CHAPTER 4

THE MAXIMUM PRINCIPLE: GENERAL INEQUALITY CONSTRAINTS

4.1 Pure State Variable Inequality Constraints: Indirect Method

It is common to require state variables to remain nonnegative:

$$x(t) \geq 0 \text{ for } t \in [0, T], \tag{1}$$

i.e., $x_i(t) \geq 0$, $i = 1, 2, ..., n$. Constraints exhibiting this property are called *pure state variable inequality constraints*. The general form is:

$$h(x, t) \geq 0 \text{ for } t \in [0, T]. \tag{2}$$

Let us consider (1) first. At any point where a component $x_i(t) > 0$, the corresponding constraint $\dot{x}_i(t) \geq 0$ is not binding and can be ignored.

In any interval where $x_i(t) = 0$, we must have $\dot{x}_i(t) \geq 0$ so that x_i does not become negative. The control must be constrained to satisfy $\dot{x}_i = f_i \geq 0$, making $f_i \geq 0$ as a constraint of the mixed type (3.3) over the interval. We can add the constraint

$$f_i(x, u, t) \geq 0, \text{ whenever } x_i(t) = 0, \tag{3}$$

We associate multipliers η_i with (3) whenever (3) must be imposed, i.e., whenever $x_i(t) = 0$. A convenient way to do this is to impose an “either or” condition $\eta_i x_i = 0$. This will make $\eta_i = 0$ whenever $x_i > 0$.

We can now form the Lagrangian

$$L = H + \mu g + \eta f, \tag{4}$$

where the Hamiltonian H is as defined in (3.7) and $\eta = (\eta_1, \eta_2, ..., \eta_n)$, and apply the maximum principle in (3.11) with the additional necessary conditions satisfied by the multiplier η, namely,

$$\eta(t) \geq 0, \ \eta(t)x^*(t) = 0, \ \dot{\eta}(t) \leq 0, \tag{5}$$

and the modified transversality condition

$$\lambda(T) = S_x(x^*(T), T) + \alpha a_x(x^*(T), T) + \beta b_x(x^*(T), T) + \gamma \tag{6}$$

where γ is a constant vector satisfying

$$\gamma \geq 0, \ \gamma x^*(T) = 0. \tag{7}$$

Since the constraints are adjoined indirectly (in this case via their first time derivative) to form the Lagrangian, the method is called the *indirect adjoining method*. If on the other hand, the Lagrangian is formed by adjoining directly the constraints (1), i.e.,

$$L^d = F + \lambda^d f + \mu^d g + \eta^d x,$$

where η^d is a multiplier associated with (1), then the method is referred to as the *direct adjoining method*.

Remark 4.1

The first two conditions in (5) are complementary slackness conditions on the multiplier η. The last condition $\dot{\eta} \leq 0$ is difficult to motivate. We know, however, that the direct maximum principle multiplier η^d is related to η as $\eta^d = -\dot{\eta}$. The complementary slackness conditions for the direct multiplier η^d are $\eta^d \geq 0$ and $\eta^d x^* = 0$. Since $\eta^d \geq 0$, it follows that $\dot{\eta} \leq 0$.

In the presence of (1), the marginal value of a change in the state at time t, whenever it is defined, is given by $\lambda(t) + \eta(t)$. It is also reasonable to expect the marginal value to be continuous even at a time when the state variable hits the constraint boundary. But $\eta(t)$ usually jumps at that time, requiring $\lambda(t)$ to jump. Fortunately, the maximum principle in presence of state constraints allows for a jump in $\lambda(t)$ at a point in time when the state $x(t)$ enters its constraint boundary.

The jump must satisfy certain conditions, which in the case of the state constraints (1) are

$$\lambda(\tau^-) = \lambda(\tau^+) + \zeta(\tau),\ \zeta(\tau) \geq 0 \tag{8}$$

and

$$H[x^*(\tau),\ u^*(\tau^-),\ \lambda(\tau^-),\ \tau] = H[x^*(\tau),\ u^*(\tau^+), \lambda(\tau^+),\ \tau], \tag{9}$$

at a time τ at which one of the state variables has just reached its boundary value zero.

Consider the problem:

$$\max \left\{ J = \int_0^2 -x dt \right\}$$

subject to

$$\dot{x} = u, \; x(0) = 1, \tag{10}$$

$$u + 1 \geq 0, \; 1 - u \geq 0, \tag{11}$$

$$x \geq 0. \tag{12}$$

The Hamiltonian is

$$H = -x + \lambda u,$$

which implies the optimal control to be

$$u^* = \text{ bang}[-1, 1; \lambda], \text{ whenever } x > 0. \quad (13)$$

When $x = 0$, we impose $\dot{x} = u \geq 0$ in order to insure that (12) holds. Therefore, the optimal control on the state constraint boundary is

$$u^* = \text{ bang}[0, 1; \lambda], \text{ whenever } x = 0. \quad (14)$$

Now we form the Lagrangian

$$L = H + \mu_1(u+1) + \mu_2(1-u) + \eta u,$$

where μ_1, μ_2, and η satisfy the complementary slackness conditions

$$\mu_1 \geq 0, \quad \mu_1(u+1) = 0, \tag{15}$$

$$\mu_2 \geq 0, \quad \mu_2(1-u) = 0, \tag{16}$$

$$\eta \geq 0, \quad \eta x = 0, \ \dot{\eta} \leq 0. \tag{17}$$

Furthermore, the optimal trajectory must satisfy

$$\frac{\partial L}{\partial u} = \lambda + \mu_1 - \mu_2 + \eta = 0. \tag{18}$$

From the Lagrangian we also get

$$\dot{\lambda} = -\frac{\partial L}{\partial x} = 1, \ \lambda(2) = \gamma \geq 0, \ \gamma x(2) = \ \lambda(2)x(2) = 0. \quad (19)$$

We guess that the optimal control u^* will be the one that keeps x^* as small as possible, subject to the state constraint (1). Thus,

$$u^*(t) = \begin{cases} -1, & t \in [0,1], \\ 0, & t \in (1,2]. \end{cases} \quad (20)$$

This gives

$$x^*(t) = \begin{cases} 1-t, & t \in [0,1], \\ 0, & t \in (1,2]. \end{cases}$$

Let us first try $\lambda(2) = \gamma = 0$. Then, since $x^*(t)$ enters the boundary zero at $t = 1$, there are no jumps in the interval $(1, 2]$, and the solution for $\lambda(t)$ is

$$\lambda(t) = t - 2, \quad t \in (1, 2]. \tag{21}$$

Since $\lambda(t) \leq 0$ and $x^*(t) = 0$ on $(1, 2]$, we have $u^*(t) = 0$ by (14), as stipulated. Now let us see what must happen at $t = 1$. We know from (21) that $\lambda(1^+) = -1$. We now apply (9) to obtain $\lambda(1^-)$. For this, we see that

$$H(1^+) = -x^*(1^+) + \lambda(1^+)u^*(1^+) = 0,$$

$$H(1^-) = -x^*(1^-) + \lambda(1^-)u^*(1^-) = -\lambda(1^-).$$

By equating $H(1^-)$ to $H(1^+)$ as required in (9), we obtain $\lambda(1^-) = 0$. This, by (8) gives the value of the jump

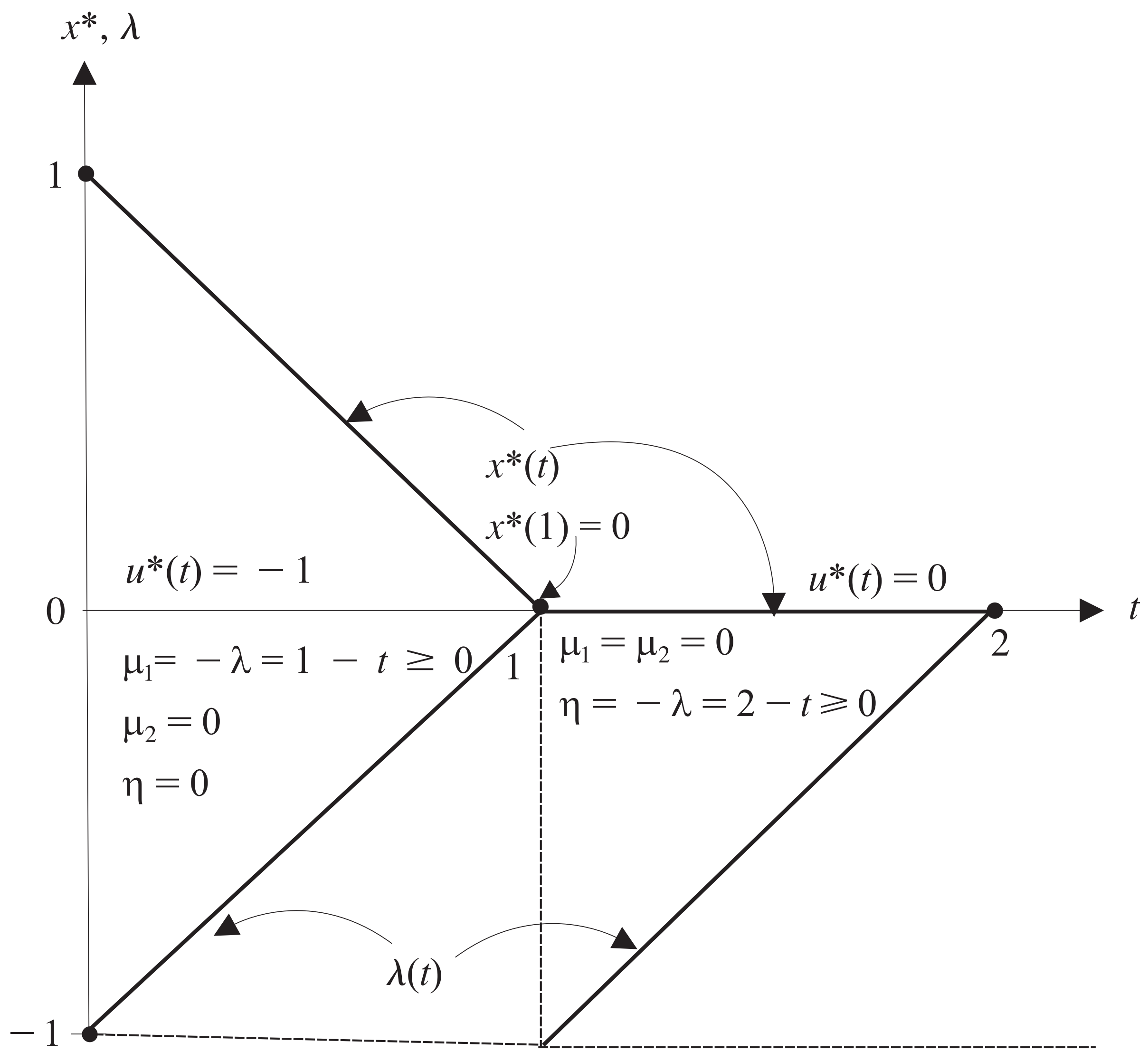
$$\zeta(1) = \lambda(1^-) - \lambda(1^+) = 1 \geq 0.$$

Since $\lambda(t) \leq -1$ and $x^*(t) = 1 - t > 0$ is positive on [0,1], we can use (13) to obtain

$$u^*(t) = -1 \text{ for } 0 \leq t < 1.$$

In the time interval [0,1) by (16), $\mu_2 = 0$ since $u^* < 1$, and by (17), $\eta = 0$ because $x > 0$. Therefore, $\mu_1(t) = -\lambda(t) = 1 - t > 0$ for $0 \leq t < 1$, and this with $u^* = -1$ satisfies (15).

Figure 4.1: State and Adjoint Trajectories in Example 4.1



To complete the solution, we calculate the Lagrange multipliers in the interval [1,2]. Since $u^*(t) = 0$ on $t \in [1, 2]$, we have $\mu_1(t) = \mu_2(t) = 0$. Then, from (18) we obtain $\eta(t) = -\lambda(t) = 2 - t \geq 0$ which, with $x^*(t) = 0$ satisfies (17). This completes the solution. The graphs of $x^*(t)$ and $\lambda(t)$ are shown in Figure 4.1.

Remark 4.2 In instances where the initial state or the final state or both are on the constraint boundary, the maximum principle may *degenerate* in the sense that there is no nontrivial solution of the necessary conditions, i.e., $\lambda(t) \equiv 0$, $t \in [0, T]$, where T is the terminal time.

Consider Example 4.1 with $T = 3$ and the terminal state constraint:

$$x(3) = 1.$$

Clearly, the optimal control u^* will be the one that keeps x as small as possible, subject to the state constraint (1) and the boundary condition $x(0) = x(3) = 1$. Thus,

$$u^*(t) = \begin{cases} -1, & t \in [0,1), \\ 0, & t \in [1,2], \\ 1, & t \in (2,3], \end{cases} \quad x^*(t) = \begin{cases} 1-t, & t \in [0,1), \\ 0, & t \in [1,2], \\ t-2, & t \in (2,3]. \end{cases}$$

We shall only compute the adjoint function and the multipliers that satisfy the optimality conditions. These are

$$\lambda(t) = \begin{cases} t-1, & t \in [0,1], \\ t-2, & t \in (1,3], \end{cases} \tag{22}$$

$$\mu_1(t) = \mu_2(t) = 0, \quad t \in [1,2], \tag{23}$$

$$\eta(t) = -\lambda(t), \quad t \in [1,2], \tag{24}$$

and the jump $\zeta(1) = 1 \geq 0$ so that

$$\lambda(1^-) = \lambda(1^+) + \zeta(1) \text{ and } H(1^-) = H(1^+). \tag{25}$$

4.2 A Maximum Principle: Indirect Method

Mixed inequality constraints and the pure state variable inequality constraints:

$$h(x, t) \geq 0, \tag{26}$$

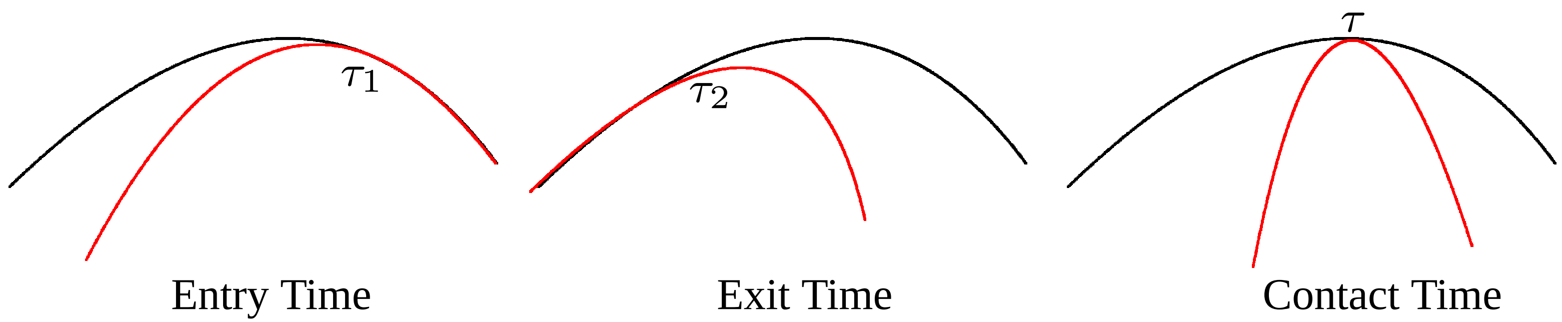
where we assume function $h: \ E^n \times E^1 \rightarrow E^p$ to be as many times continuously differentiable as required. By the definition of function h, (26) represents a set of p constraints $h_i(x, t) \geq 0$, $i = 1, 2, ..., p$. It is noted that the constraint $h_i \geq 0$ is called a constraint of rth order if the rth time derivative of h_i is the first time a term in control u appears in the expression by putting $f(x, u, t)$ for $\dot{x}$ after each differentiation.

In the case of first-order constraints, we need to define $h^1(x, u, t)$ as follows:

$$h^1 = \frac{dh}{dt} = \frac{\partial h}{\partial x} f + \frac{\partial h}{\partial t}. \tag{27}$$

With respect to the ith constraint $h_i(x, t) \geq 0$, an interval $(\theta_1, \theta_2) \subset [0, T]$ with $\theta_1 < \theta_2$ is called an *interior interval* if $h_i(x(t), t) > 0$ for all $t \in (\theta_1, \theta_2)$. If the optimal trajectory "hits the boundary," i.e., satisfies $h_i(x(t), t) = 0$ for $\tau_1 \leq t \leq \tau_2$ for some i, then $[\tau_1, \tau_2]$ is called a *boundary interval*.

An instant τ_1 is called an *entry time* if there is an interior interval ending at $t = \tau_1$ and a boundary interval starting at τ_1. Correspondingly, τ_2 is called an *exit time* if a boundary interval ends and an interior interval starts at τ_2. If the trajectory just touches the boundary at time τ, i.e., $h(x(\tau), \tau) = 0$ and if the trajectory is in the interior just before and just after τ, then τ is called a *contact time*. Taken together, entry, exit, and contact times are called *junction times*.



The following full-rank condition on any boundary interval $[\tau_1, \tau_2]$ hold:

$$\text{rank}\begin{bmatrix} \partial h_1^1/\partial u \\ \partial h_2^1/\partial u \\ \vdots \\ \partial h_{\hat{p}}^1/\partial u \end{bmatrix} = \hat{p},$$

where for $t \in [\tau_1, \tau_2]$,

$$h_i(x^*(t), t) = 0,\ i = 1, 2, ..., \hat{p} \le p$$

and

$$h_i(x^*(t), t) > 0,\ i = \hat{p} + 1, ..., p.$$

To formulate the maximum principle for the problem with mixed constraints as well as first-order pure state constraints, we form the Lagrangian as

$$L(x,u,\lambda,\mu,\eta,t) = H(x,u,\lambda,t) + \mu g(x,u,t) + \eta h^1(x,u,t), \quad (28)$$

where the Hamiltonian as defined in (3.7), μ satisfies the complementary slackness conditions in (3.9), and $\eta \in E^p$ (a row vector) satisfies the conditions

$$\eta \geq 0, \;\; \eta h(x,t) = 0, \;\; \dot{\eta} \leq 0.$$

The maximum principle states that the necessary conditions for u^* (with the state trajectory x^*) to be an optimal control for the problem defined above are that there exist adjoint variable λ, multipliers $\mu, \alpha, \beta, \gamma, \eta$, and the jump parameter ζ, which satisfy (29) that follows.

$\dot{x}^* = f(x^*, u^*, t),\ x^*(0) = x_0,$
satisfying constraints
$g(x^*, u^*, t) \geq 0,\ h(x^*, t) \geq 0$, and
the terminal constraints
$a(x^*(T), T) \geq 0$ and $b(x^*(T), T) = 0$;
$\dot{\lambda} = -L_x[x^*, u^*, \lambda, \mu, \eta, t]$
with the transversality conditions
$\lambda(T^-) = S_x(x^*(T), T) + \alpha a_x(x^*(T), T) + \beta b_x(x^*(T), T)$
$\quad +\gamma h_x(x^*(T), T)$, and
$\alpha \geq 0,\ \alpha a(x^*(T), T) = 0,\ \gamma \geq 0,\ \gamma h(x^*(T), T) = 0$;
the Hamiltonian maximizing condition
$H[x^*(t), u^*(t), \lambda(t), t] \geq H[x^*(t), u, \lambda(t), t]$
at each $t \in [0, T]$ for all u satisfying
$g[x^*(t), u, t] \geq 0$, and
$h_i^1(x^*(t), u, t) \geq 0$ whenever $h_i(x^*(t), t) = 0, i = 1, 2, \cdots, p$;
the jump conditions at any entry/contact time τ are
$\lambda(\tau^-) = \lambda(\tau^+) + \zeta(\tau) h_x(x^*(\tau), \tau)$ and
$H[x^*(\tau), u^*(\tau^-), \lambda(\tau^-), \tau] = H[x^*(\tau), u^*(\tau^+), \lambda(\tau^+), \tau]$
$\quad -\zeta(\tau) h_t(x^*(\tau), \tau)$;
the Lagrange multipliers $\mu(t)$ are such that
$\partial L/\partial u|_{u=u^*(t)} = 0$, $dH/dt = dL/dt = \partial L/\partial t$,
and the complementary slackness conditions
$\mu(t) \geq 0,\ \mu(t) g(x^*, u^*, t) = 0,$
$\eta(t) \geq 0,\ \dot{\eta}(t) \leq 0,\ \eta(t) h(x^*(t), t) = 0$, and
$\zeta(\tau) \geq 0,\ \zeta(\tau) h(x^*(\tau), \tau) = 0$ hold. Also (3.14) if T is free. (29)

Note that the jump conditions on the adjoint variables in (29) generalize the jump condition (25) encountered in Example 4.2. The jump condition on H in (29) requires that the Hamiltonian should be continuous at τ if $h_t = 0$. The continuity of the Hamiltonian (in case $h_t = 0$) makes intuitive sense when considered in the light of its interpretation given in Section 2.2.4. Hartl, Sethi, and Vickson (1995) and Feichtinger and Hartl (1986) also add the following condition

$$\zeta(\tau_1) \geq \eta(\tau_1^+) \tag{30}$$

at each entry time τ_1 to the maximum principle necessary conditions (29).

Consider the following problem with the discount rate $\rho \geq 0$:

$$\max \left\{ J = \int_0^3 -e^{-\rho t} u dt \right\} \tag{31}$$

subject to

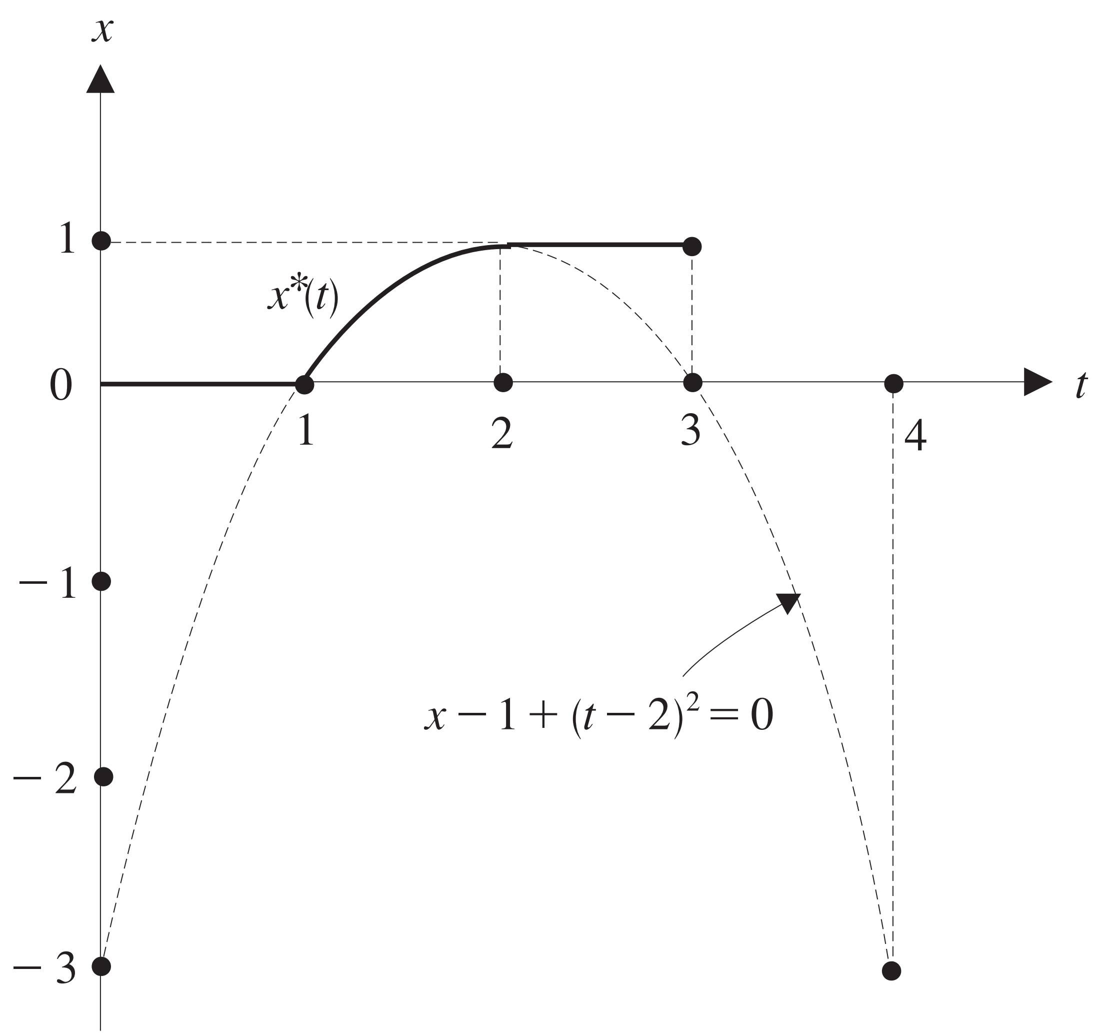
$$\dot{x} = u, \quad x(0) = 0, \tag{32}$$

$$0 \leq u \leq 3, \tag{33}$$

$$x - 1 + (t-2)^2 \geq 0. \tag{34}$$

From the objective function (31), it is good to have low values of u. If we use $u = 0$ to begin with, we see that $x(t) = 0$ as long as $u(t) = 0$. But continuing with $u(t) = 0$ beyond $t = 1$ is not feasible since $x(t) = 0$ would not satisfy the constraint (34) just after $t = 1$. At $t = 1$, the constraint (34) is satisfied with an equality; see Figure 4.2. In order not to violate the constraint, its first derivative $u - 2(2 - t)$ must be nonnegative. This gives us $u(t) = 2(2 - t)$ to be the lowest feasible value for the control. This value of control will make the state $x(t)$ ride on the constraint boundary until $t = 2$, at which point $u(2) = 0$; see Figure 4.2. Continuing with $u(t) = 2(2 - t)$ beyond $t = 2$ will make $u(t)$ negative, and violate the lower bound in (33).

Figure 4.2: Feasible State Space and Optimal State Trajectory for Example 4.3



It is easy to see that $u(t) = 0$, $t \geq 2$, is the lowest feasible value, which can be followed all the way to the terminal time $t = 3$. We can now restate the values of the state and the control variables that we have obtained:

$$x^*(t) = \begin{cases} 0, & t \in [0,1), \\ 1 - (t-2)^2, & t \in [1,2], \\ 1, & t \in (2,3], \end{cases} \quad u^*(t) = \begin{cases} 0, & t \in [0,1), \\ 2(2-t), & t \in [1,2], \\ 0, & t \in (2,3]. \end{cases}$$

The control u^* and, therefore, $h^{1*} = u^* + 2(t-2)$ is discontinuous, i.e., the entry is non-tangential. On the other hand, u^* and h^{1*} are continuous at $t = 2$ so that the exit is tangential.

With u^* and x^* thus obtained, we must obtain $\lambda,\ \mu_1,\ \mu_2,\ \eta$, and ζ so that the necessary optimality conditions (29) hold, i.e.,

$$H = -e^{-\rho t}u + \lambda u, \tag{35}$$

$$L = H + \mu_1 u + \mu_2(3-u) + \eta[u + 2(t-2)], \tag{36}$$

$$L_u = -e^{-\rho t} + \lambda + \mu_1 - \mu_2 + \eta = 0, \tag{37}$$

$$\dot{\lambda} = -L_x = 0,\ \lambda(3) = 0, \tag{38}$$

$$\mu_1 \geq 0,\ \mu_1 u = 0,\ \mu_2 \geq 0,\ \mu_2(3-u) = 0, \tag{39}$$

$$\eta \geq 0, \quad \dot{\eta} \leq 0, \quad \eta[x - 1 + (t-2)^2] = 0, \tag{40}$$

$$\lambda(1^-) = \lambda(1^+) + \zeta(1), H(1^-) = H(1^+) - \zeta(1)(-2), \; \zeta(1) \geq 0. \tag{41}$$

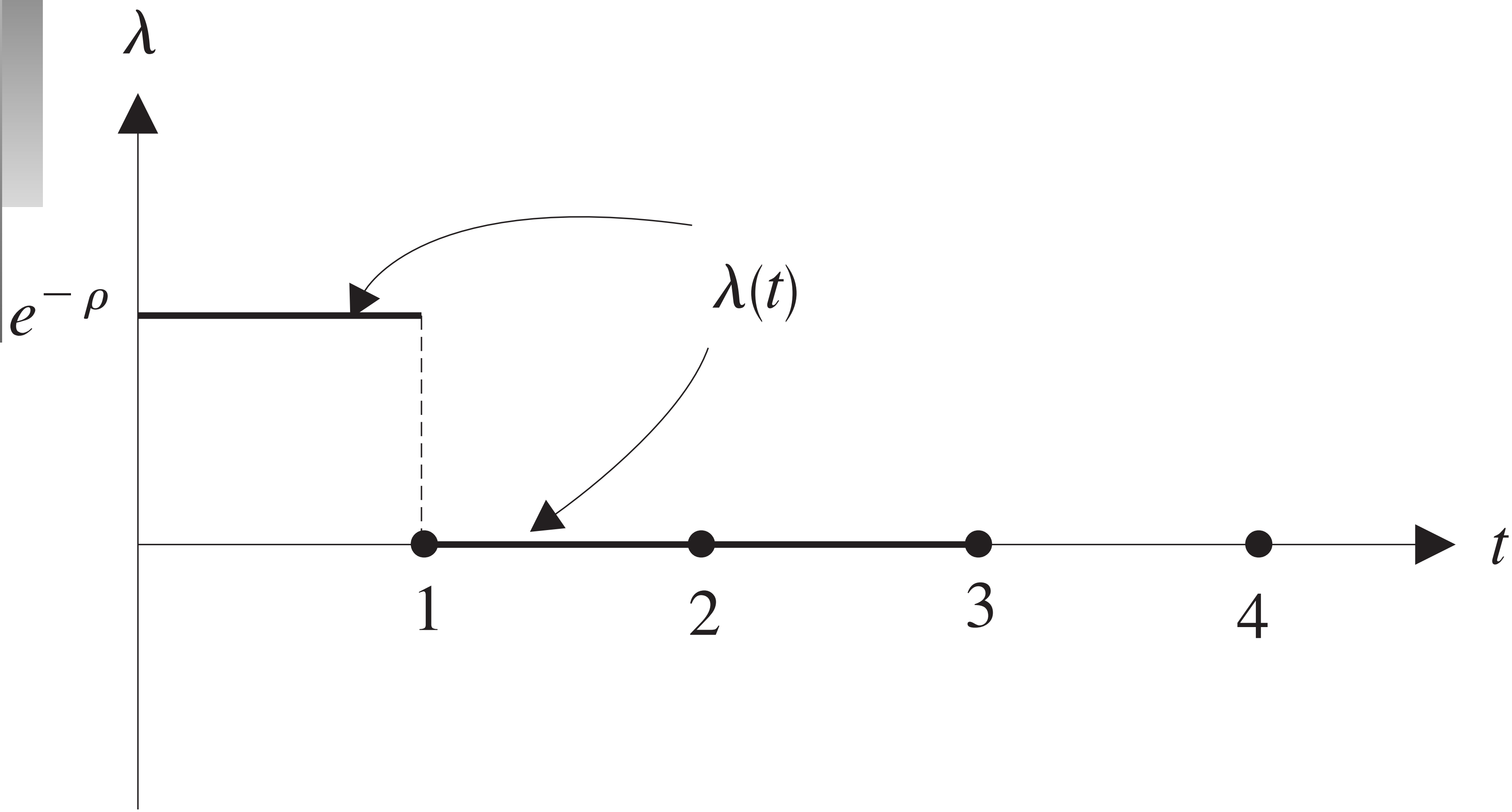
From (41), we obtain $\lambda(1^-) = e^{-\rho}$. This with (38) gives

$$\lambda(t) = \begin{cases} e^{-\rho}, & 0 \leq t < 1, \\ 0, & 1 \leq t \leq 3, \end{cases}$$

as shown in Figure 4.3,

$$\mu_1(t) = \begin{cases} e^{-\rho t} - e^{-\rho}, & 0 \leq t < 1, \\ 0, & 1 \leq t \leq 2, \\ e^{-\rho t}, & 2 < t \leq 3, \end{cases} \quad \mu_2(t) = 0, \; 0 \leq t \leq 3,$$

Figure 4.3: Adjoint Trajectory for Example 4.3



SOLUTION OF EXAMPLE 4.3 CONT.

$$\eta(t) = \begin{cases} 0, & 0 \le t < 1, \\ e^{-\rho t}, & 1 \le t \le 2, \\ 0, & 2 < t \le 3, \end{cases}$$

which, along with u^* and x^*, satisfy (29). Note, furthermore, that λ is continuous at the exit time $t = 2$. At the entry time $\tau_1 = 1$, $\zeta(1) = e^{-\rho} \ge \eta(1^+) = e^{-\rho}$, so that (30) also holds.

With the Hamiltonian H as defined in (3.33), we can write the Lagrangian

$$L[x, u, \lambda, \mu, \eta] \equiv H + \mu g + \eta h^1 = \phi + \lambda f + \mu g + \eta h^1.$$

We can now state the current-value form of the maximum principle, which states that the necessary conditions for u^* (with the state trajectory x^*) to be an optimal control are that there exist $\lambda, \mu, \alpha, \beta, \gamma, \eta,$ and ζ, which satisfy (42) that follows:

$$
\begin{aligned}
&\dot{x}^* = f(x^*, u^*, t),\ x^*(0) = x_0,\\
&\text{satisfying constraints}\\
&g(x^*, u^*, t) \geq 0,\ h(x^*(t), t) \geq 0,\\
&\text{and the terminal constraints}\\
&a(x^*(T), T) \geq 0 \text{ and } b(x^*(T), T) = 0;\\
&\dot{\lambda} = \rho\lambda - L_x[x^*, u^*, \lambda, \mu, \eta, t]\\
&\text{with the transversality conditions}\\
&\lambda(T^-) = \sigma_x(x^*(T), T) + \alpha a_x(x^*(T), T) + \beta b_x(x^*(T), T)\\
&\qquad\qquad +\gamma h_x(x^*(T), T), \text{ and}\\
&\alpha \geq 0,\ \alpha a(x^*(T), T) = 0,\ \gamma \geq 0,\ \gamma h(x^*(T), T) = 0;\\
&\text{the Hamiltonian maximizing condition}\\
&H[x^*(t), u^*(t), \lambda(t), t] \geq H[x^*(t), u, \lambda(t), t]\\
&\text{at each } t \in [0, T] \text{ for all } u \text{ satisfying}\\
&g[x^*(t), u, t] \geq 0, \text{ and}\\
&h_i^1(x^*(t), u, t) \geq 0 \text{ whenever } h_i(x^*(t), t) = 0, i = 1, 2, \cdots, p;\\
&\text{the jump conditions at any entry/contact time } \tau \text{ are}\\
&\lambda(\tau^-) = \lambda(\tau^+) + \zeta(\tau) h_x(x^*(\tau), \tau) \text{ and}\\
&H[x^*(\tau), u^*(\tau^-), \lambda(\tau^-), \tau] = H[x^*(\tau), u^*(\tau^+), \lambda(\tau^+), \tau]\\
&\qquad\qquad\qquad\qquad -\zeta(\tau) h_t(x^*(\tau), \tau);\\
&\text{the Lagrange multipliers } \mu(t) \text{ are such that}\\
&\partial L/\partial u|_{u=u^*(t)} = 0,\ dH/dt = dL/dt = \partial L/\partial t + \rho\lambda f,\\
&\text{and the complementary slackness conditions}\\
&\mu(t) \geq 0,\ \mu(t) g(x^*, u^*, t) = 0,\\
&\eta(t) \geq 0,\ \dot{\eta}(t) \leq \rho\eta(t),\ \eta(t) h(x^*(t), t) = 0, \text{ and}\\
&\zeta(\tau) \geq 0,\ \zeta(\tau) h(x^*(\tau), \tau) = 0 \text{ hold. Also (3.40) if } T \text{ is free.}
\end{aligned}
\tag{42}
$$

The sufficiency results can be stated in the indirect adjoining framework. In order to do so, let us define the Hamiltonian H and the Lagrangian L^d in the direct method as

$$H(x, u, \lambda^d, t) = F(x, u, t) + \lambda^d f(x, u, t) \tag{43}$$

and

$$L^d(x, u, \lambda^d, \mu^d, \eta^d, t) = H(x, u, \lambda^d, t) + \mu^d g(x, u, t) + \eta^d h(x, t), \tag{44}$$

where λ^d, μ^d and η^d are multipliers in the direct formulation, corresponding to λ, μ and η in the indirect formulation. It can be shown that:

$$\lambda^d(t) = \lambda(t) + \eta(t) h_x(x^*(t), t) \text{ and } \mu^d(t) = \mu(t). \tag{45}$$

Let $(x^*, u^*, \lambda, \mu, \alpha, \beta, \gamma, \zeta, \eta)$ *satisfy the necessary conditions in* (29) *and let* $\lambda^d(t) = \lambda(t) + \eta(t)h_x(x^*(t), t)$. *If* $H(x, u, \lambda^d(t), t)$ *is concave in* (x, u) *at each* $t \in [0, T]$, S *in* (3.2) *is concave in* x, g *in* (3.3) *is quasiconcave in* (x, u), h *in* (4.26) *and* a *in* (3.4) *are quasiconcave in* x, *and* b *in* (3.5) *is linear in* x, *then* (x^*, u^*) *is optimal.*

Theorem 4.1 *remains valid if the concavity of $H(x, u, \lambda^d(t), t)$ in (x, u) at each t is replaced by the concavity of the maximized Hamiltonian $H^0(x, \lambda^d(t), t)$ in x at each t, where*

$$H^0(x, \lambda^d, t) = \max_{\{u|g(x,u,t)\geq 0\}} H(x, u, \lambda^d, t). \tag{46}$$

Theorems 4.1 and 4.2 are written for finite horizon problems. For infinite horizon problems, these theorems remain valid if the transversality conditions on the adjoint variables in (29) is replaced by the following limiting transversality condition

$$\lim_{t \to \infty} \lambda(t)[x(t) - x^*(t)] \geq 0 \tag{47}$$

for every feasible state trajectory $x(t),\ t \geq 0$.

First, we obtain the direct adjoint variable

$$\lambda^d(t) = \lambda(t) + \eta(t) h_x(x^*(t), t) = \begin{cases} t - 1, & t \in [0, 1), \\ 0, & t \in [1, 2]. \end{cases}$$

It is easy to see that

$$H(x, u, \lambda^d(t), t) = \begin{cases} -x + (t-1)u, & t \in [0, 1), \\ -x, & t \in [1, 2], \end{cases}$$

is linear and hence concave in (x, u) at each $t \in [0, 2]$.

Functions

$$g(x,u,t) = \begin{pmatrix} u+1 \\ 1-u \end{pmatrix}$$

and

$$h(x) = x$$

are linear and hence quasiconcave in (x,u) and x, respectively. Functions $S \equiv 0$, $a \equiv 0$ and $b \equiv 0$ satisfy the conditions of Theorem 4.1 trivially. Thus, the solution obtained for Example 4.1 satisfies all the conditions of Theorem 4.1, and is therefore optimal.