are certainly nonnegative. To insure that $|g(\theta)|^2$ is always at most 1, we must have this quantity nonnegative; i.e., the Lax–Wendroff scheme is stable if and only if $|a\lambda| \le 1$.

For the Crank–Nicolson scheme from (3.1.3) we have

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^{n+1} - v_{m-1}^{n+1} + v_{m+1}^n - v_{m-1}^n}{4h} = 0,$$

where we have set $f = 0$ as required in obtaining the amplification factor. Substituting $g^{n'} e^{im'\theta}$ for $v_{m'}^{n'}$ and then canceling, we obtain

$$\frac{g-1}{k} + a\frac{ge^{i\theta} - ge^{-i\theta} + e^{i\theta} - e^{-i\theta}}{4h} = 0.$$

Or,

$$g - 1 + a\lambda\frac{g+1}{2} i \sin\theta = 0,$$

from which we obtain the following expression for the amplification factor:

$$g(\theta) = \frac{1 - i\frac{1}{2}a\lambda\sin\theta}{1 + i\frac{1}{2}a\lambda\sin\theta}.$$

As the ratio of a complex number and its conjugate we have immediately that $|g(\theta)| = 1$. Alternatively,

$$|g(\theta)|^2 = \frac{1 + (\frac{1}{2}a\lambda\sin\theta)^2}{1 + (\frac{1}{2}a\lambda\sin\theta)^2} = 1.$$

This scheme is stable for any value of λ; it is *unconditionally stable.*

Exercises

3.2.1. Show that the (forward-backward) MacCormack scheme

$$\tilde{v}_m^{n+1} = v_m^n - a\lambda(v_{m+1}^n - v_m^n) + kf_m^n,$$

$$v_m^{n+1} = \tfrac{1}{2}(v_m^n + \tilde{v}_m^{n+1} - a\lambda(\tilde{v}_m^{n+1} - \tilde{v}_{m-1}^{n+1}) + kf_m^{n+1})$$

is a second-order accurate scheme for the one-way wave equation (1.1.1). Show that for $f = 0$ it is identical to the Lax–Wendroff scheme (3.1.1).

3.2.2. Show that the backward-time central-space scheme (1.6.1) is unconditionally stable.

3.2.3. Show that the box scheme

$$\begin{aligned}\frac{1}{2k}&\left[(v_m^{n+1} + v_{m+1}^{n+1}) - (v_m^n + v_{m+1}^n)\right]\\ &+ \frac{a}{2h}\left[(v_{m+1}^{n+1} - v_m^{n+1}) + (v_{m+1}^n - v_m^n)\right] \qquad (3.2.3)\\ &= \frac{1}{4}\left(f_{m+1}^{n+1} + f_m^{n+1} + f_{m+1}^n + f_m^n\right)\end{aligned}$$

is an approximation to the one-way wave equation $u_t + au_x = f$ that is accurate of order $(2, 2)$ and is stable for all values of λ.

3.2.4. Using the box scheme (3.2.3), solve the one-way wave equation

$$u_t + u_x = \sin(x - t)$$

on the interval $[0, 1]$ for $0 \leq t \leq 1.2$ with $u(0, x) = \sin x$ and with $u(t, 0) = -(1 + t)\sin t$ as the boundary condition.

Demonstrate the second-order accuracy of the solution using $\lambda = 1.2$ and $h = \frac{1}{10}, \frac{1}{20}, \frac{1}{40}$, and $\frac{1}{80}$. Measure the error in the L^2 norm (3.1.24) and the maximum norm. To implement the box scheme note that v_0^{n+1} is given by the boundary data, and then each value of v_{m+1}^{n+1} can be determined from v_m^{n+1} and the other values.

3.2.5. Show that the following modified box scheme for $u_t + au_x = f$ is accurate of order $(2, 4)$ and is unconditionally stable. The scheme is

$$\begin{aligned}
&\frac{1}{16}(-v_{m+2}^{n+1} + 9v_{m+1}^{n+1} + 9v_m^{n+1} - v_{m-1}^{n+1}) \\
&\quad + \frac{a\lambda}{48}(-v_{m+2}^{n+1} + 27v_{m+1}^{n+1} - 27v_m^{n+1} + v_{m-1}^{n+1}) \\
&= \frac{1}{16}(-v_{m+2}^{n} + 9v_{m+1}^{n} + 9v_m^{n} - v_{m-1}^{n}) \\
&\quad - \frac{a\lambda}{48}(-v_{m+2}^{n} + 27v_{m+1}^{n} - 27v_m^{n} + v_{m-1}^{n}) \\
&\quad + \frac{k}{32}(-f_{m+2}^{n+1} + 9f_{m+1}^{n+1} + 9f_m^{n+1} - f_{m-1}^{n+1} \\
&\qquad - f_{m+2}^{n} + 9f_{m+1}^{n} + 9f_m^{n} - f_{m-1}^{n}).
\end{aligned}$$

3.3 Difference Notation and the Difference Calculus

To assist in our analysis and discussion of schemes we introduce some notation for finite differences. The forward and backward difference operators are defined by

$$\delta_+ v_m = \frac{v_{m+1} - v_m}{h} \tag{3.3.1}$$

and

$$\delta_- v_m = \frac{v_m - v_{m-1}}{h}, \tag{3.3.2}$$

respectively. We will occasionally use the notation δ_{x+} and δ_{x-} for these operators and define

$$\delta_{t+} v_m^n = \frac{v_m^{n+1} - v_m^n}{k}$$