

Some Basic Classes of Subsets of a Space \mathcal{X}

- *Semiring*. A nonempty class \mathcal{C} of subsets of \mathcal{X} satisfying
 - (a) $A, B \in \mathcal{C}$ implies $A \cap B \in \mathcal{C}$.
 - (b) For $A, B \in \mathcal{C}$ with $A \subset B$, there is a finite partition of $B - A$ in \mathcal{C} : there exist disjoint C_1, \dots, C_n in \mathcal{C} such that $B - A = \cup_{i=1}^n C_i$.

EXAMPLES:

- (i) The class $\emptyset \cup \{ \{x\}, x \in \mathcal{X} \}$.
- (ii) The class of finite, left-closed, and right-open intervals in \mathbb{R} .

- *Ring*. A nonempty class \mathcal{C} of subsets of \mathcal{X} satisfying

- (a) $A, B \in \mathcal{C}$ implies $A - B \in \mathcal{C}$.
- (b) $A, B \in \mathcal{C}$ implies $A \cup B \in \mathcal{C}$.

FACTS:

- (i) The empty set \emptyset must belong to \mathcal{C} .
- (ii) A semiring closed under the formation of unions is a ring.

- σ -ring. A ring closed under the formation of countable unions.

- *Monotone class*. A nonempty class \mathcal{C} of sets containing the limits of monotone sequences in \mathcal{C} : for an increasing or decreasing sequence of sets $C_i \in \mathcal{C}$, we have $\lim_n C_n \in \mathcal{C}$.

FACTS:

- (i) Every σ -ring is a monotone class.
- (ii) Every monotone ring is a σ -ring.
- (iii) A *nonmonotone* sequence can have a limit. Take $C_{2n} \subset C_{2n+2} \subset \dots \uparrow C$ and Take $C_{2n+1} \supset C_{2n+3} \supset \dots \downarrow C$.

- π -class. A nonempty class \mathcal{C} of subsets of \mathcal{X} closed under the formation of intersections.

- λ -class (also called *d-system*). A nonempty class \mathcal{C} of subsets of \mathcal{X} containing \mathcal{X} and satisfying

- (a) $A, B \in \mathcal{C}$ implies $B \setminus A \in \mathcal{C}$.
- (b) For an increasing sequence of sets $C_i \in \mathcal{C}$, we have $\lim_n C_n \in \mathcal{C}$.

FACTS:

- (i) A λ -class is closed under the formation of complements and of finite and countable *disjoint* unions.
- (ii) *Similar events*. Let $\{P_\theta, \theta \in \Theta\}$ be a family of probability distributions. An event A is “*similar*” if $P_\theta(A)$ is independent of $\theta \in \Theta$. The class of similar events forms a λ -class.

- *Semialgebra*, or *semifield*. A semiring containing \mathcal{X} .

- *Algebra, or field.* A nonempty class \mathcal{C} of subsets of \mathcal{X} closed under the formation of complements and unions:

- (a) For $A \in \mathcal{C}$, the complement A^c is in \mathcal{C} .
- (b) $A, B \in \mathcal{C}$ implies $A \cup B \in \mathcal{C}$.

FACTS:

- (i) An algebra is a ring containing \mathcal{X} .
- (ii) Every monotone class is an algebra.

- *σ -algebra, or σ -field.* A nonempty class \mathcal{C} of subsets of \mathcal{X} closed under the formation of complements and countable unions.

FACTS:

- (i) A σ -algebra is a σ -ring containing \mathcal{X} .
- (ii) A σ -algebra is a special case of λ -class: namely, a λ -class closed under formation of pairwise intersections. Equivalently, removing the “disjoint” restriction in the definition of λ -class yields that of a σ -algebra.

* *Permutation tests* (including *rank tests*) have critical regions belong to the λ -class of similar events. Critical regions of rank tests, however, constitute a σ -algebra.

* *Probability functions* are well-defined on λ -classes. *Conditional* probability functions and expectations, however, are defined only relative to specified σ -algebras.

* *Example when it is straightforward to define probability on a λ -class but not on its extension to a σ -algebra.* Let event E occur with probability $1/2$, and let event F occur independently of E . Let A occur if E occurs, and let B occur if $(E \cap F) \cup (E^c \cap F^c)$ occurs. Then $P(A) = P(B) = 1/2$. But $P(A \cap B)$ is indeterminate from the given experiment.

(iii) A class that is both a π -class and a λ -class is a σ -algebra.

(iv) A σ -algebra is closed under countable intersections.

(v) An algebra is a σ -algebra if and only if it is a monotone class.

- *Generated σ -algebras.* For a class \mathcal{E} of subsets of \mathcal{X} , there exists a *smallest* (or *minimal*) σ -algebra containing \mathcal{E} , denoted $\sigma(\mathcal{E})$. It is also called the *σ -algebra generated by \mathcal{E}* .

- *The Borel sets of \mathbb{R} .* Start with the semiring

$$\mathcal{E}_0 = \{[a, b) : -\infty < a < b < \infty\},$$

the class of finite, left-closed, and right-open intervals in the real line. The *algebra* \mathcal{E} generated by \mathcal{E}_0 consists of all finite unions of disjoint sets of the form $[a, b)$, $(-\infty, a)$, or $[b, \infty)$. The *σ -algebra* $\sigma(\mathcal{E}_0) = \sigma(\mathcal{E})$ generated by \mathcal{E} (or \mathcal{E}_0) is called the *Borel sets in \mathbb{R}* .

The semiring of left-open, right-closed intervals also generates the Borel sets. In fact, quite simply, the smallest σ -algebra containing the class of infinite intervals $\{(-\infty, a) : -\infty < a < \infty\}$ is the class of Borel sets.

Note that, using the above-noted fact that an algebra is a σ -algebra if and only if it is a monotone class, one can pass from the algebra \mathcal{E} to $\sigma(\mathcal{E})$ by means of limiting processes.

To determine a probability on the Borel sets $\sigma(\mathcal{E}_0)$, it suffices to specify its values simply on the intervals $[a, b)$. This is because any such specification has a *unique extension* to a

probability measure on all the Borel sets. In the case of continuous probability distributions, however, extensions beyond this class of sets need not be unique. Thus the Borel sets represent a convenient level of generality for probability modeling.

- *The Borel sets of \mathbb{R}^k .* Similarly defined, replacing intervals by rectangles

$$[\mathbf{x} = (x_1, \dots, x_k) : a_i \leq x_i < b_i, i = 1, \dots, k]$$

(sometimes called hyperrectangles) in the definition of $\sigma(\mathcal{E}_0)$. Indeed, the (countable) class of rectangles with a_i and b_i all rational generates the Borel sets of \mathbb{R}^k .

– *RJS, 1/28/2012*