



CHAPTER 5

APPLICATIONS TO FINANCE

- The Simple Cash Balance Problem
- Optimal Financing of a Corporation
- A Stochastic Optimal Consumption – Investment Problem (Chapter 13)

5.1. THE SIMPLE CASH BALANCE PROBLEM

To determine optimal cash levels to meet the demand for cash at minimum total discounted cost.

Too much cash \Rightarrow opportunity loss of not being able to earn higher returns by buying securities.

Too little cash \Rightarrow higher transaction costs when securities are sold to meet the demand for cash.

- T = the time horizon,
- $x(t)$ = the cash balance in dollars at time t ,
- $y(t)$ = the security balance in dollars at time t ,
- $d(t)$ = the instantaneous rate of demand for cash;
 $d(t)$ can be positive or negative,
- $u(t)$ = the rate of sale of securities in dollars; a negative sales rate means a rate of purchase,
- $r_1(t)$ = the interest rate earned on the cash balance,
- $r_2(t)$ = the interest rate earned on the security balance,
- α = the broker's commission in dollars per dollar's worth of securities bought or sold; $0 < \alpha < 1$.

The state equations are

$$\dot{x} = r_1 x - d + u - \alpha|u|, \quad x(0) = x_0, \quad (1)$$

$$\dot{y} = r_2 y - u, \quad y(0) = y_0, \quad (2)$$

and the control constraints are

$$-U_2 \leq u(t) \leq U_1, \quad (3)$$

The objective function is to

$$\text{maximize } \{J = [x(T) + y(T)]\} \quad (4)$$

subject to (1)-(3). Note that the problem is in the linear Mayer form.

SOLUTION BY THE MAXIMUM PRINCIPLE

The Hamiltonian function

$$H = \lambda_1(r_1x - d + u - \alpha|u|) + \lambda_2(r_2y - u). \quad (5)$$

The adjoint variables satisfy the differential equations

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x} = -\lambda_1 r_1, \quad \lambda_1(T) = 1, \quad (6)$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial y} = -\lambda_2 r_2, \quad \lambda_2(T) = 1. \quad (7)$$

INTERPRETATION OF THE SOLUTION

It is easy to solve these as

$$\lambda_1(t) = e^{\int_t^T r_1(\tau) d\tau}, \quad (8)$$

$$\lambda_2(t) = e^{\int_t^T r_2(\tau) d\tau}. \quad (9)$$

$\lambda_1(t)$ is the future value (at time T) of one dollar held in the cash account from time t to T and, likewise, $\lambda_2(t)$ is the future value of one dollar invested in securities from time t to T . Thus, the adjoint variables have natural interpretations as the actuarial evaluations of competitive investments at each point of time.

In order to deal with the absolute value function, we write the control variable u as the difference of two nonnegative variables, as suggested in Remark 3.3, i.e.,

$$u = u_1 - u_2, \quad u_1 \geq 0, \quad u_2 \geq 0. \quad (10)$$

We also impose the quadratic constraint

$$u_1 u_2 = 0, \quad (11)$$

so that at most one of u_1 and u_2 can be nonzero. Given (10) and (11) we can write

$$|u| = u_1 + u_2. \quad (12)$$

We can now substitute (10) and (12) into the Hamiltonian (5) and reproduce below the part which depends on control variables u_1 and u_2 , and denote it by W . Thus,

$$W = u_1[(1 - \alpha)\lambda_1 - \lambda_2] - u_2[(1 + \alpha)\lambda_1 - \lambda_2]. \quad (13)$$

W is linear in u_1 and u_2 so that the optimal strategy is bang-bang and is as follows:

$$u^* = u_1^* - u_2^*, \quad (14)$$

$$u_1^* = \text{bang}[0, U_1; (1 - \alpha)\lambda_1 - \lambda_2], \quad (15)$$

$$u_2^* = \text{bang}[0, U_2; -(1 + \alpha)\lambda_1 + \lambda_2]. \quad (16)$$

$$(1 - \alpha)\lambda_1(t) > \lambda_2(t) \Rightarrow u_1^* = U_1$$

Sell at the maximum allowable rate if the future value of a dollar less the broker's commission (i.e., the future value of $(1 - \alpha)$ dollars) is greater than the future value of a dollar's worth of securities; and do not sell if these future values are in reverse order.

$$(1 + \alpha)\lambda_1(t) < \lambda_2(t) \Rightarrow u_2^* = U_2$$

Purchase securities at the maximum rate, if the future value of a dollar plus the commission is less than the future value of a dollar's worth of securities.

FIGURE 5.1: OPTIMAL POLICY SHOWN IN (λ_1, λ_2) SPACE

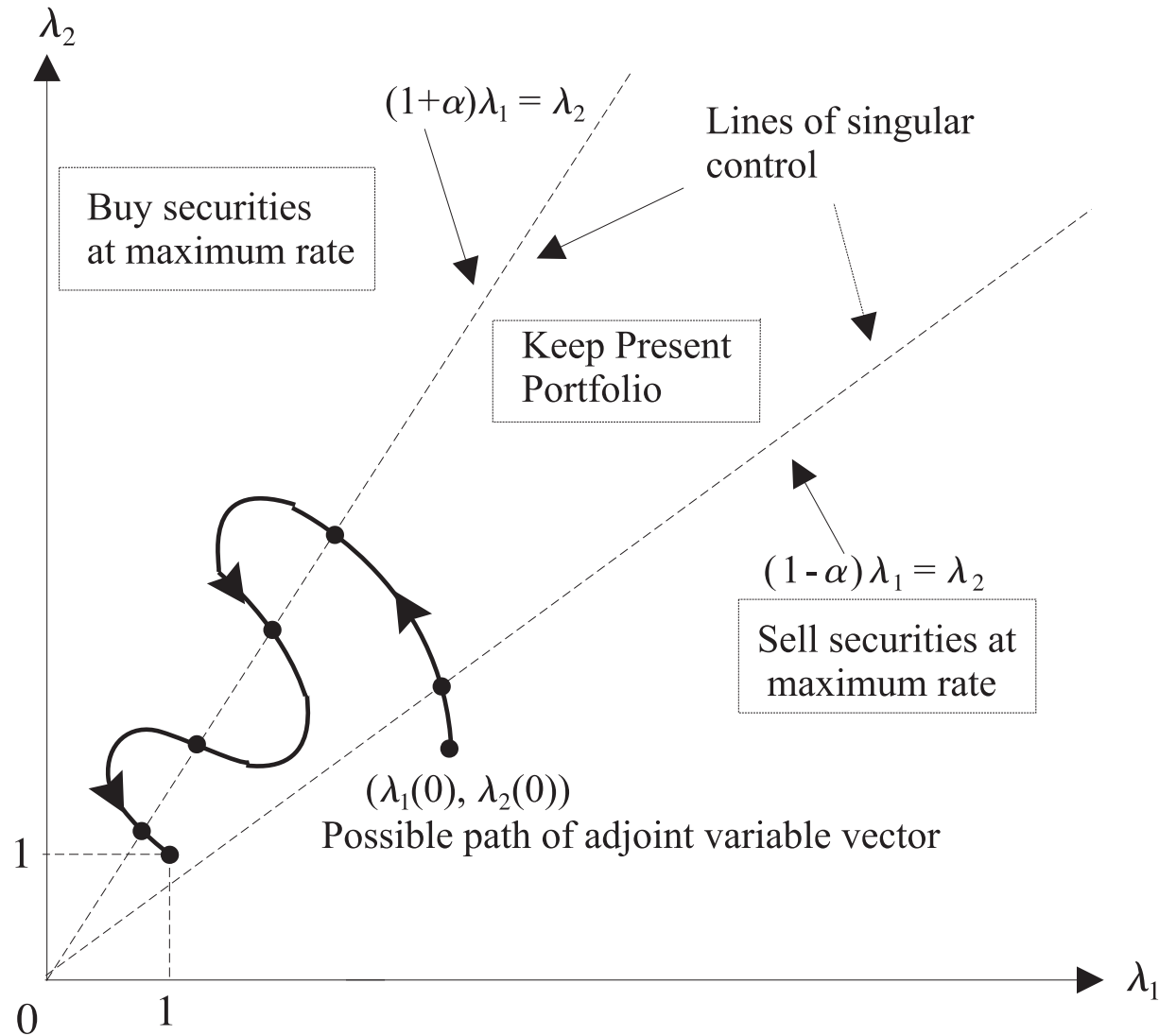
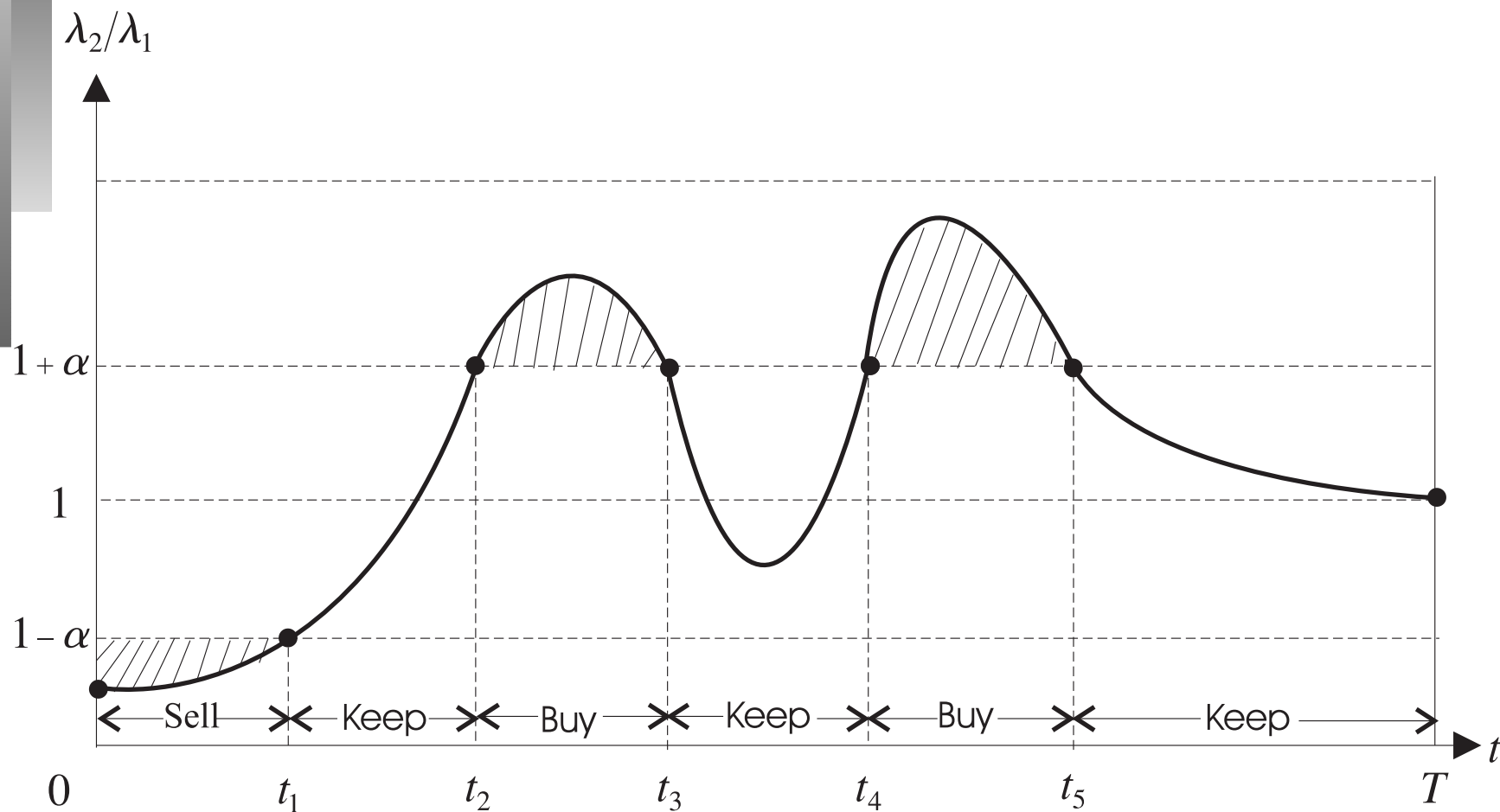


FIGURE 5.2: OPTIMAL POLICY SHOWN IN $(t, \lambda_2/\lambda_1)$ SPACE



$$x(t) \geq 0 \quad \text{and} \quad y(t) \geq 0. \quad (17)$$

$$L = H + \eta_1 \dot{x} + \eta_2 \dot{y} = \lambda_1(r_1 x - d + u - \alpha|u|) + \lambda_2(r_2 y - u) \\ + \eta_1(r_1 x - d + u - \alpha|u|) + \eta_2(r_2 y - u), \quad (18)$$

$$\dot{\lambda}_1 = -\frac{\partial L}{\partial x} = -(\lambda_1 + \eta_1)r_1, \quad \lambda_1(T^-) \geq 1, \quad [\lambda_1(T) - 1]x(T) = 0, \quad (19)$$

$$\dot{\lambda}_2 = -\frac{\partial L}{\partial y} = -(\lambda_2 + \eta_2)r_2, \quad \lambda_2(T^-) \geq 1, \quad [\lambda_2(T) - 1]y(T) = 0, \quad (20)$$

$$\eta_1(t) \geq 0, \quad \eta_1(t)x(t) = 0, \quad \dot{\eta}_1(t) \leq 0, \quad (21)$$

$$\eta_2(t) \geq 0, \quad \eta_2(t)y(t) = 0, \quad \dot{\eta}_2(t) \leq 0, \quad (22)$$

$$\frac{\partial L}{\partial u} = 0 \quad (23)$$

EXAMPLE 5.1

$$r_1(t) = \begin{cases} 0 & \text{for } 0 \leq t < 5, \\ 0.3 & \text{for } 5 \leq t \leq 10, \end{cases} \quad (24)$$

$$r_2(t) = 0.1 \quad \text{for } 0 \leq t \leq 10. \quad (25)$$

$$u^*(t) = 0 \text{ for } 0 \leq t < 5, \quad u^*(5) = \text{imp}(3e^{0.5}, 0, 5) \quad (26)$$

$$u^*(t) = 0 \text{ for } 5 < t \leq 10. \quad (27)$$

$$y(5 + \delta t) = y(5 - \delta t)e^{0.2\delta t} - \int_{5-\delta t}^{5+\delta t} e^{0.1(5+\delta t-\tau)} \frac{3e^{0.5}}{2\delta t} d\tau \quad (28)$$

$$= y(5 - \delta t)e^{0.2\delta t} - \frac{3e^{0.5}}{0.2\delta t} \left[e^{0.2\delta t} - 1 \right]$$

$$= y(5 - \delta t)e^{0.2\delta t} - \frac{3e^{0.5}}{0.2\delta t} [0.2\delta t + o(\delta t)].$$

EXAMPLE 5.1 CONT.

$$\lambda_1 + \eta_1 = \lambda_2 + \eta_2. \quad (29)$$

$$\lambda_1(t) = e^{0.3(10-t)},$$

$$\lambda_2(t) = 1 - \frac{1}{3}[1 - e^{0.3(10-t)}] = \frac{2}{3} + \frac{1}{3}e^{0.3(10-t)},$$

$$\eta_2(t) = \lambda_1 - \lambda_2 = \frac{2}{3}[e^{0.3(10-t)} - 1],$$

$$\dot{\lambda}_1 = -(\lambda_1 + \eta_1)r_1(t) = 0, \quad \lambda_1(5^-) = e^{1.5},$$

$$\dot{\lambda}_2 = -(\lambda_2 + \eta_2)r_2(t) = -0.1\lambda_2, \quad \lambda_2(5^-) = e^{1.5}.$$

OPTIMAL FINANCING MODEL

- $y(t)$ = the value of the firm's assets or invested capital at time t ,
- $x(t)$ = the current earnings rate in dollars per unit time at time t ,
- $u(t)$ = the external or new equity financing expressed as a multiple of current earnings; $u \geq 0$,
- $v(t)$ = the fraction of current earnings retained, i.e., $1 - v(t)$ represents the rate of dividend payout; $0 \leq v(t) \leq 1$,
- $1 - c$ = the proportional floatation (i.e., transaction) cost for external equity; c a constant, $0 \leq c < 1$,
- ρ = the continuous discount rate (assumed constant); known commonly as the stockholder's required rate of return,
- r = the actual rate of return (assumed constant) on the firm's invested capital $r > \rho$,
- g = the upper bound on the growth rate of the firm's assets,
- T = the planning horizon; $T < \infty$ ($T = \infty$ in Section 5.2.4).

The rate of change in the current earnings rate is given by

$$\dot{x} = r\dot{y} = r(cu + v)x, \quad x(0) = x_0. \quad (30)$$

The upper bound on the rate of growth of the assets implies the following constraint on the control variables:

$$\dot{y}/y = (cu + v)x/(x/r) = r(cu + v) \leq g. \quad (31)$$

Maximize the net present value of the total future dividends that accrue to the initial shares. That is maximize

$$J = \int_0^T e^{-\rho t} (1 - v - u)x dt; \quad (32)$$

assume no salvage value at T for the time being.

For convenience, we restate this problem as

$$\left\{ \begin{array}{l} \max_{u,v} \left\{ J = \int_0^T e^{-\rho t} (1 - v - u)x dt \right\} \\ \text{subject to} \\ \dot{x} = r(cu + v)x, \quad x(0) = x_0, \\ \text{and the control constraints} \\ cu + v \leq g/r, \quad u \geq 0, \quad 0 \leq v \leq 1. \end{array} \right. \quad (33)$$

APPLICATION OF THE MAXIMUM PRINCIPLE

The current-value Hamiltonian is

$$H = (1 - v - u)x + \lambda r(cu + v)x, \quad (34)$$

where the current-value adjoint variable λ satisfies

$$\dot{\lambda} = \rho\lambda - (1 - v - u) - \lambda r(cu + v) \quad (35)$$

with the transversality condition

$$\lambda(T) = 0. \quad (36)$$

We rewrite the Hamiltonian as

$$H = [W_1 u + W_2 v + 1]x, \quad (37)$$

where

$$W_1 = cr\lambda - 1, \quad (38)$$

$$W_2 = r\lambda - 1. \quad (39)$$

The optimal policy is a combination of generalized bang-bang and singular controls. The characterization of these optimal controls will require solving a parametric linear programming problem at each instant of time t .

The Hamiltonian maximization problem can be stated as follows:

$$\begin{cases} \max_{u,v} \{W_1u + W_2v\} \\ \text{subject to} \\ u \geq 0, \quad 0 \leq v \leq 1, \quad cu + v \leq g/r. \end{cases} \quad (40)$$

We have two cases:

Case A: $g \leq r$ and **Case B:** $g > r$,

under each of which, we can solve the linear programming problem (40) graphically in a closed form. This is done in Figures 5.4 and 5.5.

Since $c < 1$ by assumption, the following subcases shown in Figures 5.4 and 5.5 can be ruled out:

- (i) $W_1 > cW_2, W_1 > 0 \Rightarrow c > 1$, ruling out Subcases A2 and B2.
- (ii) $W_1 = 0, W_2 < 0 \Rightarrow c > 1$, ruling out Subcases A4 and B5.
- (iii) $W_1 = cW_2 > 0 \Rightarrow c = 1$, ruling out Subcases A5 and B6.
- (iv) $W_1 = W_2 = 0 \Rightarrow c = 1$, ruling out Subcases A7 and B9.

Remaining subcases are shown with darkened lines.

FIGURE 5.4: CASE A: $g \leq r$

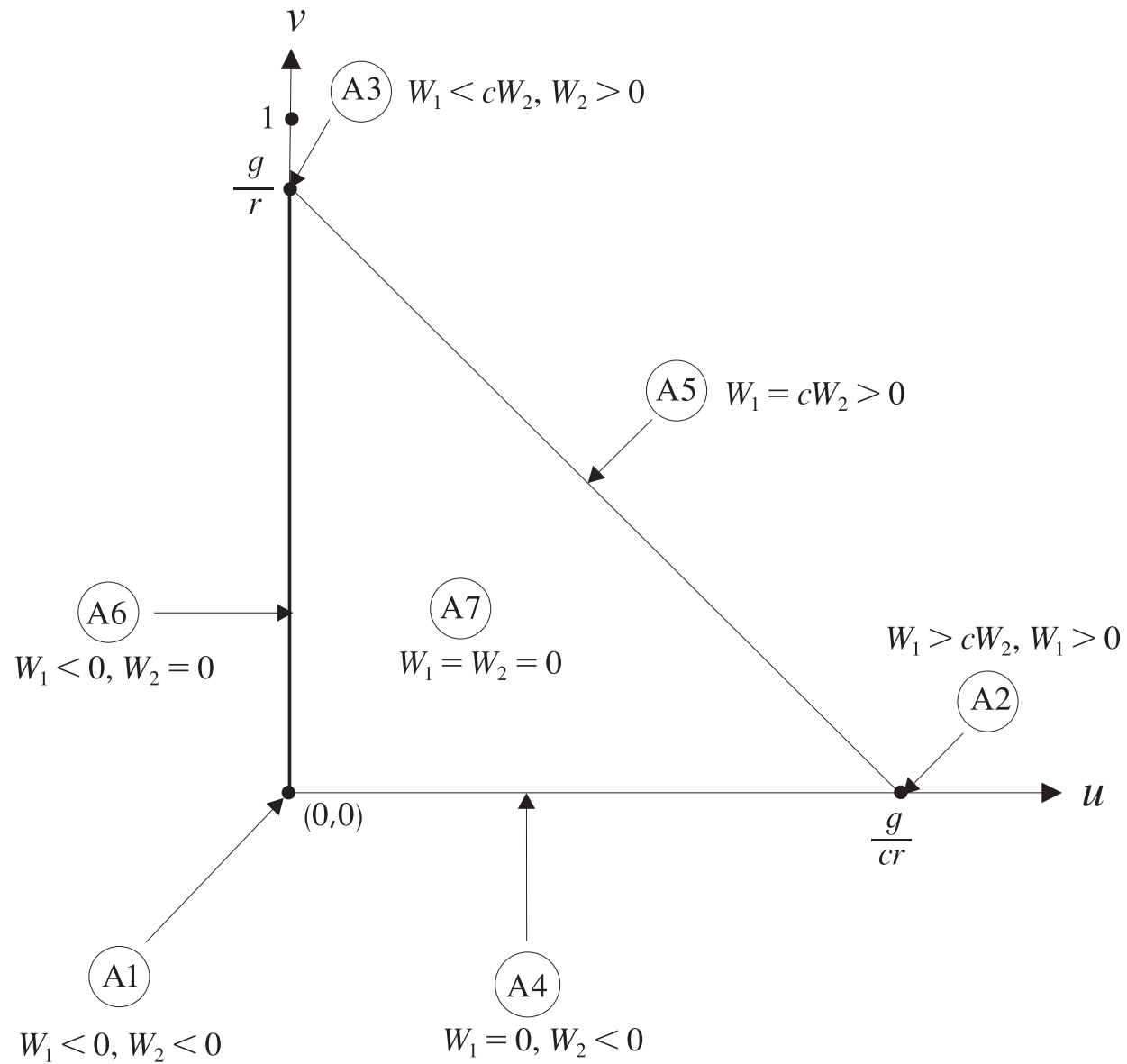
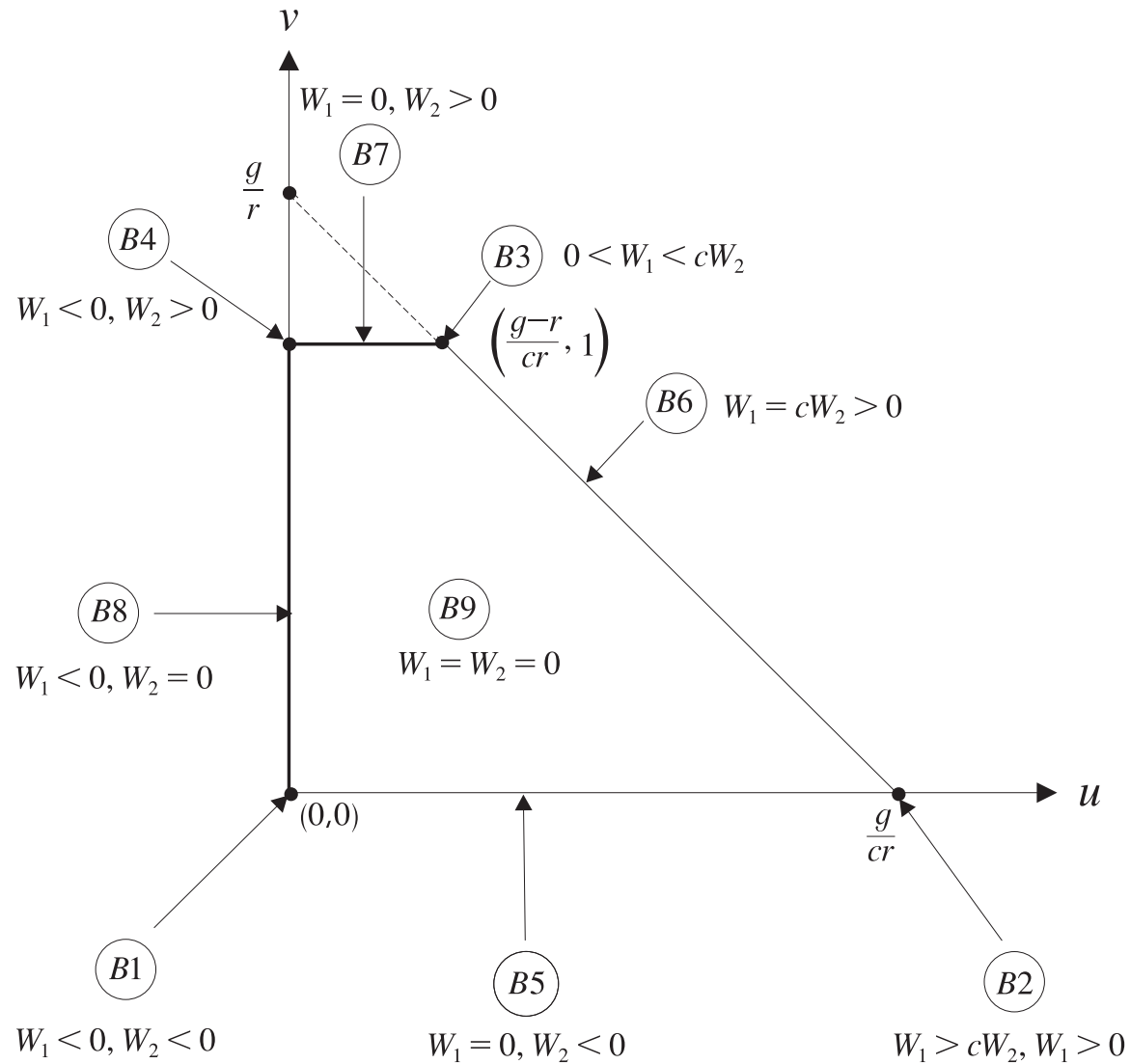


FIGURE 5.5: CASE A: $g > r$



Before proceeding with our synthesis, we note that since we have assumed $c < 1$ in this chapter, we have $W_1 < cW_2$ from (38) and (39). We can, therefore, characterize Subcase A3 simply by $W_2 > 0$ and Subcase B3 simply by $W_1 > 0$. It is these simpler characterizations of Subcases A2 and B2 that we shall use in our subsequent discussion.

TABLE 5.1: CHARACTERIZATION OF OPTIMAL CONTROLS

| Row | Conditions on W_1, W_2 | Case A: $g \leq r$ | Case B: $g > r$ | Optimal Controls | Characterization |
|-----|--------------------------|-----------------------|--------------------|--|-----------------------|
| | | Subcases | Subcases | | |
| (1) | $W_1 < 0, W_2 < 0$ | A1 | B1 | $u^* = 0, v^* = 0$ | generalized bang-bang |
| (2) | $W_1 < cW_2, W_2 > 0$ | A3 | - | $u^* = 0, v^* = g/r$ | generalized bang-bang |
| (3) | $W_1 < 0, W_2 = 0$ | A6 | B8 | $u^* = 0,$ $0 \leq v^* \leq \min[1, g/r]$ | singular |
| (4) | $0 < W_1 < cW_2$ | - | B3 | $u_* = (g - r)/rc, v^* = 1$ | generalized bang-bang |
| (5) | $W_1 < 0, W_2 > 0$ | - | B4 | $u^* = 0, v^* = 1$ | generalized bang-bang |
| (6) | $W_1 = 0, W_2 > 0$ | - | B7 | $0 \leq u^* \leq (g - r)/rc,$ $v^* = 1$ | singular |

Define the reverse-time variable τ as

$$\tau = T - t,$$

so that

$$\overset{\circ}{y} = \frac{dy}{d\tau} = \frac{dy}{dt} \frac{dt}{d\tau} = -\dot{y}.$$

The transversality condition on the adjoint variable

$$\lambda(t = T) = \lambda(\tau = 0) = 0. \quad (41)$$

Let us parameterize the terminal state by assuming that

$$x(t = T) = x(\tau = 0) = \alpha_A. \quad (42)$$

Using the definitions of $\overset{\circ}{x}$ and $\overset{\circ}{\lambda}$ and the conditions (42) and (41), we can write reverse-time versions of (30) and (35) as follows:

$$\overset{\circ}{\dot{x}} = -r(cu + v)x, \quad x(0) = \alpha_A, \quad (43)$$

$$\overset{\circ}{\dot{\lambda}} = (1 - u - v) - \lambda\{\rho - r(cu + v)\}, \quad \lambda(0) = 0. \quad (44)$$

Case A: $g \leq r$. Feasible subcases are A1, A3, and A6. Since $\lambda(0) = 0$, we have $W_1(0) = W_2(0) = -1$, and Subcase A1 obtains.

Subcase A1: $W_1 = cr\lambda - 1 < 0$ and $W_2 = r\lambda - 1 < 0$.

We have: $u^* = v^* = 0$

$$\dot{x} = 0 \quad \text{and} \quad \dot{\lambda} = 1 - \rho\lambda. \quad (45)$$

$$x(\tau) = \alpha_A \quad \text{and} \quad \lambda(\tau) = (1/\rho)[1 - e^{-\rho\tau}]. \quad (46)$$

Since $0 \leq c < 1$, it follows that if $W_2 = r\lambda - 1 < 0$, then $W_1 = cr\lambda - 1 < 0$. To remain in this subcase as τ increases, $W_2(\tau)$ must remain negative for some time as τ increases.

From (46), $\lambda(\tau)$ is increasing asymptotically toward the value $1/\rho$ and $W_2(\tau)$ is increasing asymptotically toward the value $r/\rho - 1$. Since, we have assumed $r > \rho$, there exists a τ_1 such that $W_2(\tau_1) = (1 - e^{-\rho\tau_1})r/\rho - 1 = 0$. It is easy to compute

$$\tau_1 = (1/\rho) \ln[r/(r - \rho)]. \quad (47)$$

It is clear that the firm leaves Subcase A1 provided $\tau_1 < T$.

- When T is not sufficiently large, i.e., when $T \leq \tau_1$ in Case A, the firm stays in Subcase A1. The optimal solution in this case is $u^* = 0$ and $v^* = 0$, i.e., a policy of no investment.
- For $r < \rho$, the firm never exits from Subcase A1 regardless of the value of T . Obviously, there is no use investing if the rate of return is less than the discount rate.
- At reverse time τ_1 , we have $W_2 = 0$ and $W_1 < 0$ and the firm, therefore, is in Subcase A6.

Subcase A6: $W_1 = cr\lambda - 1 < 0$ and $W_2 = r\lambda - 1 = 0$.

The optimal controls

$$u^* = 0, \quad 0 \leq v^* \leq g/r \quad (48)$$

The optimal controls are obtained by conditions required to sustain $W_2 = 0$ for a finite time interval. We substitute (48) into (44) and obtain

$$\dot{\lambda} = (1 - v^*) - \lambda[\rho - rv^*]. \quad (49)$$

Substituting $\lambda = 1/r$ (since $W_2 = 0$) in (49) and equating the right-hand side to zero we obtain

$$r = \rho \quad (50)$$

SUBCASE A6 CONT.

We have assumed $r > \rho$. Thus, the firm will not stay in Subcase A6 for a finite time interval. Since $r > \rho$, we have $\dot{\lambda}(\tau_1) = (r - \rho/r) > 0$. Therefore, W_2 is increasing from zero and becomes positive after τ_1 . Thus, at τ_1^+ the firm switches to Subcase A3.

Subcase A3: $W_2 = r\lambda - 1 > 0$.

The optimal controls in this subcase from Row (2) of Table (1) are

$$u^* = 0, \quad v^* = g/r. \quad (51)$$

The state and the adjoint equations are

$$\dot{x} = -gx, \quad x(\tau_1) = \alpha_A. \quad (52)$$

$$\dot{\lambda} = (1 - g/r) - \lambda(\rho - g), \quad \lambda(\tau_1) = 1/r. \quad (53)$$

Since $\dot{\lambda}(\tau_1) > 0$, λ is increasing at τ_1 from its value of $1/r$.

(i) $\rho > g$: As λ increases, $\dot{\lambda}$ decreases and becomes zero at a value obtained by equating the right-hand side of (53) to zero, i.e., at

$$\bar{\lambda} = \frac{1 - g/r}{\rho - g}. \quad (54)$$

Since $r > \rho > g$ in this case,

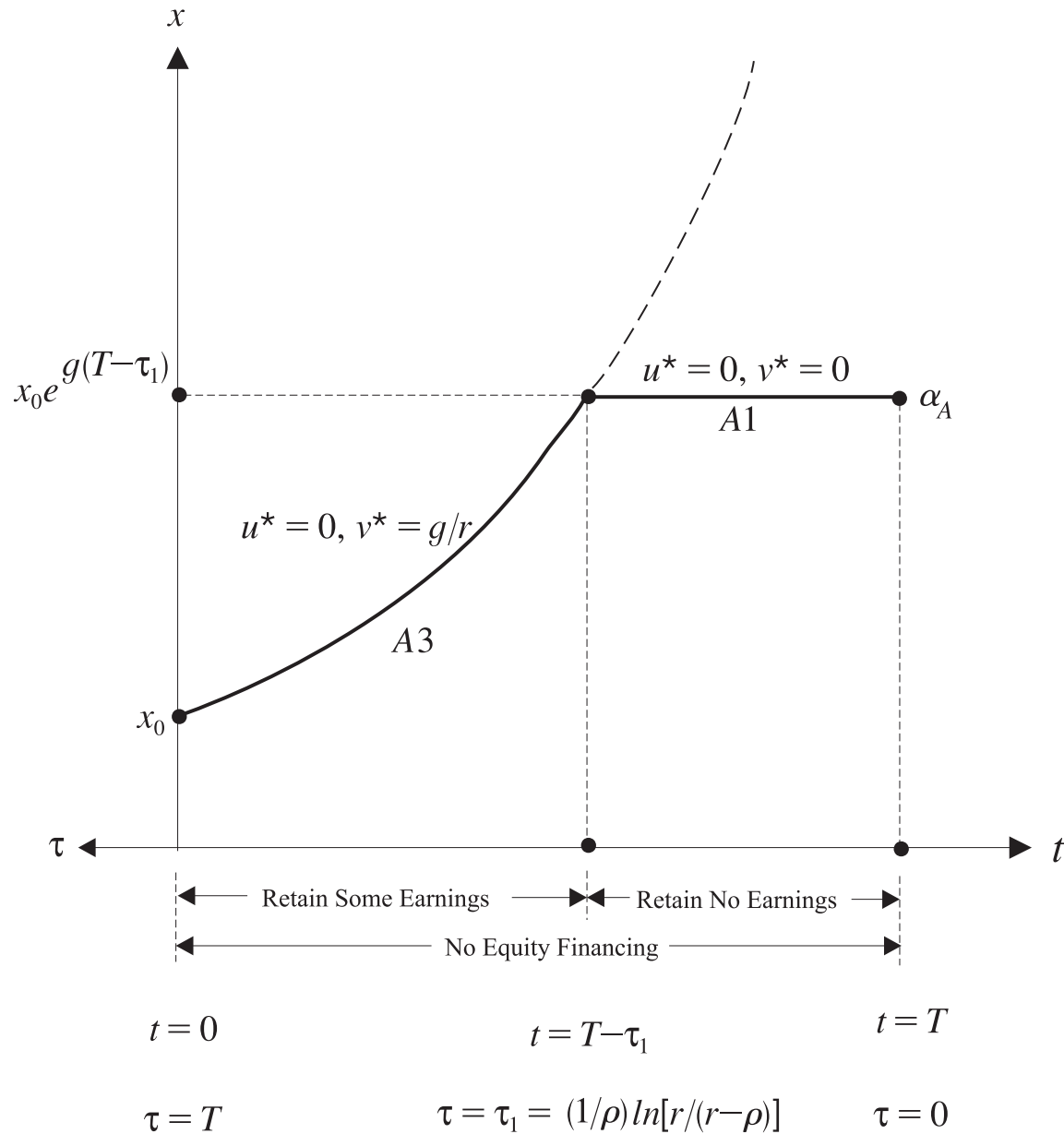
$$\overline{W}_2 = r\bar{\lambda} - 1 = \frac{r(1 - g/r)}{\rho - g} - 1 = \frac{r - \rho}{\rho - g} > 0, \quad (55)$$

which implies that the firm continues to stay in Subcase A3.

(ii) $\rho \leq g$: As $\lambda(\tau)$ increases, $\dot{\lambda}(\tau)$ increases. So $W_2(\tau) = r\lambda(\tau) - 1$ continues to be greater than zero and the firm continues to remain in Subcase A3.

FIGURE 5.6: OPTIMAL PATH FOR CASE A:

$g \leq r$ FOR EXAMPLE 4.3



SWITCHING TIME INTERPRETATION

The switching time $t = T - \tau_1$ has an interesting economic interpretation. Namely, it requires at least τ_1 units of time to retain a dollar of earnings (for investment) to be worthwhile. That means, it pays to invest as much earnings as feasible before $T - \tau_1$, and it does not pay to invest any earnings after $T - \tau_1$. Thus, $T - \tau_1$ is the point of indifference between retaining earnings or paying dividends out of earnings. Suppose the firm retains one dollar of earnings at $T - \tau_1$. Since this is the last time any earnings invested will be worthwhile, it is obvious (because all earnings are paid out) that the dollar just invested at $T - \tau_1$ yields dividends at the rate r from $T - \tau_1$ to T .

SWITCHING TIME INTERPRETATION CONT.

The value of this dividend stream in terms of $(T - \tau_1)$ -dollars is

$$\int_0^{\tau_1} r e^{-\rho t} dt = \frac{r}{\rho} [1 - e^{-\rho \tau_1}], \quad (56)$$

which must be equated to one dollar to find the indifference point. Equating (56) to 1 yields precisely the value of τ_1 given in (47). So under the optimal policy, the firm grows exponentially at the maximum rate of g until $t = T - \tau_1$. After this time, all of the earnings are paid out and the firm stops growing. Since $g \leq r$ (assumed for Case A), the growth in the first part of the solution can be financed entirely from retained earnings. Thus, there is no need to resort to more expensive external equity financing.

Case B: $g > r$.

Since $g/r > 1$, the constraint $v \leq 1$ in Case B is relevant.

Subcase B1: $W_1 = cr\lambda - 1 < 0, W_2 = r\lambda - 1 < 0$.

The analysis of this subcase is the same as Subcase A1. As in that subcase the firm switches out at time $\tau = \tau_1$ to Subcase B8.

Subcase B8: $W_1 = cr\lambda - 1 < 0, W_2 = r\lambda - 1 = 0.$

The optimal controls

$$u^* = 0, \quad 0 \leq v^* \leq 1 \quad (57)$$

As before in Subcase A6, the singular case cannot be sustained for a finite time because of our assumption $r > \rho$. W_2 is increasing at τ_1 from zero and becomes positive after τ_1 . Thus, at τ_1^+ , the firm finds itself in Subcase B4.

Subcase B4: $W_1 = cr\lambda - 1 < 0, W_2 = r\lambda - 1 > 0.$

The optimal controls in this subcase are

$$u^* = 0, \quad v^* = 1, \quad (58)$$

The state and the adjoint equations are

$$\dot{x} = -rx, \quad x(\tau_1) = \alpha_B \quad (59)$$

$$\dot{\lambda} = \lambda(r - \rho), \quad \lambda(\tau_1) = 1/r. \quad (60)$$

Since $\lambda(\tau_1) = 1/r$, we have

$$\lambda(\tau) = (1/r)e^{(r-\rho)(\tau-\tau_1)} \quad \text{for } \tau \geq \tau_1. \quad (61)$$

SUBCASES B4 AND B7

As λ increases, W_1 increases and becomes zero at a time τ_2 defined by

$$W_1(\tau_2) = cr\lambda(\tau_2) - 1 = ce^{(r-\rho)(\tau_2-\tau_1)} - 1 = 0, \quad (62)$$

which gives

$$\tau_2 = \tau_1 + [1/(r - \rho)] \ln(1/c). \quad (63)$$

At τ_2^+ , the firm switches to Subcase B7.

Subcase B7: $W_1 = cr\lambda - 1 = 0, W_2 = r\lambda - 1 > 0.$

The optimal controls are

$$0 \leq u^* \leq (g - r)/rc, \quad v^* = 1. \quad (64)$$

To maintain this singular control over a finite time period, we must keep $W_1 = 0$ in the interval. This means we must

have $\dot{W}_1(\tau_2) = 0$, which implies $\lambda(\tau_2) = 0$.

To compute $\overset{\circ}{\lambda}$, we substitute (64) into (44) and obtain

$$\overset{\circ}{\lambda} = -u^* - \lambda\{\rho - r(cu^* + 1)\}. \quad (65)$$

Substituting $\lambda(\tau_2) = 1/rc$ (since $W_1(\tau_2) = 0$) in (65) and equating the right-hand side to zero, we obtain

$$r = \rho.$$

Since $r > \rho$, the firm will not stay in Subcase B7 for a finite amount of time. From (65), we have

$$\overset{\circ}{\lambda}(\tau_2) = \frac{r - \rho}{rc} > 0, \quad (66)$$

which implies that λ is increasing and therefore, W_1 is increasing. Thus at τ_2^+ , the firm switches to Subcase B3.

Subcase B3: $W_1 = cr\lambda - 1 > 0$.

The optimal controls are

$$u^* = \frac{g - r}{rc}, \quad v^* = 1. \quad (67)$$

The reverse-time state and the adjoint equations are

$$\overset{\circ}{x} = -gx, \quad (68)$$

$$\overset{\circ}{\lambda} = -\left(\frac{g - r}{rc}\right) + \lambda(g - \rho). \quad (69)$$

Since $\overset{\circ}{\lambda}(\tau_2) > 0$, $\lambda(\tau)$ is increasing. We assume $g > r$. But $r > \rho$ has been assumed throughout the chapter. Therefore, $\rho < g$ and the second term in the right-hand side of (69) is increasing. That means $\overset{\circ}{\lambda}(\tau) > 0$ and $\lambda(\tau)$ continues to increase. The firm continues to stay in Subcase B3.

INTERPRETATION OF CASE B

Since external equity is more expensive than retained earnings as a source of financing, investment financed by external equity requires more time to be worthwhile. Thus,

$$\tau_2 - \tau_1 = \frac{1}{r - \rho} \ln(1/c) \quad (70)$$

should be the time required to compensate for the floatation cost of external equity.

INTERPRETATION OF CASE B CONT.

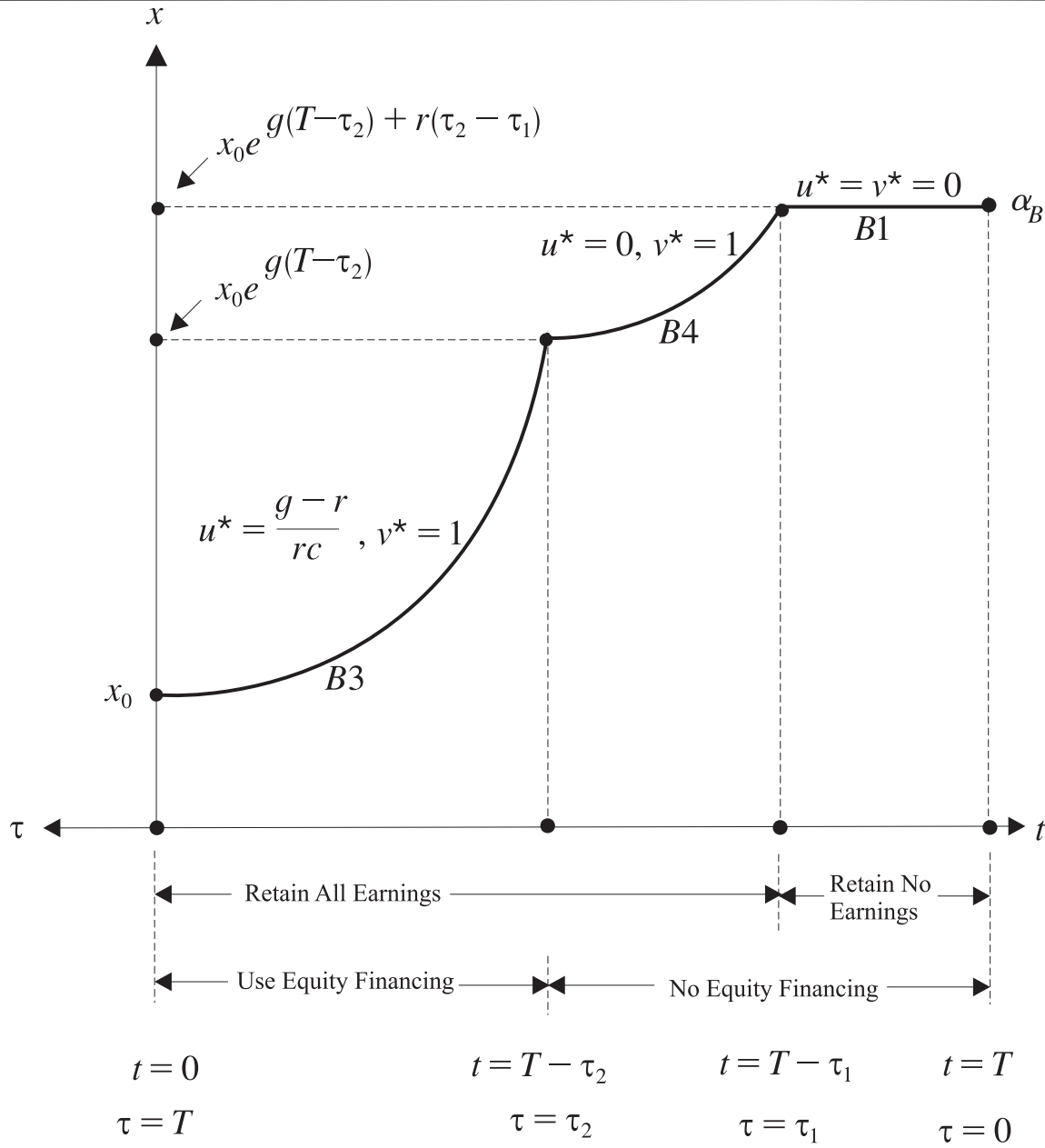
Suppose the firm issues a dollar's worth of stock at $t = T - \tau_2$. While the cost of this issue is one dollar, the capital acquired is c dollars because of the floatation cost $(1 - c)$. Since we are attempting to find the breakeven time for external equity, it is obvious that retaining all of the earnings for investment is still profitable. Thus, there is no dividend from $T - \tau_2$ to $T - \tau_1$ and the firm is growing at the rate r . Therefore, the value of this investment at $(T - \tau_2)$ measured in $(T - \tau_2)$ -dollars is

$$ce^{(r-\rho)(\tau_2-\tau_1)} = ce^{\ln(1/c)} = 1. \quad (71)$$

INTERPRETATION OF CASE B CONT.

Equation (71) states that one $(T - \tau_2)$ -dollar of external equity at time $(T - \tau_2)$, which brings in c dollars of capital at time $T - \tau_2$, is equivalent to one $(T - \tau_2)$ -dollar investment at $(T - \tau_1)$. But the firm is indifferent between investing or not investing the costless retained earnings at $(T - \tau_1)$. To summarize, the firm is indifferent between issuing a dollar's worth of stock at $(T - \tau_2)$ or not issuing it. Before $(T - \tau_2)$, it pays to issue stocks at as large a rate as is feasible. After $(T - \tau_2)$, it does not pay to issue any external equity at all.

FIGURE 5.7: OPTIMAL PATH FOR CASE B: $g > r$



5.2.4 SOLUTION FOR THE INFINITE HORIZON PROBLEM

For the infinite horizon case the transversality condition must be changed to

$$\lim_{t \rightarrow \infty} e^{-\rho t} \lambda(t) = 0. \quad (72)$$

This condition is a sufficient condition but may no longer be a necessary condition.

5.2.4 SOLUTION FOR THE INFINITE HORIZON

PROBLEM CONT.

Case A: $g \leq r$.

The limiting solution in this case is given as Subcase A3,

$$u = 0, \quad v = g/r, \quad (73)$$

$$\dot{x} = gx, \quad x(0) = x_0, \quad (74)$$

and

$$\dot{\lambda} = -(1 - g/r) - \lambda(g - \rho), \quad \lim_{t \rightarrow \infty} e^{-\rho t} \lambda(t) = 0. \quad (75)$$

For $\rho > g$ in which case $r > \rho > g$, $\lambda = \bar{\lambda}$ clearly satisfies (75). Furthermore,

$$\bar{W}_2 = r\bar{\lambda} - 1 = \frac{r - \rho}{\rho - g} > 0,$$

which implies that the firm stays in Subcase A3, i.e., the maximum principle holds.

5.2.4 SOLUTION FOR THE INFINITE HORIZON

PROBLEM CONT.

Result: For $\rho > g$ in Case A,

$$u^* = 0, v^* = g/r,$$

along with the corresponding state trajectory

$$x^*(t) = x_0 e^{gt}$$

and the adjoint trajectory

$$\bar{\lambda} = \frac{1 - g/r}{\rho - g} \quad (76)$$

$\bar{\lambda}$ is a constant and its form is reminiscent of the Gordon's classic formula. $\bar{\lambda}$ represents the marginal worth per additional unit of earnings. Obviously, a unit increase in earnings will mean an increase of $1 - v^*$ or $1 - g/r$ units in dividends. This, of course, should be capitalized at a rate equal to the discount rate less the growth rate (i.e., $\rho - g$).

5.2.4 SOLUTION FOR THE INFINITE HORIZON

PROBLEM CONT.

For $\rho \leq g$, the reverse-time construction in Subcase A3 implies that $\lambda(\tau)$ increases without bound as τ increases. Thus, we cannot find any λ which satisfies (75). Note that for $\rho \leq g$, the objective function can be made infinite. For example, any control policy with earnings growing at rate q , $\bar{\rho} \leq q \leq g$, coupled with a partial dividend payout, i.e. a constant v such that $0 < v < 1$, gives an infinite value for the objective function. That is, with $u^* = 0$, $v^* = q/r < 1$, we have

$$J = \int_0^{\infty} e^{-\rho t} (1 - u - v)x dt = \int_0^{\infty} e^{-\rho t} (1 - v)x_0 e^{qt} dt = \infty.$$

Case B: $g > r$.

The limit of the finite horizon optimal solution is to grow at the maximum allowable growth rate with

$$u = \frac{g - r}{rc} \text{ and } v = 1$$

Since τ_1 disappears in the limit, the stockholders will never collect dividends. The firm has become an infinite *sink* for investment.

REMARK 5.7

Let (u_T^*, v_T^*) denote the optimal control for the finite horizon problem in Case B. Let (u_∞^*, v_∞^*) denote any optimal control for the infinite horizon problem in Case B. We already know that $J(u_\infty^*, v_\infty^*) = \infty$. Define an infinite horizon control (u_∞, v_∞) by extending (u_T^*, v_T^*) as follows:

$$(u_\infty, v_\infty) = \lim_{T \rightarrow \infty} (u_T^*, v_T^*).$$

We now note that for our model in Case B, we have

$$\lim_{T \rightarrow \infty} J(u_T^*, v_T^*) = \infty \text{ and } J(u_\infty, v_\infty) = 0. \quad (77)$$

Obviously (u_∞, v_∞) is *not* an optimal control for the infinite horizon problem.

REMARK 5.7 CONT.

If we introduce a salvage value $Bx(T)$, $B > 0$, however, for the finite horizon problem, then the new objective function,

$$J(u, v) = \begin{cases} \int_0^T e^{-\rho t} (1 - u - v)x dt + Bx(T)e^{-\rho T}, & \text{if } T < \infty, \\ \int_0^\infty e^{-\rho t} (1 - u - v)x dt + \lim_{T \rightarrow \infty} \{ Bx(T)e^{-\rho T} \}, & \text{if } T = \infty, \end{cases}$$

is a closed mapping in the sense that

$$\lim_{T \rightarrow \infty} J(u_T^*, v_T^*) = \infty \text{ and } J(u_\infty, v_\infty) = \infty$$

for the modified model.

$$\begin{aligned}x_0 &= 1000/\text{month}, T = 60 \text{ months}, \\r &= 0.15, \rho = 0.10, g = 0.05, c = 0.98.\end{aligned}$$

Solution: Since $g \leq r$, the problem belongs to Case A.

$$\tau_1 = \frac{1}{\rho} \ln[r/(r - \rho)] = 10 \ln 3 \approx 11 \text{ months}.$$

The optimal controls for the problem are

$$\begin{aligned}u^* &= 0, \quad v^* = g/r = 1/3, \quad t \in [0, 49), \\u^* &= 0, \quad v^* = 0, \quad t \in [49, 60],\end{aligned}$$

SOLUTION OF EXAMPLE 5.2

The optimal state trajectory is

$$x(t) = \begin{cases} 1000e^{0.05t}, & t \in [0, 49), \\ 1000e^{2.45}, & t \in [49, 60]. \end{cases}$$

The value of the objective function is

$$\begin{aligned} J^* &= \int_0^{49} e^{-0.1t} (1 - 1/3)(1000)e^{0.05t} dt && (78) \\ &+ \int_{49}^{60} 1000e^{2.45} \cdot e^{-0.1t} dt \\ &= 12,578.75. \end{aligned}$$

The infinite horizon

Since $g < \rho$ and $g < r$. The optimal controls are

$$u^* = 0, v^* = g/r = 1/3,$$

and

$$\begin{aligned} J &= \int_0^{\infty} e^{-0.1t} (2/3)(1000)e^{0.05t} dt \\ &= 2000/0.15 = 13,333\frac{1}{3}. \end{aligned}$$