The state variable x is affected by the obsolescence factor, the amount of preventive maintenance, and the maintenance effectiveness function. Thus,

$$\dot{x}(t) = -d(t) + g(t)u(t), \quad x(0) = x_0. \tag{9.3}$$

In the interests of realism we assume that

$$-d(t) + g(t)U \le 0, \; t \ge 0. \tag{9.4}$$

The assumption implies that preventive maintenance is not so effective as to enhance the resale value of the machine over its previous values; rather it can at most slow down the decline of the resale value, even when preventive maintenance is performed at the maximum rate U. A modification of (9.3) is given in Arora and Lele (1970). See also Hartl (1983b).

The optimal control problem is to maximize (9.2) subject to (9.1) and (9.3).

9.1.2 Solution by the Maximum Principle

This problem is similar to Model Type (a) of Table 3.3 with the free-end-point condition as in Row 1 of Table 3.1. Therefore, we follow the steps for solution by the maximum principle stated for it in Chapter 3.

The standard Hamiltonian as formulated in Section 2.2 is

$$H = (\pi x - u)e^{-\rho t} + \lambda(-d + gu), \tag{9.5}$$

where the adjoint variable λ satisfies

$$\dot{\lambda} = -\pi e^{-\rho t}, \quad \lambda(T) = e^{-\rho T}. \tag{9.6}$$

Since T is unspecified, the required additional terminal condition (3.14) for this problem is

$$-\rho e^{-\rho T} x(T) = -H, \tag{9.7}$$

which must hold on the optimal path at time T.

The adjoint variable λ can be easily obtained by integrating (9.6), i.e.,

$$\lambda(t) = e^{-\rho T} + \frac{\pi}{\rho}[e^{-\rho t} - e^{-\rho T}]. \tag{9.8}$$

The interpretation of $\lambda(t)$ is as follows. It gives in present value terms, the marginal profit per dollar of gain in resale value at time t. The first term represents the present value of one dollar of additional salvage value at T brought about by one dollar of additional resale value at the current time t. The second term represents the present value of incremental production from t to T brought about by the extra productivity of the machine due to the additional one dollar of resale value at time t.

Since the Hamiltonian is linear in the control variable u, the optimal control for a problem with any fixed T is bang-bang as in Model Type (a) in Table 3.3. Thus,

$$u^*(t) = \text{bang}\left[0, U; e^{-\rho T}g(t) + \frac{\pi}{\rho}(e^{-\rho t} - e^{-\rho T})g(t) - e^{-\rho t}\right]. \tag{9.9}$$

To interpret this optimal policy, we see that the term

$$e^{-\rho T}g(t) + \frac{\pi}{\rho}(e^{-\rho t} - e^{-\rho T})g(t)$$

is the present value of the marginal return from increasing the preventive maintenance by one dollar at time t. The last term $e^{-\rho t}$ in the argument of the bang function is the present value of that one dollar spent for preventive maintenance at time t. Thus, in words, the optimal policy means the following: If the marginal return of one dollar of additional preventive maintenance is more than one dollar, then perform the maximum possible preventive maintenance, otherwise do not carry out any at all.

To find how the optimal control switches, we need to examine the switching function in (9.9). Rewriting it as

$$e^{-\rho t}\left[\frac{\pi g}{\rho} - (\frac{\pi}{\rho} - 1)e^{\rho(t-T)}g - 1\right] \tag{9.10}$$

and taking the derivative of the bracketed terms with respect to t, we can conclude that the expression inside the square brackets in (9.10) is monotonically decreasing with time t on account of the assumptions that $\pi/\rho > 1$ and that g is nonincreasing with t. It follows that there will not be any singular control for any finite interval of time. Furthermore, since $e^{-\rho t} > 0$ for all t, we can conclude that the switching function can only go from positive to negative and not vice versa. Thus, the optimal control will be either U, or zero, or U followed by zero. The switching time t^s is obtained as follows: equate (9.10) to zero and solve for t. If the solution is negative, let $t^s = 0$, and if the solution is greater than T, let $t^s = T$, otherwise set t^s equal to the solution. It is clear that the

T, the optimal control is

$$u^*(t) = \text{sat}[0, U; u^0(t)]. \tag{9.22}$$

To determine the direction of change in $u^*(t)$, we obtain $\dot{u}^0(t)$. For this, we use (9.21) and the value $\lambda(t)$ from (9.8) to obtain

$$g_u = \frac{e^{-\rho t}}{\lambda(t)} = \frac{1}{\frac{\pi}{\rho} - (\frac{\pi}{\rho} - 1)e^{\rho(t-T)}}. \tag{9.23}$$

Since $\pi > \rho$, the denominator on the right-hand side of (9.23) is monotonically decreasing with time. Therefore, the right-hand side of 9.23) is increasing with time. Taking the time derivative of (9.23), we have

$$g_{ut} + g_{uu}\dot{u}^0 = \rho^2(\pi - \rho)e^{\rho(t-T)}/[\pi - (\pi - \rho)e^{\rho(t-T)}]^2 > 0.$$

But $g_{ut} \leq 0$ and $g_{uu} < 0$, it is therefore obvious that $\dot{u}^0(t) < 0$. In order now to sketch the optimal control $u^*(t)$ specified in (9.22), let us define $0 \leq t_1 \leq t_2 \leq T$ such that $u^0(t) \geq U$ for $t \leq t_1$ and $u^0(t) \leq 0$ for $t \geq t_2$. Then, we can rewrite the sat function in (9.22) as

$$u^*(t) = \begin{cases} U & \text{for } t \in [0, t_1], \\ u^0(t) & \text{for } t \in (t_1, t_2), \\ 0 & \text{for } t \in [t_2, T]. \end{cases} \tag{9.24}$$

In (9.24), it is possible to have $t_1 = 0$ and/or $t_2 = T$. In Figure 9.2 we have sketched a case when $t_1 > 0$ and $t_2 < T$.

Note that while $u^0(t)$ in Figure 9.2 is decreasing over time, the way it will decrease will depend on the nature of the function g. Indeed, the shape of $u^0(t)$, while always decreasing, can be quite general. In particular, you will see in Exercise 9.5 that the shape of $u^0(t)$ is concave and, furthermore, $u^0(t) > 0,\ t \geq 0$, so that $t_2 = T$ in that case.

9.2 Maintenance and Replacement for a Machine Subject to Failure

In Kamien and Schwartz (1971a), a related model is developed which has somewhat different assumptions. They assume that the production rate of the machine is independent of its age, while its probability of failure

ρ	$=$	the constant discount rate,
L	$=$	the constant positive junk value of the failed machine independent of its age at failure,
$B(t)$	$=$	the (exogenously specified) resale value of the machine at time t, if it is still functioning; $\dot{B}(t) \leq 0$,
$h(t)$	$=$	the natural failure rate (also termed the natural hazard rate in the reliability theory); $h(t) \geq 0$, $\dot{h}(t) \geq 0$,
$F(t)$	$=$	the cumulative probability that the machine has failed by time t,
$C(u,h)$	$=$	the cost function depending on the preventive maintenance u when the natural failure rate is h.

To make economic sense, an operable machine must be worth at least as much as an inoperable machine and its resale value should not exceed the present value of the potential revenue generated by the machine if it were to function forever. Thus,

$$0 \leq L \leq B(t) \leq R/\rho, \quad t \geq 0. \tag{9.25}$$

Also for all $t > 0$,

$$u(t) \in \Omega = [0, 1]. \tag{9.26}$$

Finally, the cost of reducing the natural failure rate is assumed to be proportional to the natural failure rate. Specifically, we assume that $C(u,h) = C(u)h$ denotes the cost of preventive maintenance u when the natural failure rate is h. In other words, when the natural failure rate is h and a controlled failure rate of $h(1-u)$ is sought, the action of achieving this reduction will cost $C(u)h$ dollars. It is assumed that

$$C(0) = 0, \; C_u > 0, \; C_{uu} > 0, \; \text{for } u \in [0, 1]. \tag{9.27}$$

Thus, the cost of reducing the failure rate increases more than proportionately as the fractional reduction increases. But the cost of a given fractional reduction increases linearly with the natural failure rate. Hence, these conditions imply that a given absolute reduction becomes increasingly more costly as the machine gets older.

To derive the state equation for $F(t)$, we note that $\dot{F}/(1-F)$ denotes the conditional probability density for the failure of machine at time t given that it has survived to time t. This is assumed to depend on two things, namely (i) the natural failure rate that governs the machine in the absence of preventive maintenance, and (ii) the current rate of preventive maintenance.

Thus,

$$\frac{\dot{F}(t)}{1-F(t)} = h(t)[1-u(t)], \tag{9.28}$$

which gives the state equation

$$\dot{F} = h(1-u)(1-F),\ F(0) = 0. \tag{9.29}$$

Thus, the controlled failure rate at time t is $h(t)(1-u(t))$. If $u=0$, the failure rate assumes its natural value h. As u increases, the failure rate decreases and drops to zero when $u=1$.

The expected present value of the machine is the sum of the expected present values of (i) the total revenue it produces less the total cost of maintenance, (ii) its junk value if it should fail, and (iii) the salvage value if it does not fail and is sold. That is,

$$J = \int_0^T e^{-\rho t}\left\{[R - C(u)h](1-F) + L\dot{F}\right\} dt + e^{-\rho T}B(T)[1-F(T)].$$

Using (9.29), we can rewrite J as follows:

$$J = \int_0^T e^{-\rho t}[R - C(u)h + L(1-u)h]\,(1-F)dt + e^{-\rho T}B(T)\,[1-F(T)]\,. \tag{9.30}$$

The optimal control problem is to maximize J in (9.30) subject to (9.29) and (9.26).

9.2.2 Optimal Policy

The problem is similar to Model Type (f) in Table 3.3 subject to the free-end-point condition as in Row 1 of Table 3.1. Therefore, we follow the steps for solution by the maximum principle stated in Chapter 3. The standard Hamiltonian is

$$H = e^{-\rho t}[R - C(u)h + L(1-u)h](1-F) + \lambda[(1-u)h(1-F)], \tag{9.31}$$

and the adjoint variable satisfies

$$\begin{cases} \dot{\lambda} & = e^{-\rho t}[R - C(u)h + L(1-u)h] + \lambda(1-u), \\ \lambda(T) & = -e^{-\rho T}B(T). \end{cases} \tag{9.32}$$

Since T is unspecified, we apply the additional terminal condition (3.14) to obtain (see Exercise 9.6)

$$\begin{aligned} R - C[u^*(T^*)]h(T^*) + L[1 - u^*(T^*)]h[T^*] \\ -[\rho + \{1 - u^*(T^*)\}h(T^*)]B(T^*) = -B_T(T^*). \end{aligned} \tag{9.33}$$

Consider keeping the machine to time $T^* + \delta$. The first two terms in (9.33) when multiplied by δ give the incremental net cash inflow (revenue − cost of preventive maintenance), to which is added the junk value L multiplied by the probability $[1 - u^*(T^*)]h(T^*)\delta$ that the machine fails during the short time δ. From this, we subtract the third term which is the sum of loss of interest $\rho B(T^*)\delta$ on the resale value and the loss of the entire resale value, when the machine fails, with probability $[1 - u^*(T^*)]h(T^*)\delta$. Thus, the LHS of (9.33) represents the marginal benefit of keeping the machine to $T^* + \delta$. On the other hand, the RHS term $-B_T(T^*)\delta$ is decrease in the resale value from time T^* to $T^* + \delta$. Hence, equation (9.33) determining the optimal sale date is the usual economic condition equating marginal benefit to marginal cost.

Next, we analyze the problem to obtain the optimal maintenance policy for a fixed T. If the optimal solution is in the interior, i.e., $u^* \in (0, 1)$, then the Hamiltonian maximizing condition gives

$$H_u = -e^{-\rho t}h(1 - F)[C_u + L + e^{\rho t}\lambda] = 0. \tag{9.34}$$

In the trivial cases in which the natural failure rate $h(t)$ is zero or when the machine fails with certainty by time t (i.e., $F(t) = 1$), then $u^*(t) = 0$. Assume therefore $h > 0$ and $F < 1$. Under these conditions, we can infer from (9.27) and (9.34) that

$$\left.\begin{array}{lll} \text{(i)} & C_u(0) + L + \lambda e^{\rho t} > 0 \Rightarrow u^*(t) = 0, \\ \text{(ii)} & C_u(1) + L + \lambda e^{\rho t} < 0 \Rightarrow u^*(t) = 1, \\ \text{(iii)} & \text{Otherwise,}\ \ C_u + L + \lambda e^{\rho t} = 0 \text{ determines } u^*(t). \end{array}\right\} \tag{9.35}$$

Using the terminal condition $\lambda(T) = -e^{-\rho T}B(T)$ from (9.32), we can derive $u^*(T)$ satisfying (9.35):

$$\left.\begin{array}{ll} \text{(i)} & C_u(0) > B(T) - L \text{ and } u^*(T) = 0, \\ \text{(ii)} & C_u(1) < B(T) - L \text{ and } u^*(T) = 1, \\ \text{(iii)} & \text{Otherwise, } C_u = B(T) - L \Rightarrow u^*(T). \end{array}\right\} \tag{9.36}$$

Next we determine how $u^*(t)$ changes over time. Kamien and Schwartz (1971a, 1998) have shown that $u^*(t)$ is non-increasing; see Exercise 9.7. Thus there exists $T \geq t_2 \geq t_1 \geq 0$ such that

$$u^*(t) = \begin{cases} 1 & \text{for } t \in [0, t_1], \\ u^0(t) & \text{for } t \in (t_1, t_2), \\ 0 & \text{for } t \in (t_2, T]. \end{cases} \tag{9.37}$$

Here $u^0(t)$ is the solution of (9.35)(iii), and it is easy to show that $\dot{u}^0(t) \leq 0$. Of course, $u^*(T)$ is immediately known from (9.36). If $u^*(T) \in (0,1)$, it implies $t_2 = T$; and if $u^*(T) = 1$, it implies $t_1 = t_2 = T$.

For this model, the sufficiency of the maximum principle follows from Theorem 2.1; see Exercise 9.8.

9.2.3 Determination of the Sale Date

For a fixed T, we know that the terminal optimal control $u^*(T)$ is determined by (9.36). If this $u^*(T)$ also satisfies (9.33), we have determined an optimal trajectory as well as the optimal life of the machine. This, of course, is subject to the second-order condition since (9.33) is only a necessary condition for an optimal T^* to satisfy. It is clear that the determination of T^*, in most cases, will require numerical computations. The algorithm needs only be a simple search method because it requires consideration of the single variable T.

Before we go to the next section, we remark that a business is usually a continuing entity and does not end at the sale date of one machine. Normally, an existing machine will be replaced by another which, in turn, will be replaced by another, and so on. The technology of the newer machines will in general be different from that of the existing machine. In what follows, we address these issues. We shall choose the discrete-time setting and illustrate the use of the discrete-time maximum principle developed in Chapter 8.