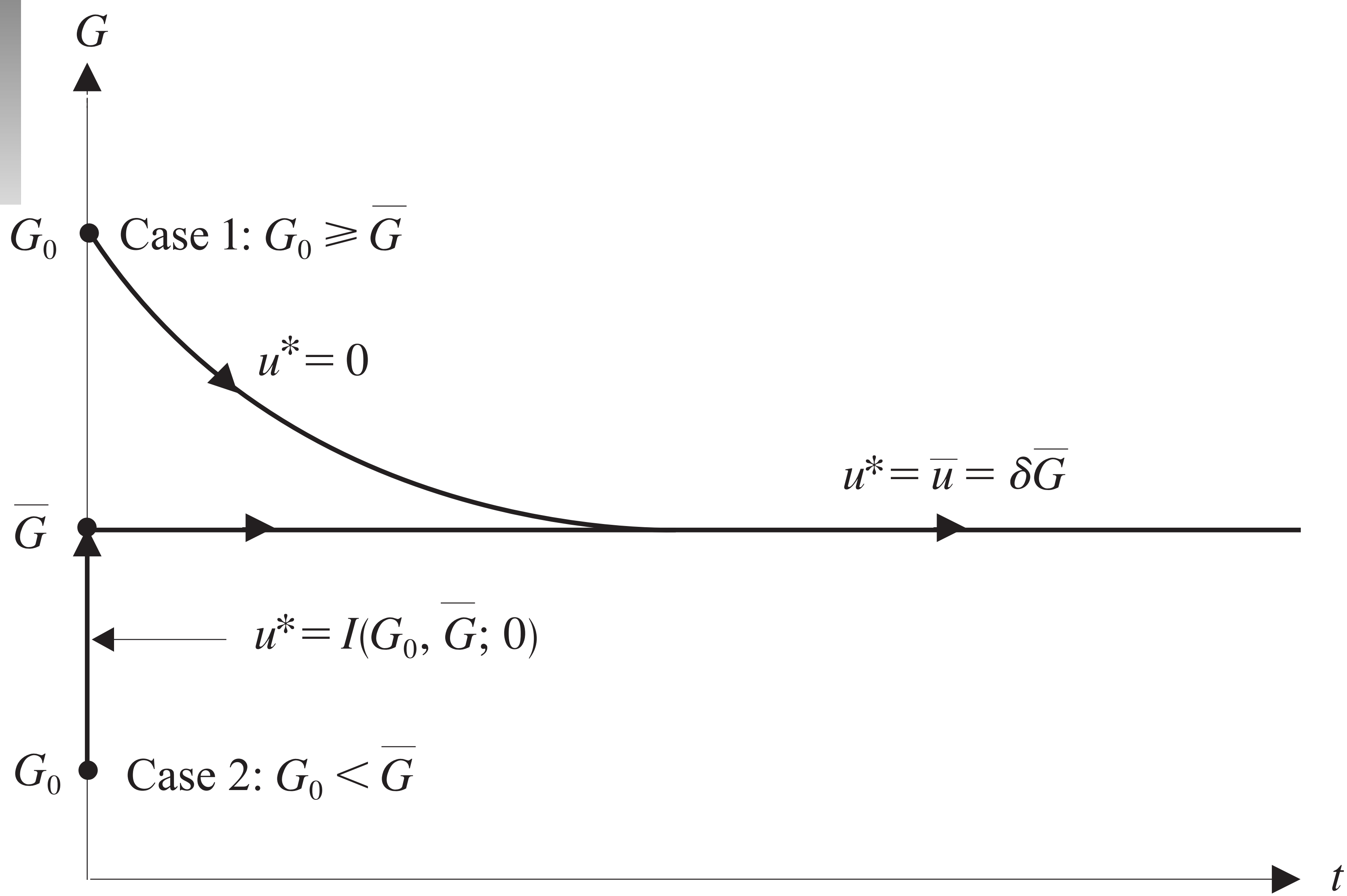
$$G^* = \frac{\beta p S}{\eta[(\rho+\delta)\lambda - \dot{\lambda}]}. \tag{11}$$

We use (3.74) to obtain the optimal long-run stationary equilibrium or turnpike $\{\bar{G}, \bar{u}, \bar{\lambda}\}$. That is, we obtain $\lambda = \bar{\lambda} = 1$ from (8) by using $\partial H/\partial u = 0$. We then set $\lambda = \bar{\lambda} = 1$ and $\dot{\lambda} = 0$ in (9). Finally, from (11) and (9), $\bar{G}$ or also the singular level G^s can be obtained as

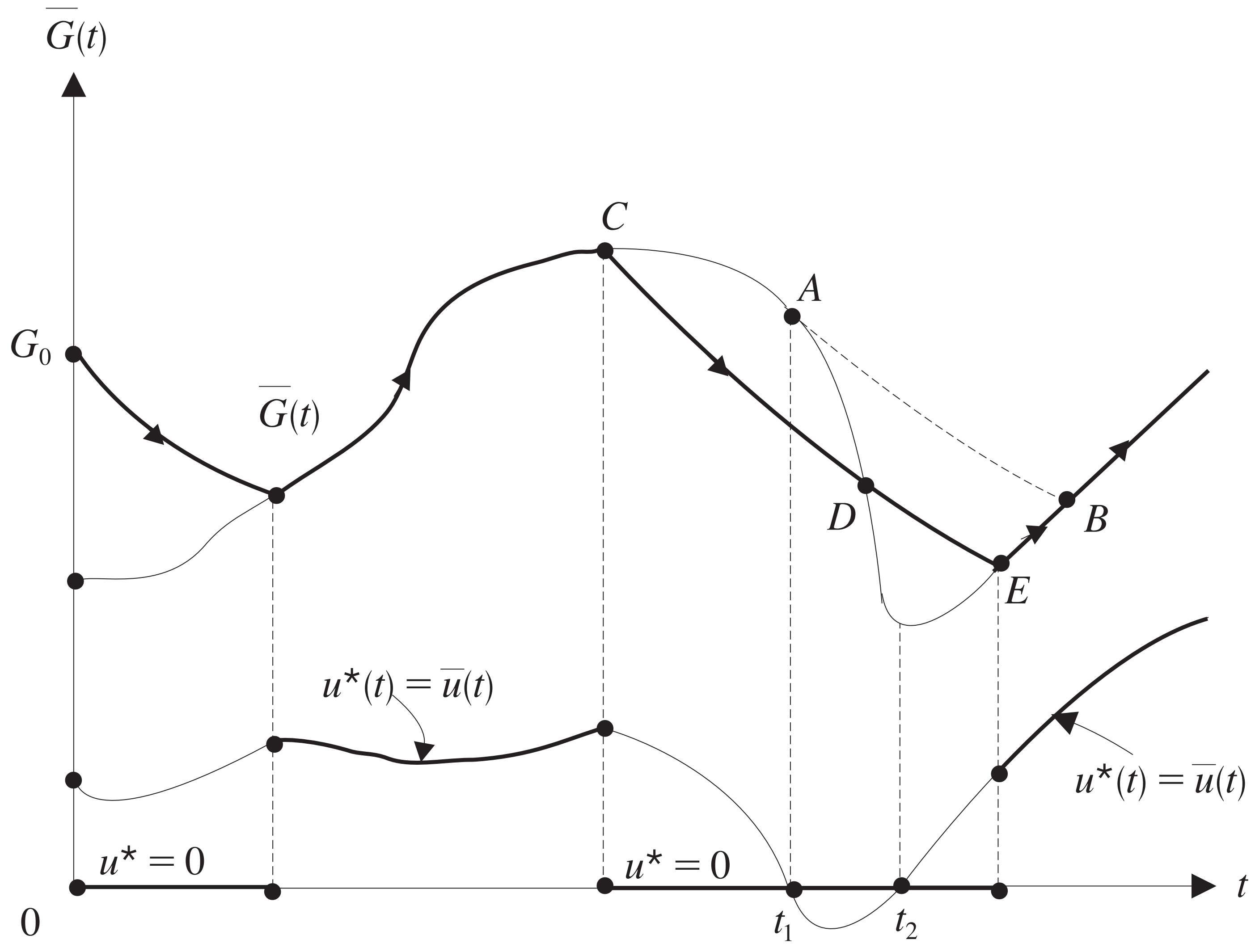
$$\bar{G} = G^s = \frac{\beta p S}{\eta(\rho+\delta)}. \tag{12}$$

The property of $\bar{G}$ is that the optimal policy is to go to $\bar{G}$ as fast as possible. If $G_0 < \bar{G}$, it is optimal to jump instantaneously to $\bar{G}$ by applying an appropriate impulse at $t = 0$ and then set $u^*(t) = \bar{u} = \delta\bar{G}$ for $t > 0$. If $G_0 > \bar{G}$, the optimal control $u^*(t) = 0$ until the stock of goodwill depreciates to the level $\bar{G}$, at which time the control switches to $u^*(t) = \delta\bar{G}$ and stays at this level to maintain the level $\bar{G}$ of goodwill. This optimal policy is graphed in Figure 7.1 for these two different initial conditions.

Figure 7.1: Optimal Policies in the Nerlove-Arrow Model



For a time-dependent Z, however, $\bar{G}(t) = G(Z(t))$ will be a function of time. To maintain this level of $\bar{G}(t)$, the required control is $\bar{u}(t) = \delta\bar{G}(t) + \dot{\bar{G}}(t)$. If $\bar{G}(t)$ is decreasing sufficiently fast, then $\bar{u}(t)$ may become negative and thus infeasible. If $\bar{u}(t) \geq 0$ for all t, then the optimal policy is as before. However, suppose $\bar{u}(t)$ is infeasible in the interval $[t_1, t_2]$ shown in Figure 7.2. In such a case, it is feasible to set $u(t) = \bar{u}(t)$ for $t \leq t_1$; at $t = t_1$ (which is point A in Figure 7.2) we can no longer stay on the turnpike and must set $u(t) = 0$ until we hit the turnpike again (at point B in Figure 7.2). However, such a policy is not necessarily optimal.



$\overline{G}(t)$
C
A
G_0
$\overline{G}(t)$
D
B
E
$u^*(t) = \overline{u}(t)$
$u^*(t) = \overline{u}(t)$
$u^* = 0$
$u^* = 0$
0
t_1
t_2
t

For instance, suppose we leave the turnpike at point C anticipating the infeasibility at point A. The new path CDEB may be better than the old path CAB. Roughly the reason this may happen is that path CDEB is “closer” to the turnpike than CAB. The picture in Figure 7.2 illustrates such a case. The optimal policy is the one that is “closest” to the turnpike. This discussion will become clearer in Section 7.2.2, when a similar situation arises in connection with the Vidale-Wolfe model. For further details, see Sethi (1977b) and Breakwell (1968).

The new optimal control problem is:

$$\begin{cases} \max_{u \geq 0} \left\{ J = \int_0^\infty e^{-\rho t} [\pi_1(u) + \pi_2(G) - u] dt \right\} \\ \text{subject to} \\ \dot{G} = u - \delta G, \ G(0) = G_0. \end{cases} \quad (13)$$

$$H = \pi_1(u) + \pi_2(G) - u + \lambda(u - \delta G), \quad (14)$$

$$\dot{\lambda} = \rho\lambda - \pi_2'(G) + \delta\lambda, \quad (15)$$

$$\frac{\partial H}{\partial u} = \pi_1'(u) - 1 + \lambda = 0. \quad (16)$$

Since $\pi_1'' < 0$, we can invert π_1' to solve (16) for u as a function of λ. Thus,

$$u = f_1(\lambda). \quad (17)$$

The function f_1 in (17) permits us to rewrite the state equation in (13) and the adjoint equation (15) as the following initial value problem:

$$\begin{cases} \dot{G} + \delta G = f_1(\lambda), & G(0) = G_0, \\ -\dot{\lambda} + (\rho + \delta)\lambda = \pi_2'(G), & \lambda(0) = \lambda_0. \end{cases} \tag{18}$$

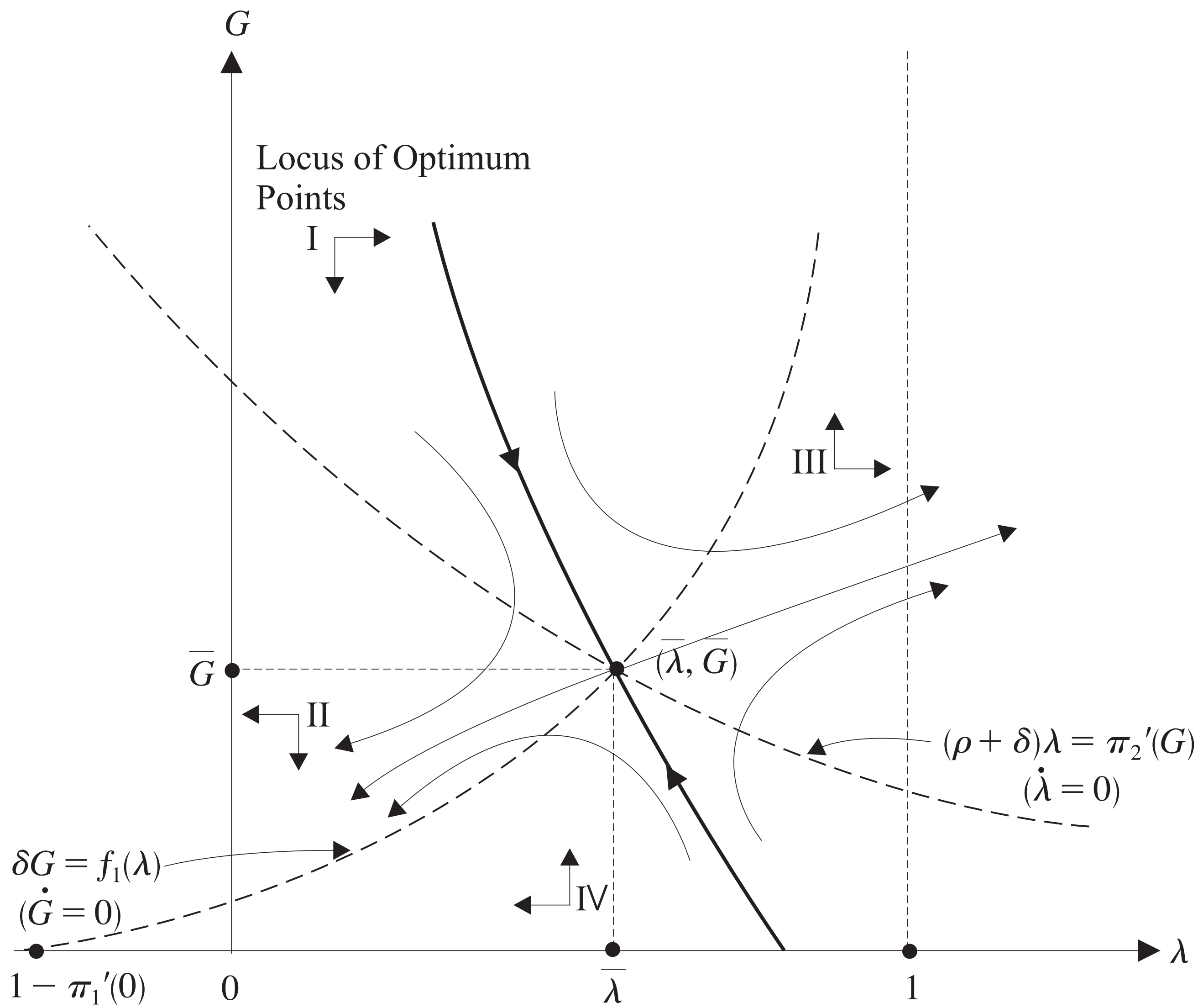
We note that

$$\begin{cases} \text{in Regions I and II}, & f_1(\lambda) - \delta G < 0, \\ \text{in Regions III and IV}, & f_1(\lambda) - \delta G > 0, \\ \text{in Regions I and III}, & (\rho + \delta)\lambda - \pi_2'(G) > 0, \\ \text{in Regions II and IV}, & (\rho + \delta)\lambda - \pi_2'(G) < 0. \end{cases}$$

The above implies that

$$\begin{cases} \text{in Region I}, & \text{G decreasing and } \lambda \text{ increasing}, \\ \text{in Region II}, & \text{G decreasing and } \lambda \text{ decreasing}, \\ \text{in Region III}, & \text{G increasing and } \lambda \text{ increasing}, \\ \text{in Region IV}, & \text{G increasing and } \lambda \text{ decreasing}. \end{cases}$$

Figure 7.3: Phase Diagram of System (18) for Problem (13)



Because of these conditions it is clear that for a given G_0, a choice of λ_0 such that (λ_0, G_0) is in Regions II and III, will *not* lead to a path converging to the turnpike point $(\bar{\lambda}, \bar{G})$. On the other hand, the choice of (λ_0, G_0) in Region I when $G_0 > \bar{G}$ or (λ_0, G_0) in Region IV when $G_0 < \bar{G}$, can give a path that converges to $(\bar{\lambda}, \bar{G})$. From a result in Coddington and Levinson (1955), it can be shown that at least in the neighborhood of $(\bar{\lambda}, \bar{G})$, there exists a locus of optimum starting points (λ_0, G_0). A detailed analysis leads to the saddle point path as shown darkened in Figure 7.3. Clearly the initial control $u^*(0) = f_1(\lambda_0)$.

Given $G_0 > \bar{G}$, we choose λ_0 on the saddle point path in Region I of Figure 7.3. Clearly the initial control $u^*(0) = f_1(\lambda_0)$. Furthermore, $\lambda(t)$ is increasing and by (17), $u(t)$ is increasing, so that in this case the optimal policy is to advertise at a low rate initially and gradually increase advertising to the turnpike level $\bar{u} = \delta\bar{G}$. If $G_0 < \bar{G}$, it can be shown similarly that the optimal policy is to advertise most heavily in the beginning and gradually decrease it to the turnpike level $\bar{u}$ as G approaches $\bar{G}$.

Note that the approach to the equilibrium $\bar{G}$ is no longer via the bang-bang control as in the Nerlove-Arrow model. This, of course, is what one would expect when a model is made nonlinear with respect to the control variable u.

$$\dot{S} = au(1 - \frac{S}{M}) - bS. \tag{19}$$

$$x = \frac{S}{M}. \tag{20}$$

$$r = \frac{a}{M}, \ \delta = b + \frac{\dot{M}}{M}. \tag{21}$$

Now we can rewrite (19) as

$$\dot{x} = ru(1 - x) - \delta x, \ x(0) = x. \tag{22}$$

From now on we assume M, and hence δ and r, to be positive constants.

The optimal control problem can be stated as follows:

$$\begin{cases} \max\left\{ J = \int_0^T e^{-\rho t}(\pi x - u)dt \right\} \\ \text{subject to} \\ \dot{x} = ru(1-x) - \delta x,\ x(0) = x_0, \\ \text{the terminal state constraint} \\ x(T) = x_T, \\ \text{and the control constraint} \\ 0 \le u \le Q. \end{cases} \quad (23)$$

Here Q can be finite or infinite and the target market share x_T is in $[0, 1]$.

To make use of Green's theorem, it is convenient to consider times τ and θ, where $0 \leq \tau < \theta \leq T$, and the problem:

$$\max \left\{ J(\tau, \theta) = \int_{\tau}^{\theta} e^{-\rho t}(\pi x - u) dt \right\} \tag{24}$$

subject to

$$\dot{x} = ru(1-x) - \delta x, \;\; x(\tau) = A, \;\; x(\theta) = B, \tag{25}$$

$$0 \leq u \leq Q. \tag{26}$$

To change the objective function in (24) into a line integral along any feasible arc Γ_1 from (τ, A) to (θ, B) in (t, x)-space as shown in Figure 7.4, we multiply (25) by dt and obtain the formal relation

$$udt = \frac{dx + \delta x dt}{r(1-x)},$$

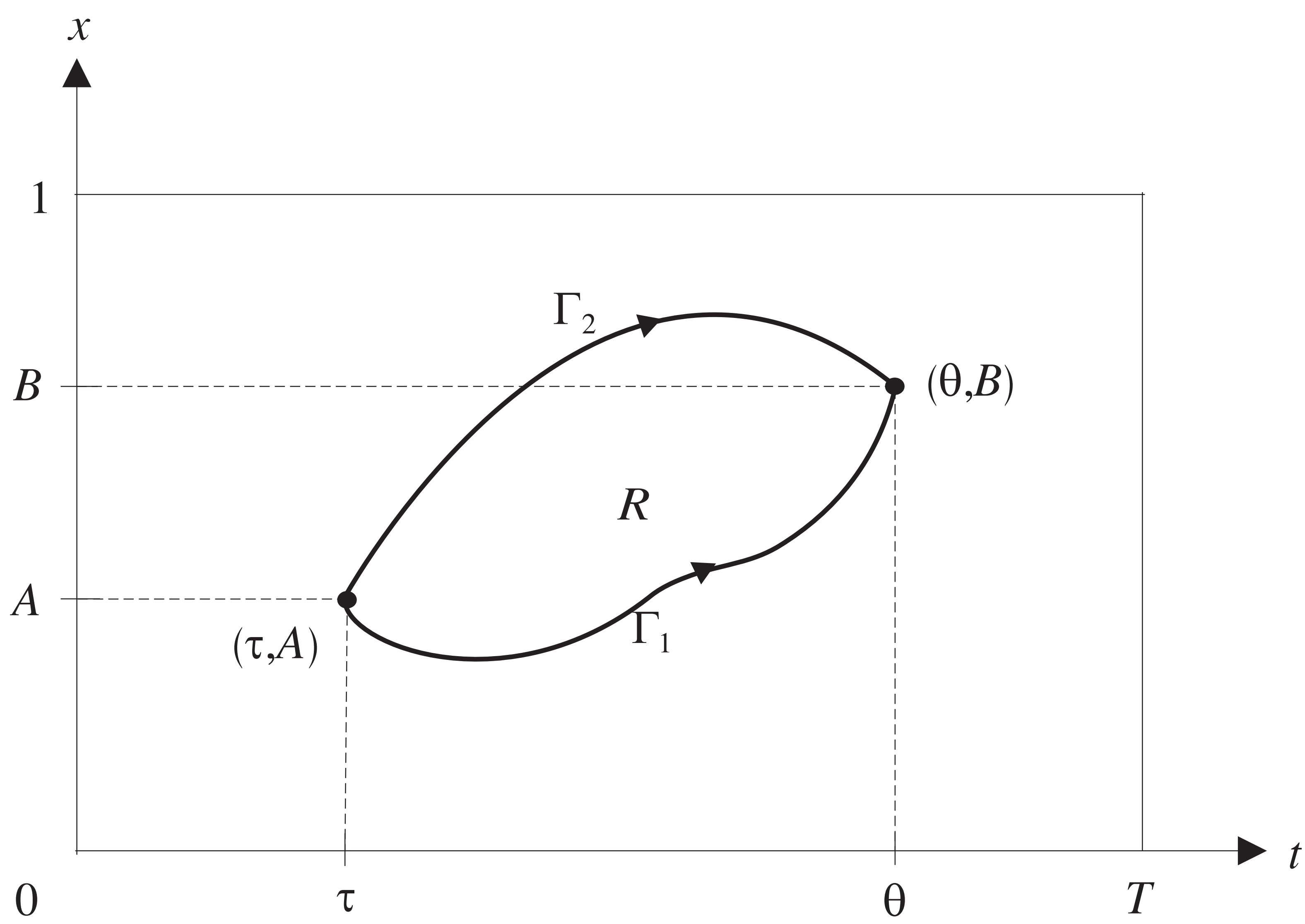
which we substitute into the objective function (24). Thus,

$$J_{\Gamma_1} = \int_{\Gamma_1} \left\{ \left[\pi x - \frac{\delta x}{r(1-x)} \right] e^{-\rho t} dt - \frac{1}{r(1-x)} e^{-\rho t} dx \right\}.$$

Consider another feasible arc Γ_2 from (τ, A) to (θ, B) lying above Γ_1 as shown in Figure 7.4. Let $\Gamma = \Gamma_1 - \Gamma_2$, where Γ is a simple closed curve traversed in the counter-clockwise direction. That is, Γ goes along Γ_1 in the direction of its arrow and along Γ_2 in the direction opposite to its arrow. We now have

$$J_\Gamma = J_{\Gamma_1 - \Gamma_2} = J_{\Gamma_1} - J_{\Gamma_2}. \tag{27}$$

Figure 7.4: Feasible Arcs in (t, x)-Space



Since Γ is a simple closed curve, we can use Green's theorem to express J_Γ as an area integral over the region R enclosed by Γ. Thus, treating x and t as independent variables, we can write

$$\begin{aligned} J_\Gamma &= \oint_\Gamma \left\{ \left[\pi x - \frac{\delta x}{r(1-x)} \right] e^{-\rho t} dt - \frac{1}{r(1-x)} e^{-\rho t} dx \right\} \\ &= \int\int_R \left\{ \frac{\partial}{\partial t} \left[\frac{-e^{-\rho t}}{r(1-x)} \right] - \frac{\partial}{\partial x} \left[(\pi x - \frac{\delta x}{r(1-x)}) e^{-\rho t} \right] \right\} dt dx \\ &= \int\int_R \left[\frac{\delta}{(1-x)^2} + \frac{\rho}{(1-x)} - \pi r \right] \frac{e^{-\rho t}}{r} dt dx. \end{aligned} \quad (28)$$

Denote the term in brackets of the integrand of (28) by

$$I(x) = \frac{\delta}{(1-x)^2} + \frac{\rho}{(1-x)} - \pi r. \quad (29)$$

Note that the sign of the integrand is the same as the sign of $I(x)$.

Lemma 7.1 (Comparison Lemma). *Let* Γ_1 *and* Γ_2 *be the lower and upper feasible arcs as shown in Figure* 7.4. *If* $I(x) \geq 0$ *for all* $(x,t) \in R$, *then the lower arc* Γ_1 *is at least as profitable as the upper arc* Γ_2. *Analogously, if* $I(x) \leq 0$ *for all* $(x,t) \in R$, *then* Γ_2 *is at least as profitable as* Γ_1.

Proof. If $I(x) \geq 0$ for all $(x,t) \in R$, then $J_\Gamma \geq 0$ from (28) and (29). Hence from (27), $J_{\Gamma_1} \geq J_{\Gamma_2}$. The proof of the other statement is similar. □

To make use of this lemma to find the optimal control for the problem stated in (23), we need to find regions where $I(x)$ is positive and where it is negative. For this, note first that $I(x)$ is an increasing function of x in $[0, 1]$. Solving $I(x) = 0$ will give that value of x, above which $I(x)$ is positive and below which $I(x)$ is negative. Since $I(x)$ is quadratic in $1/(1-x)$, we can use the quadratic formula (see Exercise 7.15) to get

$$x = 1 - \frac{2\delta}{-\rho \pm \sqrt{\rho^2 + 4\pi r\delta}}.$$

To keep x in the interval $[0, 1]$, we must choose the positive sign before the radical.

The optimal x must be nonnegative so we have

$$x^s = \max\left\{1 - \frac{2\delta}{-\rho + \sqrt{\rho^2 + 4\pi r\delta}}, 0\right\}, \qquad (30)$$

where the superscript s is used because this will turn out to be a singular trajectory. Since x^s is nonnegative, the control

$$u^s = \frac{\delta x^s}{r(1 - x^s)}. \qquad (31)$$

Note that $x^s = 0$ and $u^s = 0$ if, and only if, $\pi r \le \delta + \rho$.

FIGURE 7.3: OPTIMAL TRAJECTORY FOR CASE 1: $x_0 \le x^s$ AND $x^s \ge x_T$

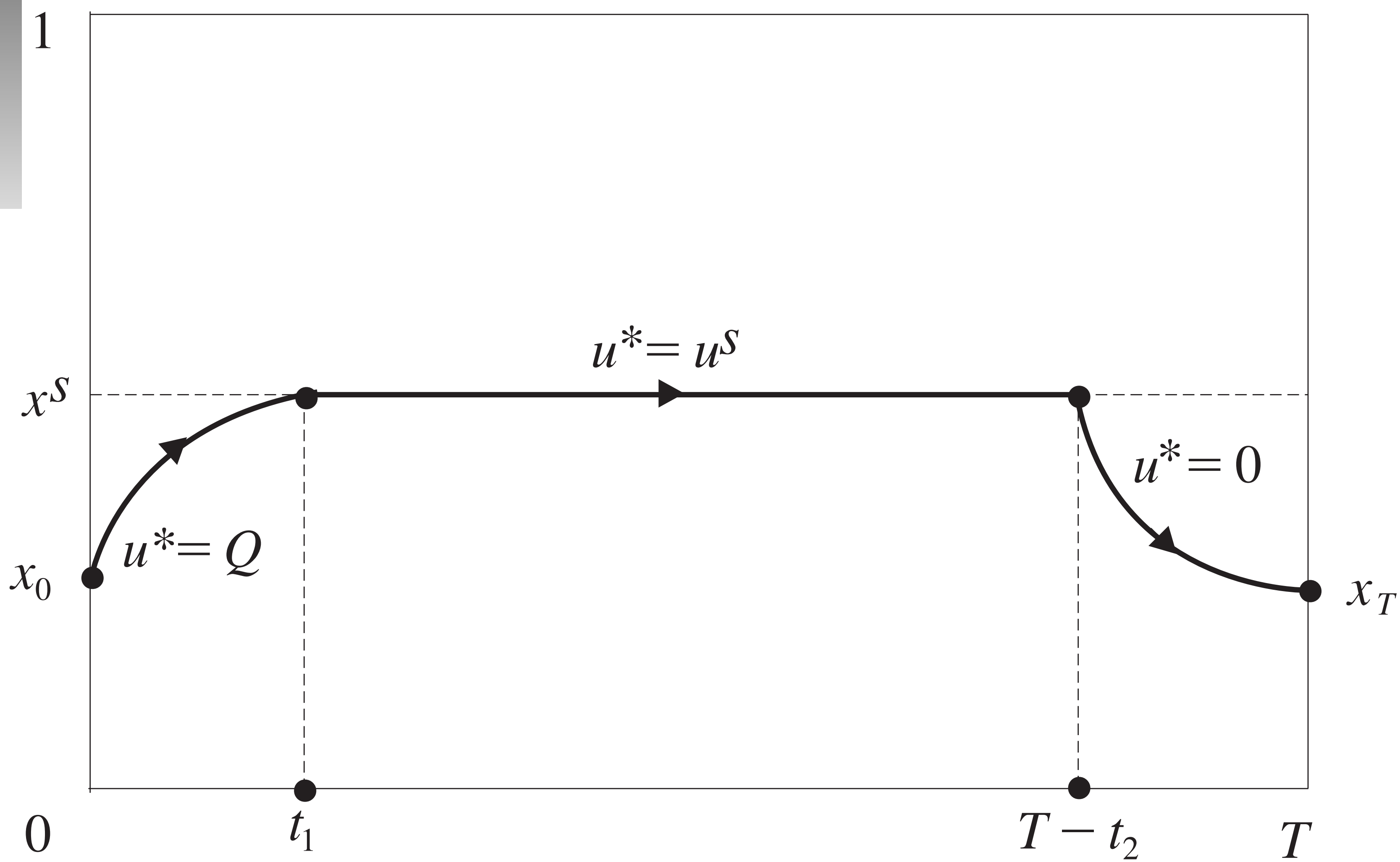


FIGURE 7.5: OPTIMAL TRAJECTORY FOR CASE 2: $x_0 \le x^s$ AND $x^s \le x_T$

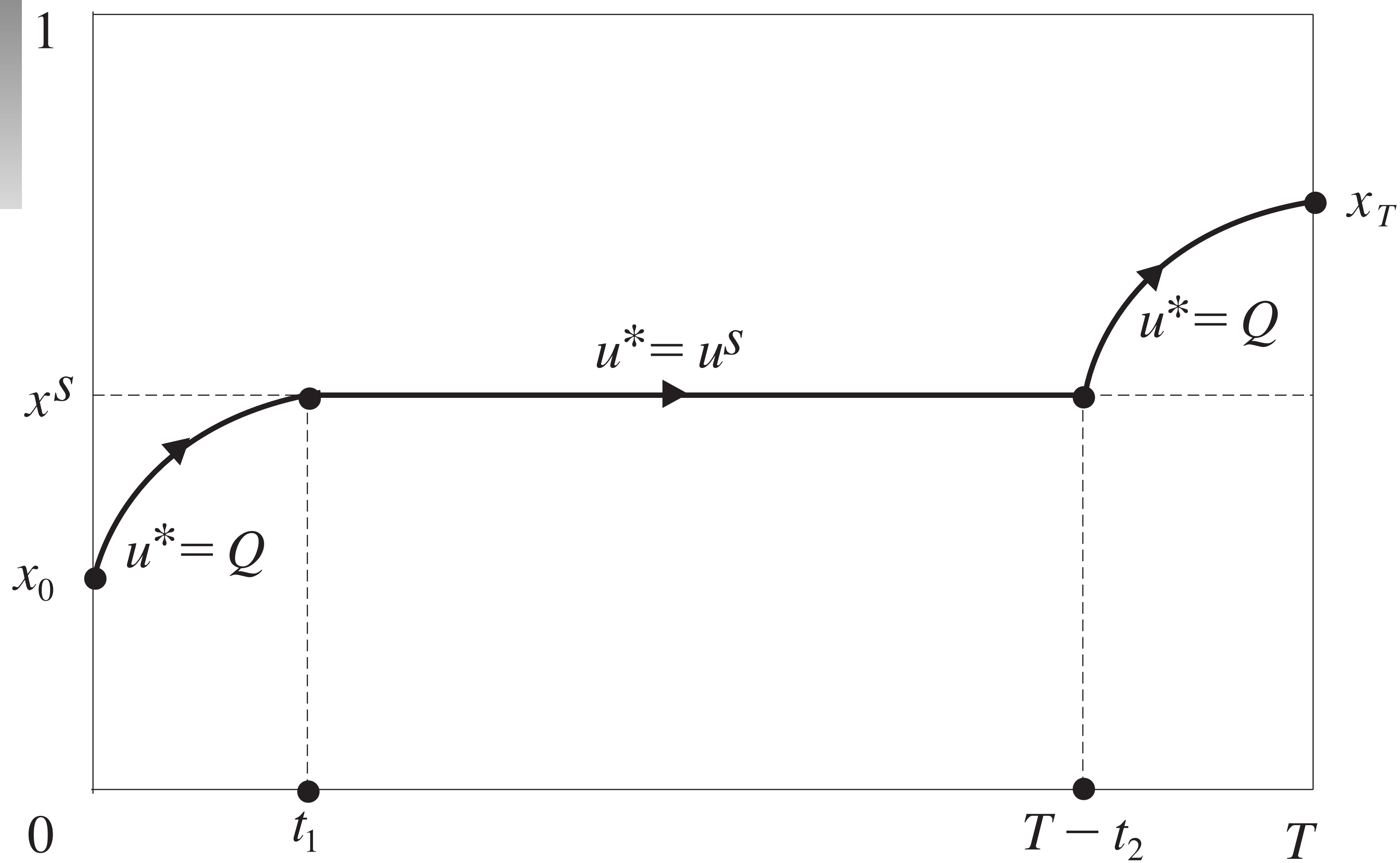


FIGURE 7.7: OPTIMAL TRAJECTORY FOR CASE 3: $x_0 \geq x^s$ AND $x^s \geq x_T$

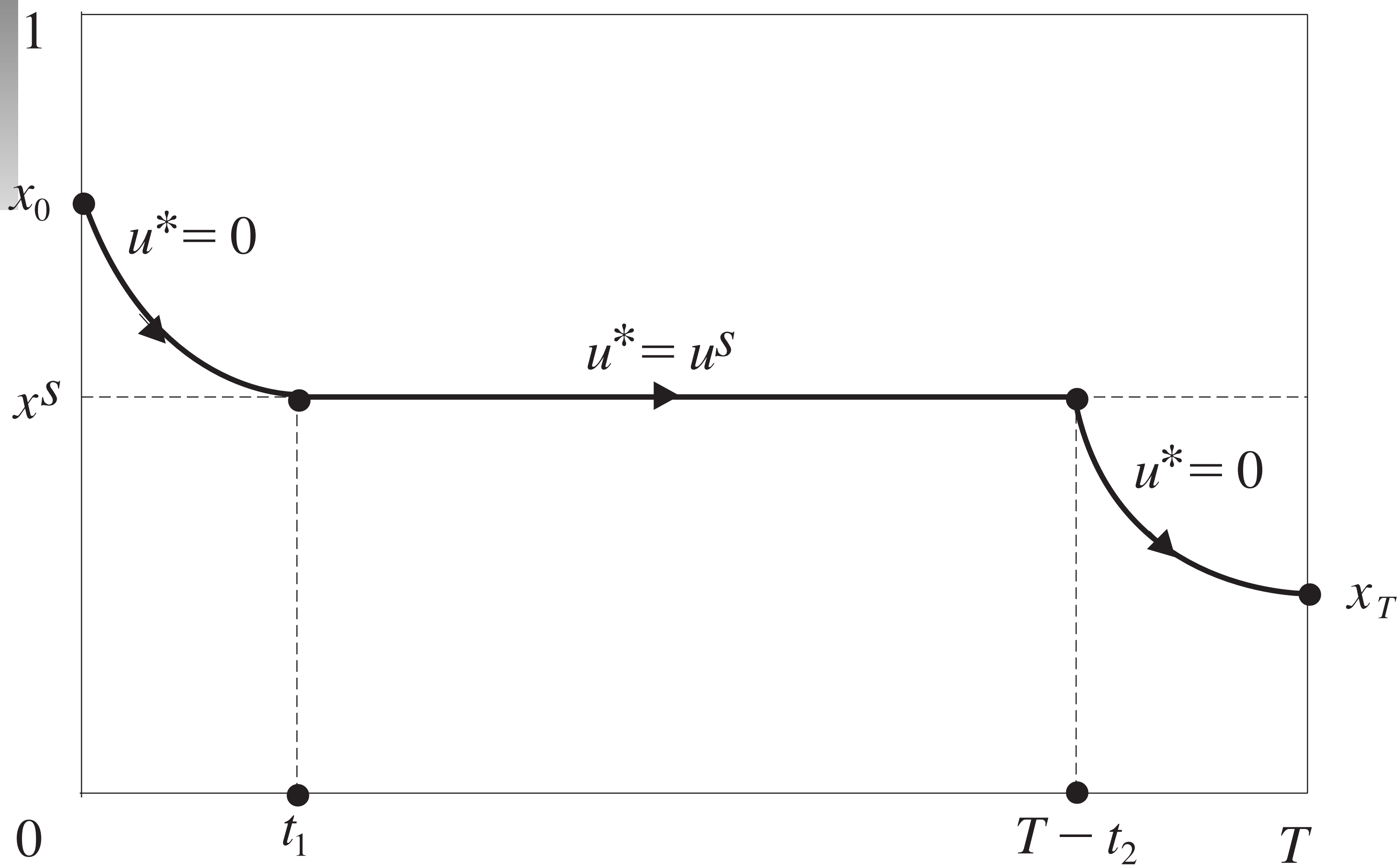
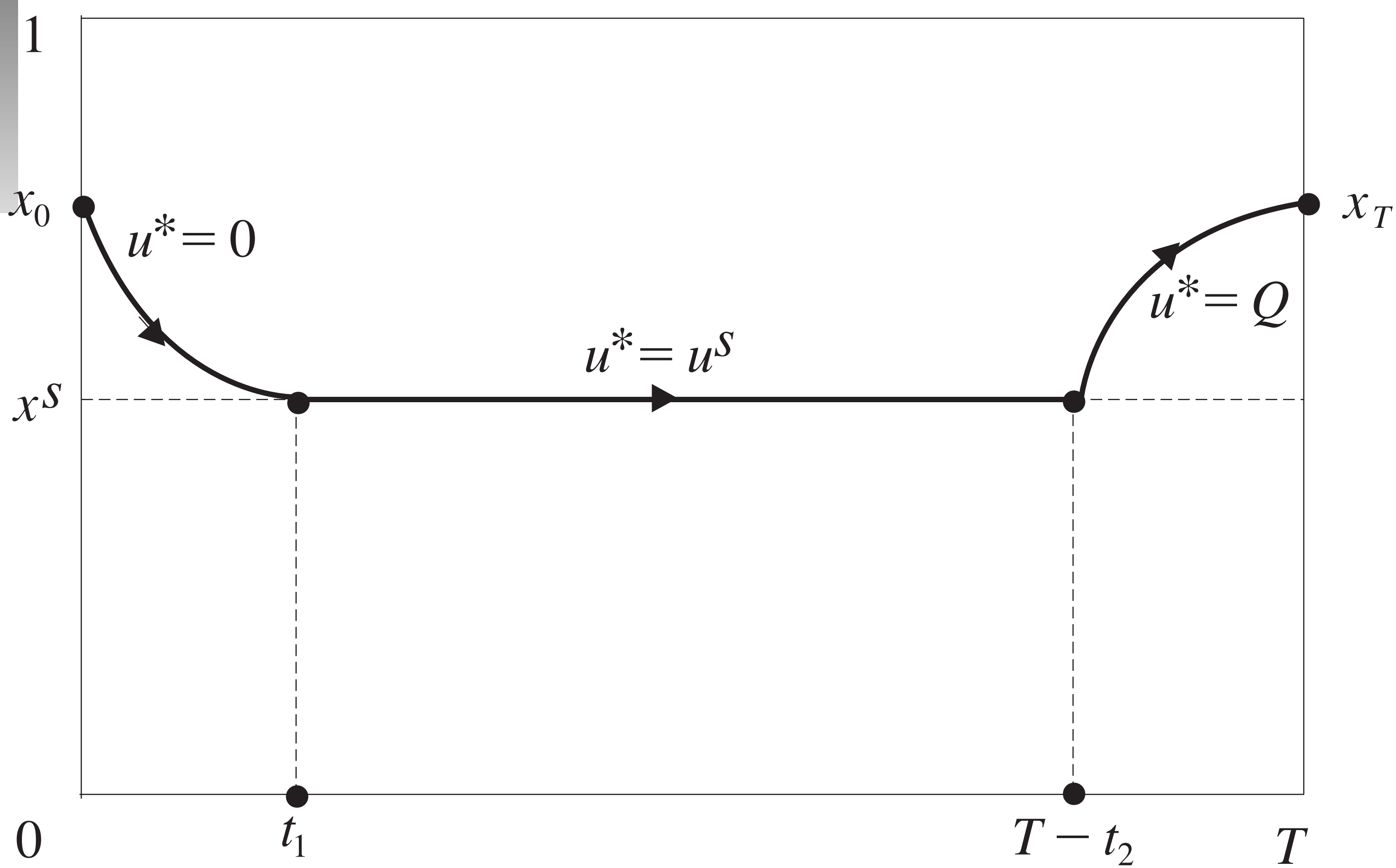


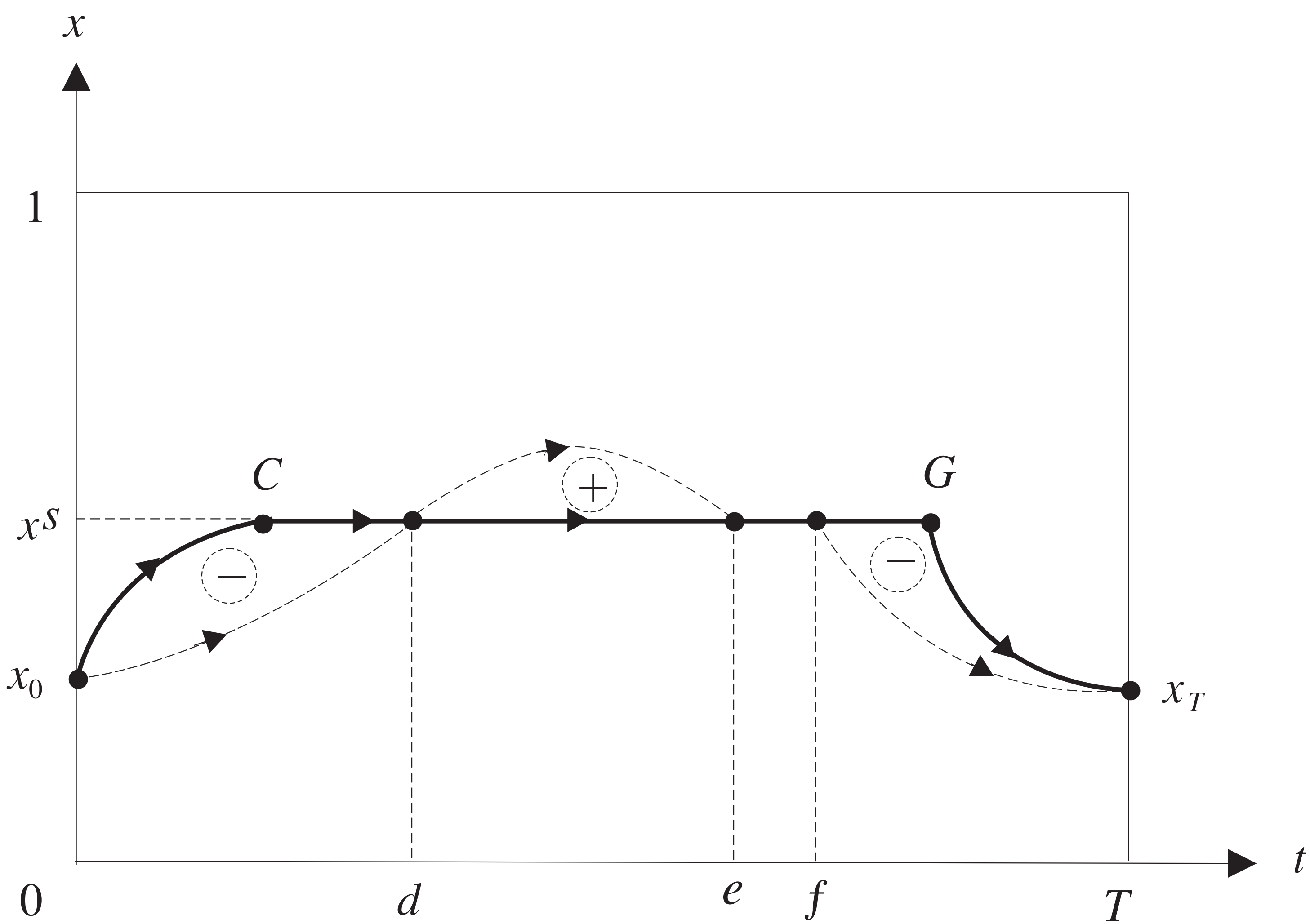
FIGURE 7.6: OPTIMAL TRAJECTORY FOR CASE 4: $x_0 \geq x^s$ AND $x^s \leq x_T$



Theorem 7.1 *Let T be large and let x_T be reachable from x_0. For the Cases* 1-4 *of inequalities relating x_0 and x_T to x^s, the optimal trajectories are given in Figures* 7.5-7.8, *respectively.*

Proof. We give details for Case 1 only. The proofs for the other cases are similar. Figure 7.9 shows the optimal trajectory for Figure 7.5 together with an arbitrarily chosen feasible trajectory, shown dotted. It should be clear that the dotted trajectory cannot cross the arc x_0 to C, since $u = Q$ on that arc. Similarly the dotted trajectory cannot cross the arc G to x_T, because $u = 0$ on that arc.

Figure 7.9: Optimal Trajectory (Solid line)



We subdivide the interval $[0, T]$ into subintervals over which the dotted arc is either above, below, or identical to the solid arc. In Figure 7.9 these subintervals are $[0, d]$, $[d, e]$, $[e, f]$, and $[f, T]$. Because $I(x)$ is positive for $x > x^s$ and $I(x)$ is negative for $x < x^s$, the regions enclosed by the two trajectories have been marked with $+$ or $-$ sign depending on whether $I(x)$ is positive or negative on the regions, respectively. By Lemma 7.1, the solid arc is better than the dotted arc in the subintervals $[0, d]$, $[d, e]$, and $[f, T]$; in interval $[e, f]$, they have identical values. Hence the dotted trajectory is inferior to the solid trajectory. This proof can be extended to any (countable) number of crossings of the trajectories; see Sethi (1977b). $\square$

Theorem 7.2 *Let T be small, i.e., $T < t_1 + t_2$, and let x_T be reachable from x_0. For the two possible Cases* 1 *and* 2 *of inequalities relating x_0 to x_T and x^s, the optimal trajectories are given in Figures* 7.10 *and* 7.11, *respectively.*

Proof. The requirement of feasibility when T is small rules out cases where x_0 and x_T are on opposite sides of or equal to x^s. The proofs of optimality of the trajectories shown in Figures 7.10 and 7.11 are similar to the proofs of the parts of Theorem 7.1, and are left as Exercise 7.23. In Figures 7.10 and 7.11, it is possible to have either $t_1 \geq T$ or $t_2 \geq T$. Try sketching some of these special cases. □

Figure 7.10: Optimal Trajectory When T Is Small in Case 1: $x_0 < x^s$ and $x_T < x^s$

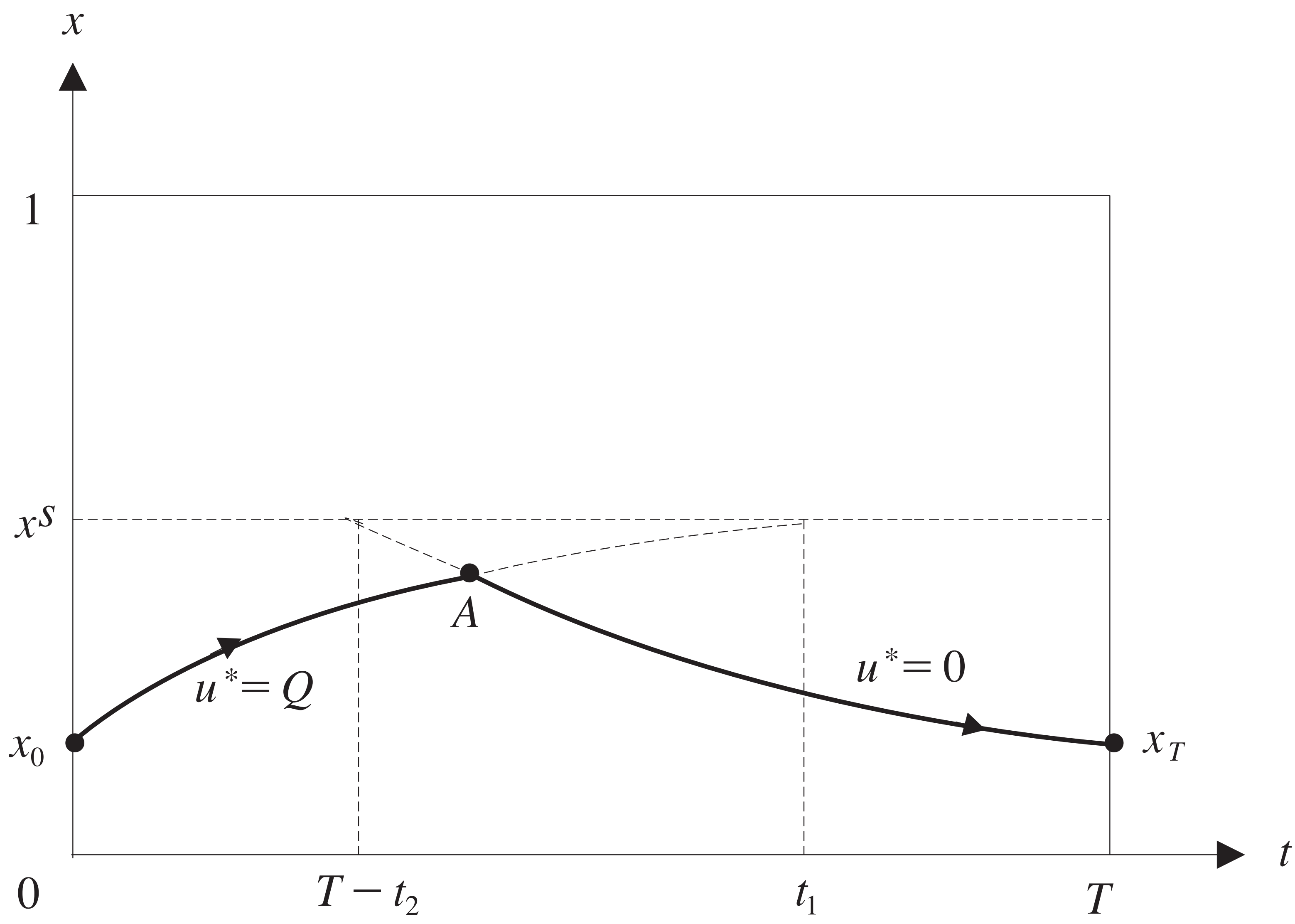


Figure 7.11: Optimal Trajectory When T Is Small in Case 2: $x_0 > x^s$ and $x_T < x^s$

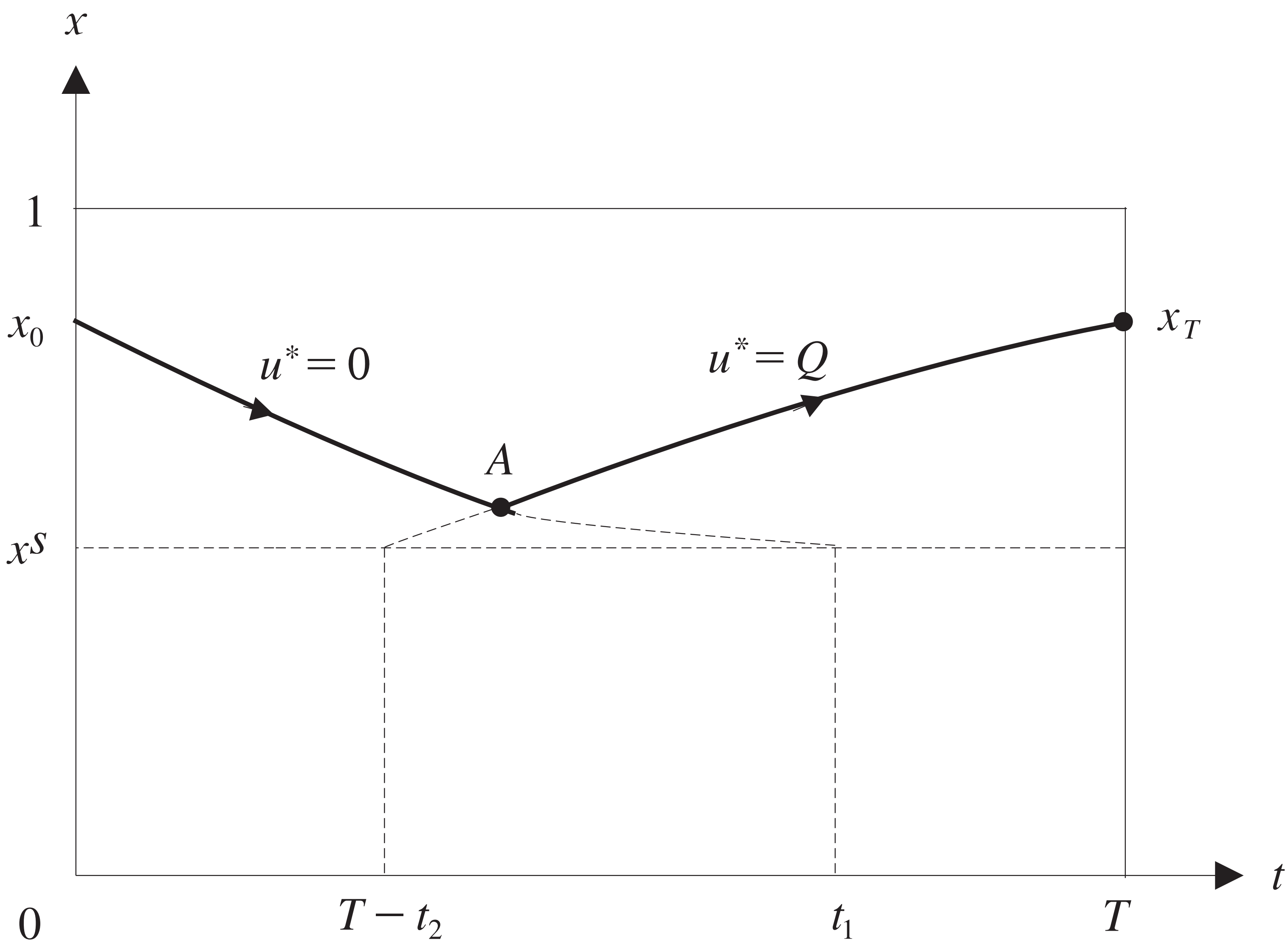
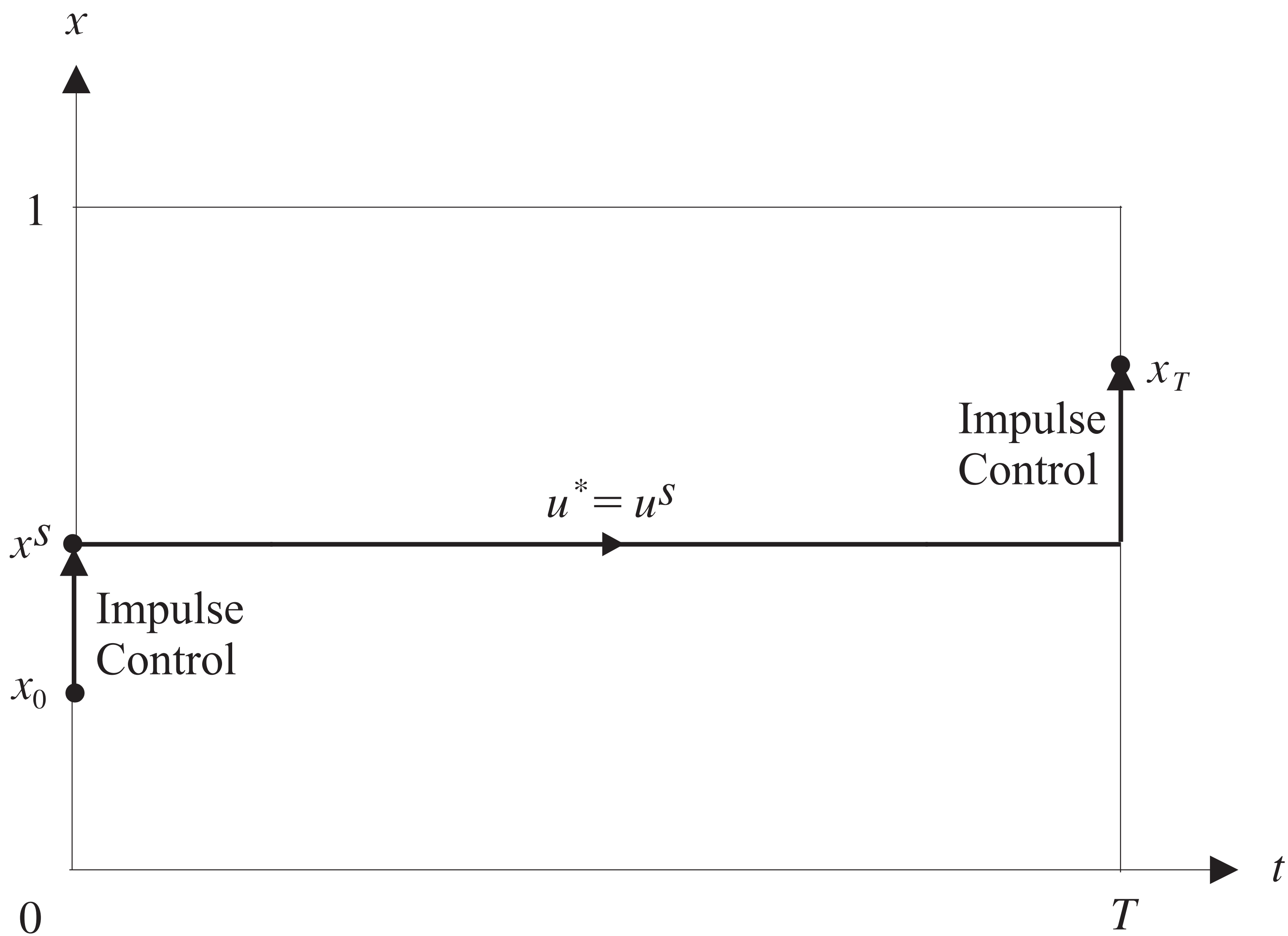


Figure 7.12: Optimal Trajectory for Case 2 of Theorem 7.1 for $Q = \infty$



The effect on the objective function of the *impulse control* $\text{imp}(x_0, x^s)$ is

$$x(\varepsilon) = \left(x_0 - \frac{ru}{\delta + ru}\right) e^{-(\delta+ru)\varepsilon} + \frac{ru}{\delta + ru}.$$

We must choose $u(\varepsilon)$ so that $x(\varepsilon)$ is x^s. It should be clear that $u(\varepsilon) \to \infty$ as $\varepsilon \to 0$. With $F(x, u, \tau) = \pi x(\tau) - u(\tau)$ and $t = 0$ in (1.23), we have the impulse

$$I = \text{imp}(x_0, x^s, 0) = \lim_{\varepsilon \to 0}[-u(\varepsilon)\varepsilon].$$

Letting $\varepsilon \to 0$, $-u(\varepsilon)\varepsilon \to I$, and $x(\varepsilon) = x^s$ in the expression for $x(\varepsilon)$ obtained above. This gives

$$x(0+) = e^{rI}(x_0 - 1) + 1, \quad \text{and}$$

$$\text{imp}(x_0, x^s, 0) = -\frac{1}{r} \ln\left[\frac{1 - x_0}{1 - x^s}\right]. \qquad (32)$$

$$
\begin{aligned}
H &= \pi x - u + \lambda[ru(1-x) - \delta x] \\
&= \pi x - \delta\lambda x + u[-1 + r\lambda(1-x)], \qquad (33)
\end{aligned}
$$

$$
L = H + \mu(Q - u), \qquad (34)
$$

$$
\dot{\lambda} = \rho\lambda - \frac{\partial L}{\partial x} = \rho\lambda + \lambda(ru + \delta) - \pi, \qquad (35)
$$

where $\lambda(T)$ is a constant, as in Row 2 of Table 3.1, that must be determined. Furthermore, the Lagrange multiplier μ in (34) must satisfy

$$
\mu \geq 0, \quad \mu(Q - u) = 0. \qquad (36)
$$

From (33) we notice that the Hamiltonian is linear in the control. So the optimal control is

$$u^*(t) = \text{bang}[0, Q; W(t)], \tag{37}$$

where

$$W(t) = W(x(t), \lambda(t)) = r\lambda(t)(1 - x(t)) - 1. \tag{38}$$

We now formulate the infinite horizon version of (23):

$$\begin{cases} \max \left\{ J = \int_0^\infty e^{-\rho t}(\pi x - u)dt \right\} \\ \text{subject to} \\ \dot{x} = ru(1-x) - \delta x, \ x(0) = x_0, \\ 0 \le u \le Q. \end{cases} \tag{39}$$

When Q is small, i.e., $Q < u^s$, it is not possible to follow the turnpike $x = x^s$, because that would require $u = u^s$, which is not a feasible control. Intuitively, it seems clear that the "closest" stationary path which we can follow is the path obtained by setting $\dot{x} = 0$ and $u = Q$, the largest possible control, in the state equation of (39).

This gives

$$\bar{x} = \frac{rQ}{rQ + \delta}, \tag{40}$$

and correspondingly we obtain

$$\bar{\lambda} = \frac{\pi}{\rho + \delta + rQ} \tag{41}$$

by setting $u = Q$ and $\dot{\lambda} = 0$ in (35) and solving for λ. More specifically, we state the following theorems which give the turnpike and optimal control when Q is small. Let us define $\hat{x}$ and $\bar{\mu}$ such that $W(\hat{x}, \bar{\lambda}) = r\bar{\lambda}(1 - \hat{x}) - 1 = 0$ and $L_u(\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu}) = W(\bar{x}, \bar{\lambda}) - \bar{\mu} = 0$. Thus,

$$\hat{x} = 1 - 1/r\bar{\lambda}, \tag{42}$$

$$\bar{\mu} = r\bar{\lambda}(1 - \bar{x}) - 1. \tag{43}$$

Theorem 7.3 *When Q is small, the following quantities*

$$\{\bar{x}, Q, \bar{\lambda}, \bar{\mu}\} \tag{44}$$

form a turnpike.

Proof. We show that the conditions in (3.73) hold for (44). The first two are obvious. By Exercise 7.28 we know $\bar{x} \leq \hat{x}$, which, from definitions (42) and (43), implies $\bar{\mu} \geq 0$. Furthermore $\bar{u} = Q$, so (36) holds and the third condition of (3.73) also holds. Finally because $W = \bar{\mu}$ from (38) and (43), it follows that $W \geq 0$, so the Hamiltonian maximizing condition of (3.73) holds with $\bar{u} = Q$, and the proof is complete. $\square$

FIGURE 7.13: OPTIMAL TRAJECTORIES FOR $x(0) < \hat{x}$

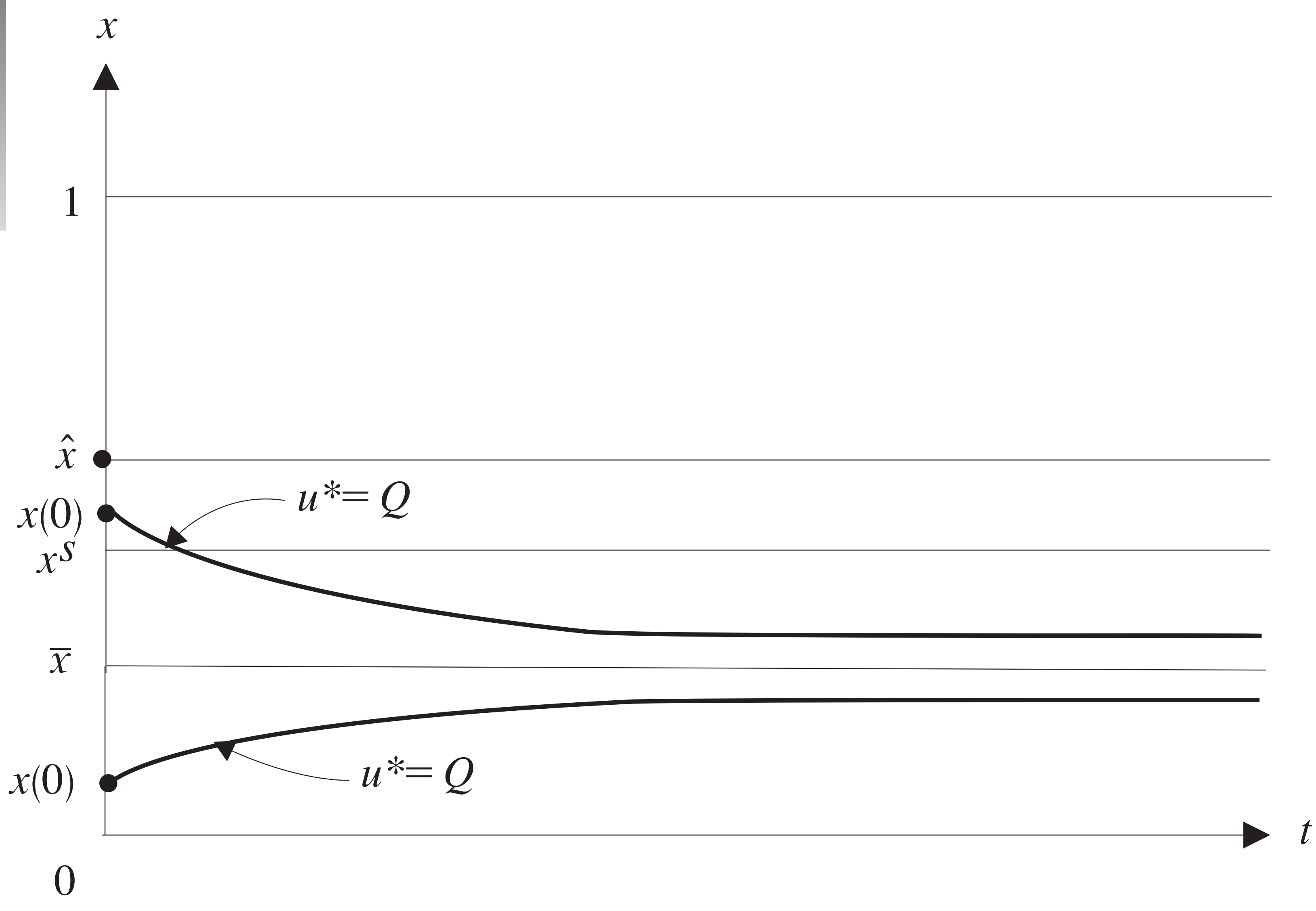
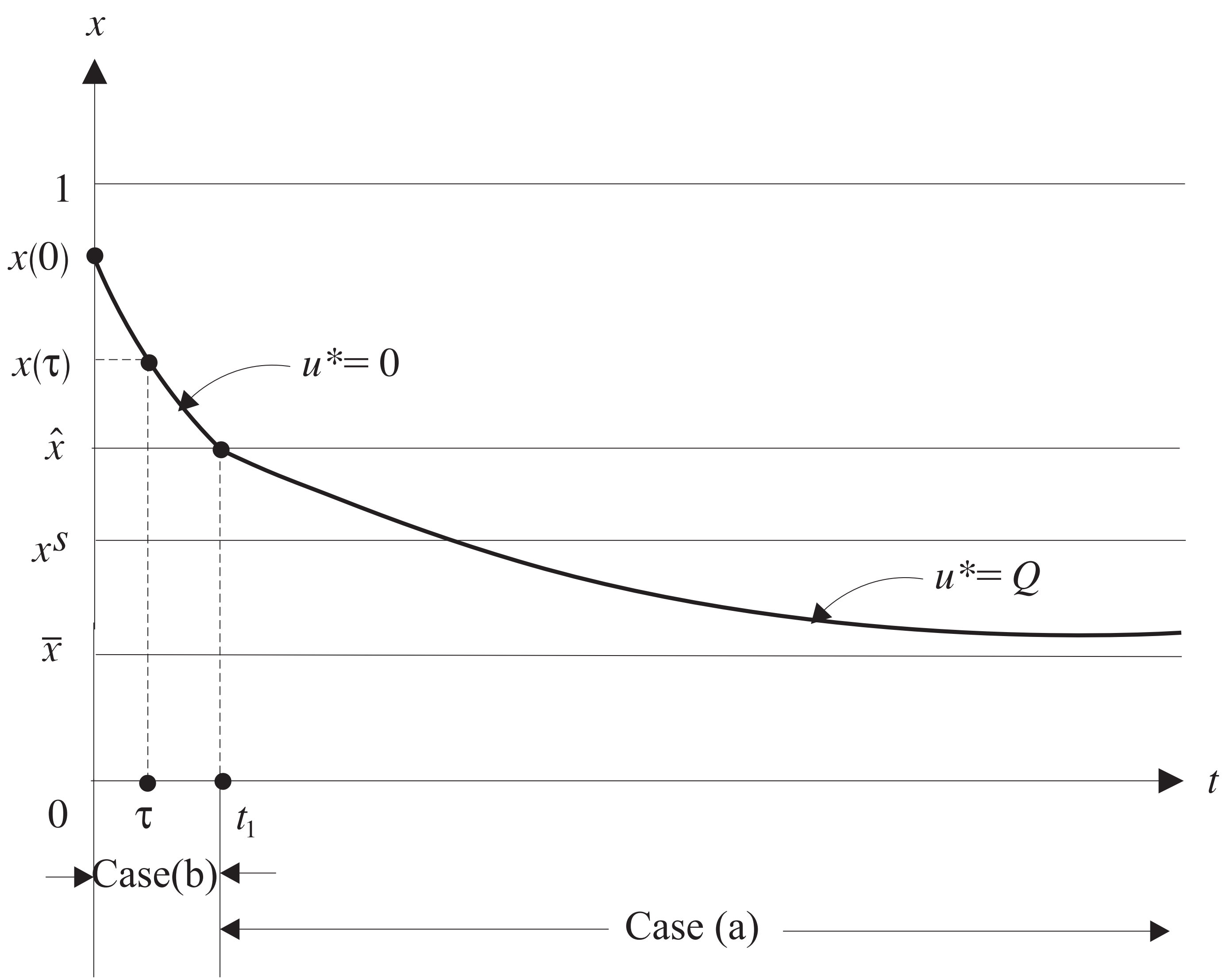


FIGURE 7.14: OPTIMAL TRAJECTORY FOR $x(0) > \hat{x}$



Theorem 7.4 *When Q is small, the optimal control at time τ is given by:*

(a) *If $x(\tau) \leq \hat{x}$, then $u^*(\tau) = Q$.*

(b) *If $x(\tau) > \hat{x}$, then $u^*(\tau) = 0$.*

Proof. (a) We set $\lambda(t) = \bar{\lambda}$ for all $t \geq \tau$ and note that λ satisfies the adjoint equation (35) and the transversality condition (3.70).

By Exercise 7.28 and the assumption that $x(\tau) \leq \hat{x}$, we know that $x(t) \leq \hat{x}$ for all t. The proof that (36) and (37) hold for all $t \geq \tau$ relies on the fact that $x(t) \leq \hat{x}$ and on an argument similar to the proof of the previous theorem.

Figure 7.13 shows the optimal trajectories for $x(0) < \hat{x}$ and two different starting values $x(0)$, one above and the other below $\hat{x}$. Note that in this figure we are always in Case (a) since $x(\tau) \leq \hat{x}$ for all $\tau \geq 0$.

(b) Assume $x(0) > \hat{x}$. For this case we will show that the optimal trajectory is as shown in Figure 7.14, which is obtained by applying $u = 0$ until $x = \hat{x}$ and $u = Q$ thereafter. Using this policy we can find the time t_1 at which $x(t_1) = \hat{x}$, by solving the state equation in (39) with $u = 0$. This gives

$$t_1 = \frac{1}{\delta} \ln \frac{x(0)}{\hat{x}}. \tag{45}$$

Clearly for $t \geq t_1$, the policy $u = Q$ is optimal because Case (a) applies. We now consider the interval $[0, t_1)$, where we set $u = 0$. Let τ be any time in this interval as shown in Figure 7.14, and let $x(\tau)$ be the corresponding value of the state variable. Then $x(\tau) = x_0 e^{-\delta\tau}$ in the interval $[0, t_1]$. With $u = 0$ in (35), the adjoint equation on $[0, t_1)$ becomes $\dot{\lambda} = (\rho + \delta)\lambda - \pi$. So we solve the adjoint equation with $\lambda(t_1) = \bar{\lambda}$ and obtain

$$\lambda(\tau) = \frac{\pi}{\rho + \delta} + \left(\bar{\lambda} - \frac{\pi}{\rho + \delta}\right) e^{(\rho+\delta)(\tau - t_1)}, \ \tau \in [0, t_1]. \tag{46}$$

In Exercise 7.31, you are asked to show that the switching function $W(t)$ defined in (38) is negative in the interval $[0, t_1)$ and $W(t_1) = 0$.

Therefore by (37), the policy $u = 0$ used in deriving (45) and (46) satisfies the maximum principle. This policy "joins" the optimal policy after t_1 because $\lambda(t_1) = \bar{\lambda}$.

In this book the sufficiency of the transversality condition (3.70) was stated under the hypothesis that the derived Hamiltonian was concave; see Theorem 2.1. In the present example, this hypothesis does not hold. However, as mentioned in Section 7.2.3, for this simple bilinear problem it can be shown that (3.70) is sufficient for optimality. Because of the technical nature of this issue we omit the details. □