Math 6313 Midterm Exam 10/16/17
Dr. Minkoff

Name:

Instructions: You may not use notes or books on this exam. Don't spend too much time on any one problem. Show your work!

NAME:							
1	/12	2	/11	3	/16	4	/12
5	/14	6	/12	7	/8	T	/85

[12 pts](1) Identify the (decimal) floating point number corresponding to the bit string $\underset{\text{sign}}{0}\underbrace{10000001}_{\text{exp}}\underbrace{01100000000000000000000}_{\text{mantissa}}$ stored on a Marc-32 machine.

Sign = 0 ⇒ positive number

exponent = $10000001 = 1\times 2^0 + 0\times 2^1 + 0\times 2^2 + 0\times 2^3 + 0\times 2^4 + 0\times 2^5 + 0\times 2^6 + 1\times 2^7 = 1+128 = 129$

$129-127 = 2$ so $\exp = 2$

mantissa = 011 0000 0000 0000 0000 0000

$= 1.011_2 = 1\times 2^0 + 0\times 2^{-1} + 1\times 2^{-2} + 1\times 2^{-3}$

$= 1 + \frac{1}{4} + \frac{1}{8} = 1.375_{10}$

So in decimal number is $1.375\times 2^2 = \boxed{5.5}$

[11 pts](2a) How would you compute

$$\sqrt{x^2+1}-1$$

to avoid loss of significance when $x \approx 0$?

$$\frac{\left(\sqrt{x^2+1}-1\right)\left(\sqrt{x^2+1}+1\right)}{\left(\sqrt{x^2+1}+1\right)} = \frac{x^2+1-1}{\sqrt{x^2+1}+1}$$

$$= \frac{x^2}{\sqrt{x^2+1}+1}$$

(b) If you are only comfortable losing two digits of accuracy in your answer to Part (a), for what range of x values should you use the formula given in Part (a)? For what range should you use your modified formula?

Using Loss of Precision Theorem, we need $x > y > 0$ and

$$2^{-p} \le 1 - \frac{y}{x} \le 2^{-q}$$

so use modified formula for $|x| \le \sqrt{7}/3$

use original formula for $|x| > \sqrt{7}/3$

Require at most 2 digits lost so $p = 2$

$\sqrt{x^2+1} > 1$ so "x" $= \sqrt{x^2+1}$ and "y" $= 1$

$$\Rightarrow \quad 1 - \frac{1}{\sqrt{x^2+1}} \geq 2^{-2} = 1/4$$

$$\Rightarrow \quad 3/4 \geq \frac{1}{\sqrt{x^2+1}} \Rightarrow \sqrt{x^2+1} \geq \frac{4}{3}$$

$$\Rightarrow \quad x^2+1 \geq \frac{16}{9} \Rightarrow x^2 \geq \frac{7}{9} \text{ or } \boxed{|x| \geq \frac{\sqrt{7}}{3}}$$

~~use modified~~

[16 pts](3a) Estimate $f(2.14)$ using the Newton interpolating polynomial for this dataset:

x	2	2.1	2.2
y	1.414214	1.449138	1.483240

x	$f(x)$	$f[\,,\,]$	$f[\,,\,,\,]$
2	1.414214		
		.34924	
2.1	1.449138		-.0411
		.34102	
2.2	1.483240		

$$P_2(x) = 1.414214 + .34924(x-2) - .0411(x-2)(x-2.1)$$

$$P_2(2.14) = 1.414214 + .34924(2.14-2)$$

$$- .0411(2.14-2)(2.14-2.1)$$

$$= 1.4629$$

(b) If you are now told that the function being interpolated in Problem (3a) is $f(x) = \sqrt{x}$, give an upper bound on the error in your approximation.

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (x - x_i)$$

Alternative: plug $x = 2.14$ into cubic: Error $\leq \frac{1}{6}(.066)(2.14-2)(2.14-2.1)(2.14-2.2) = -3.1696 \times 10^{-6}$

$n = 2$

$$f(x) - p(x) = \frac{1}{3!} f^{(3)}(\xi) (x - x_0)(x - x_1)(x - x_2)$$

$$x \in [2, 2.2]$$

$f(x) = \sqrt{x} = x^{1/2}$, $f'(x) = \frac{1}{2} x^{-1/2}$, $f''(x) = -\frac{1}{4} x^{-3/2}$

$f'''(x) = \frac{3}{8} x^{-5/2} = \frac{3}{8\sqrt{x^5}}$ which is maximized in this interval at $x = 2$.

so $|f'''(x)| \leq \frac{3}{8\sqrt{32}} = \frac{3}{32\sqrt{2}} \approx .066$

so $|f(x) - p(x)| \leq \frac{1}{6}(.066)(x - x_0)(x - x_1)(x - x_2)$

maximizing $(x - x_0)(x - x_1)(x - x_2) = g(x)$

$= x^3 + x^2(-x_0 - x_1 - x_2) + x(x_0 x_1 + x_0 x_2 + x_1 x_2) - x_0 x_1 x_2$

$\frac{dg}{dx} = 3x^2 + 2x(-x_0 - x_1 - x_2) + (x_0 x_1 + x_0 x_2 + x_1 x_2) = 0$

$= 3x^2 + 2x(-6.3) + 13.22$

Roots: $\frac{12.6 \pm \sqrt{(12.6)^2 - 4(3)(13.22)}}{2(3)} = \frac{12.6 \pm \sqrt{.12}}{6}$

maximized for pos root $x \approx 2.16$

$\boxed{|\text{Error}| \leq .02376}$

[12 pts](4) Prove the *Theorem on Bisection:* If $[a_0, b_0]$, $[a_1, b_1]$, $\ldots$, $[a_n, b_n], \ldots$ denote the intervals in the bisection method, then the limits $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = r$. If $c_n = (a_n + b_n)/2$ then $|r - c_n| \le 2^{-(n+1)}(b_0 - a_0)$ where $r = \lim_{n\to\infty} c_n$.

Proof : note: $a_0 \le a_1 \le a_2 \le \ldots \le b_0$

$b_0 \geq b_1 \geq b_2 \ldots \geq a_0$.

Since the sequence $\{a_n\}$ is nondecreasing and bounded above it converges. Same idea for $\{b_n\}$.

note that we halve the interval each step so

$$b_{n+1} - a_{n+1} = \tfrac{1}{2}(b_n - a_n) \qquad n \geq 0$$

$$\Rightarrow b_{n+1} - a_{n+1} = \tfrac{1}{2}(\tfrac{1}{2}(b_{n-1} - a_{n-1})) \ldots$$

$$= (\tfrac{1}{2})^{n+1}(b_0 - a_0)$$

so $\lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = \lim_{n\to\infty} (\tfrac{1}{2})^n (b_0 - a_0) = 0$

Let $r = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$

note $0 \geq \lim_{n\to\infty} f(a_n) f(b_n) = [f(r)]^2 \geq 0$

$\Rightarrow f(r) = 0$ so $r =$ root of f.

Error $|r - c_n|$ is bounded by

$$|r - c_n| \le \tfrac{1}{2}(b_n - a_n) = (\tfrac{1}{2})^{n+1}(b_0 - a_0).$$

[14 pts](5a) Perform one iteration of Newton's method for finding the root of the nonlinear system

$$x^2 + y^2 = 4$$
$$x^2 - y^2 = 1.$$

As an initial guess use $(x_0, y_0) = (2, 1)$. Newton for Systems: $F'(x^k)H^k = -F(x^k)$

$$x^{k+1} = x^k + H^k$$

$$\text{"}F'\text{"} = J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x & 2y \\ 2x & -2y \end{bmatrix}$$

$$J(\vec{x}^0) = \begin{bmatrix} 2(2) & 2(1) \\ 2(2) & -2(1) \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 4 & -2 \end{bmatrix}$$

$$J^{-1}(\vec{x}^0) = -\frac{1}{16}\begin{bmatrix} -2 & -2 \\ -4 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & \frac{1}{8} \\ 1/4 & -1/4 \end{bmatrix}$$

$$-F(\vec{x}^0) = -\begin{bmatrix} x^2+y^2-4 \\ x^2-y^2-1 \end{bmatrix}\Bigg|_{\vec{x}^0} = -\begin{bmatrix} 2^2+1^2-4 \\ 2^2-1^2-1 \end{bmatrix} = -\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\text{so} \quad H^0 = -J^{-1}(\vec{x}^0)F(\vec{x}^0) = -\begin{bmatrix} 1/8 & 1/8 \\ 1/4 & -1/4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = -\begin{bmatrix} 3/8 \\ -1/4 \end{bmatrix} = \begin{bmatrix} -3/8 \\ 1/4 \end{bmatrix}$$

$$\vec{X}^1 = \vec{X}^0 + H^0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -3/8 \\ 1/4 \end{bmatrix} = \boxed{\begin{bmatrix} 13/8 \\ 5/4 \end{bmatrix} = \vec{X}^1}$$

(b) Under what conditions will Newton's method converge? When Newton's method converges under these conditions, how quickly does it do so?

Good initial guess, no double root

Quadratically

[12 pts](6) If we want to find the root $x = 2$ of the function

$$f(x) = x^2 - x - 2$$

using fixed point iteration $x = g(x)$, which of the following choices for $g(x)$ should we choose? Justify your answer.

(a) $g_1(x) = x^2 - 2$

(b) $g_2(x) = \sqrt{2 + x}$

(c) $g_3(x) = 1 + \frac{2}{x}$

(a) $g_1'(x) = 2x$

$g'(2) = 4 > 1$

no, does not converge

* best choice

(b) $g_2'(x) = 1/2(2+x)^{-1/2} = \frac{1}{2\sqrt{2+x}}$

$g_2'(2) = \frac{1}{2\sqrt{2+2}} = \frac{1}{4} < 1$

yes converges at fastest rate.

(c) $g_3'(x) = \frac{-2}{x^2} = \frac{-2}{4} = -1/2 < 1$

yes, converges

[8 pts](7) If $p(x) = x^4 - x^3 + x^2 - x + 1$ is a polynomial that interpolates the following function values:

x	-2	-1	0	1	2	3
y	31	5	1	1	11	61

Find a polynomial q that interpolates these values:

x	-2	-1	0	1	2	3
y	31	5	1	1	11	30

(Hint: This polynomial can be found easily. You should only need to determine one coefficient. Do not start from scratch.)

$$q(x) = p(x) + c(x+2)(x+1)x(x-1)(x-2)$$

when $x = 3$

$$q(x) = 30 = 61 + c(3+2)(3+1)3(3-1)(3-2)$$

$$\Rightarrow -31 = c(5)(4)3(2)(1)$$

$$\Rightarrow -31 = 120c$$

$$\Rightarrow c = \frac{-31}{120}$$

$$q(x) = x^4 - x^3 + x^2 - x + 1 - \frac{31}{120}(x+2)(x+1)x(x-1)(x-2)$$

Please sign the following honor statement: *On my honor, I pledge that I have neither given nor received any aid on this exam.*