○ Math/CS 4334 Midterm Exam 3/5/18
Dr. Minkoff

Instructions: You may not use notes or books on this exam. Don't spend too much time on any one problem. Show your work!

NAME:

1 /10	2 /14	3 /14	4 /12
5 /16	6 /9	T /75	

[10 pts](1) Find an interval containing a real positive zero of the function $f(x) = 2x^3 - 3x - 4$. How many steps of bisection would be required to compute this root to three significant decimal digits for your interval?

if $x=0$ $f(x) = -4 < 0$

if $x=1$ $f(x) = 2(1)^3 - 3(1) - 4 = -5 < 0$

if $x=2$ $f(x) = 2(2)^3 - 3(2) - 4$

$= 16 - 6 - 4 = 6 > 0$

so let interval $[a,b] = [0,2]$

$$\frac{b-a}{2^{n+1}} < \underset{(\varepsilon)}{10^{-3}} \Leftrightarrow n > \frac{\log(b-a) - \log(2\varepsilon)}{\log(2)}$$

$$\Rightarrow n > \frac{\log(2-0) - \log(2 * 10^{-3})}{\log(2)}$$

$$\Rightarrow n > \frac{.30103 - (-2.69897)}{.30103} = 9.96578$$

$\boxed{n \geq 10 \text{ steps required}}$

8

[14 pts](2a) Determine the Marc-32 machine representation of the decimal number -285.75. (Hint: your final answer should be an 8 digit hexadecimal number.)

Sign=1 (negative number)

$$\begin{array}{r} 142 \\ 2\,\overline{)\,285} \\ 284 \\ \hline \textcircled{1} \end{array} \quad \begin{array}{r} 71 \\ 2\,\overline{)\,142} \\ 142 \\ \hline \textcircled{0} \end{array} \quad \begin{array}{r} 35 \\ 2\,\overline{)\,71} \\ 70 \\ \hline \textcircled{1} \end{array} \quad \begin{array}{r} 17 \\ 2\,\overline{)\,35} \\ 34 \\ \hline \textcircled{1} \end{array} \quad \begin{array}{r} 8 \\ 2\,\overline{)\,17} \\ 16 \\ \hline \textcircled{1} \end{array}$$

$$\begin{array}{r} 4 \\ 2\,\overline{)\,8} \\ 8 \\ \hline \textcircled{0} \end{array} \quad \begin{array}{r} 2 \\ 2\,\overline{)\,4} \\ 4 \\ \hline \textcircled{0} \end{array} \quad \begin{array}{r} 1 \\ 2\,\overline{)\,2} \\ 2 \\ \hline \textcircled{0} \end{array} \quad \begin{array}{r} .75 \\ 2 \\ \hline \textcircled{1}.50 \\ 2 \\ \hline \textcircled{1}0 \end{array}$$

$c_0=1$
$c_1=0$
$c_2=1$
$c_3=1$
$c_4=1$
$c_5=0$
$c_6=0$
$c_7=0$
$c_8=1$

$$285.75_{10} = 100011101.11_2 = 1.0001110111 \times 2^8$$

$C-127=8 \Rightarrow C=135$

$$\begin{array}{r} 67 \\ 2\,\overline{)\,135} \\ 134 \\ \hline \textcircled{1} \end{array} \quad \begin{array}{r} 33 \\ 2\,\overline{)\,67} \\ 66 \\ \hline \textcircled{1} \end{array} \quad \begin{array}{r} 16 \\ 2\,\overline{)\,33} \\ 32 \\ \hline \textcircled{1} \end{array} \quad \begin{array}{r} 8 \\ 2\,\overline{)\,16} \\ 16 \\ \hline \textcircled{0} \end{array} \quad \begin{array}{r} 4 \\ 2\,\overline{)\,8} \\ 8 \\ \hline \textcircled{0} \end{array} \quad \begin{array}{r} 2 \\ 2\,\overline{)\,4} \\ 4 \\ \hline 0 \end{array} \quad \begin{array}{r} 1 \\ 2\,\overline{)\,2} \\ 2 \\ \hline \textcircled{0} \end{array}$$

$c_0=1$
$c_1=1$
$c_2=1$
$c_3=0$
$c_4=0$
$c_5=0$
$c_6=0$
$c_7=1$

Exponent $= 10000111_2$

so

1	10000111	00011101110000000000000
sign	Exponent	Mantissa

(b) Does -285.75 fit exactly on the Marc-32 machine?

yes

Hex # is

$$\boxed{C38EE000_{16}}$$

[14 pts](3a) Given the data in the table below find an approximate value of $\sin 0.705$ using linear interpolation. (Note: you must use either the Newton or Lagrange form to receive credit.)

x	$\sin x$
0.70	0.644
0.71	0.651

Newton form:

$$P_1(x) = .644 + C(x - .70) \Rightarrow C = \frac{.651 - .644}{.71 - .70}$$

$$\boxed{\text{So } P_1(x) = .644 + .7(x - .7)} \Big\} = .7$$

Lagrange form: $P_1(x) = \left(\frac{x - .71}{.7 - .71}\right)(.644) + \left(\frac{x - .7}{.71 - .7}\right)(.651)$

$$\boxed{P_1(.705) = .6475}$$

(b) Using one of the formulas given in class for polynomial interpolation error, give an upper bound on the error in the approximation you found in Part (a).

$$|f(x) - p(x)| \le \frac{1}{4(n+1)} M h^{n+1}$$

here $n=1$ (degree of polynomial)

so $n+1=2$, $h = .01$

$f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$

$|f^{(2)}(x)| \le M$ $\quad |-\sin x| \le 1 = M \quad \forall x$

so $|f(x) - P_1(x)| \le \frac{1}{4(1+1)}(1)(.01)^2$

or error $|f(x) - p_1(x)| \le 1.25 \times 10^{-5}$

[12 pts](4) Consider the function

$$g(x) = \frac{e^x - 1}{x}.$$

(a) For what values of x might catastrophic cancellation occur?

valus near $x=0$.

(b) Find an alternate formula for g(x) that eliminates the catastrophic cancellation seen in Part (a) above.

Use Taylor series for $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

new formula is ↙ TS remainder

$$h(x) = \frac{1 + x + \frac{x^2}{2} + \frac{x^3}{3!}e^{\xi} - 1}{x}$$

$$\Rightarrow h(x) = 1 + \frac{x}{2} + \frac{x^2}{6}e^{\xi}$$

$\boxed{\text{new formula near } x=0 \text{ is } h(x) = 1 + \frac{x}{2}}$

(c) Give a formula for the error in using this approximation. (Hint: don't use the Loss of Precision Theorem.)

Error is $\frac{x^2}{6}e^{\xi}$ for ξ the MVT point

For example for $|x| \leq .1$,

$$|\text{error}| \leq \frac{(.1)^2}{6}e^{.1} = 1.84 \times 10^{-3}$$

[16 pts](5) Give the best method to use to find a root for each of the following problems. Your choices are bisection, Newton, and secant. (**Note: you will only receive credit for your answer if you explain your reasoning!**)

(a) Find a root of the equation $f(x) = 0$ assuming you have a good initial guess x_0 for the root.

Newton - fastest w/good initial guess

(b) Find a root of the equation $f(x) = 0$ where $f'(x)$ is difficult or impossible to evaluate but you have a good initial estimate of the root.

Secant - faster than bisection with good initial guess. Newton can't be used since you don't have good information about $f'(x)$.

(c) Find a root of the equation $f(x) = 0$ in the case where you don't know a good starting point x_0.

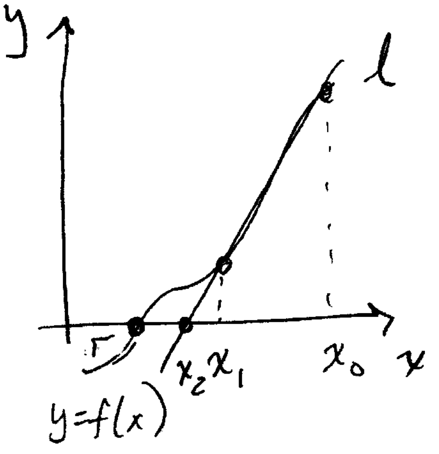
Bisection - guaranteed to converge if $f(a)f(b) < 0$, and you don't need good initial guess. Perhaps switch to Newton when close to the root.

(d) Find a root of the equation $f(x) = 0$ where $f(x)$ and $f'(x)$ are both expensive to evaluate but you do have a good initial guess x_0.

Secant - only one function evaluation per step after the first. Newton requires 2 function evaluations per step $\Rightarrow$ more expensive.

[9 pts](6) Give a very careful graphical derivation of the secant method. (Note: to receive credit you must give a well labeled graph and a corresponding algebraic derivation.)

Note that 2 pts determine a line
If we have 2 pts x_0, x_1 which we think are good guesses as roots of f, we can draw a line between points $(x_0, f(x_0))$ and $(x_1, f(x_1))$.
If x_2 is new root guess we get after an iteration of secant we know 3 points on secant line l: $(x_0, f(x_0))$, $(x_1, f(x_1))$, $(x_2, 0)$.



Line has one slope so

$$\frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{0-f(x_1)}{x_2-x_1}$$

Solving for x_2 gives

$$(x_2-x_1)\left[\frac{f(x_1)-f(x_0)}{x_1-x_0}\right] = -f(x_1)$$

$$\Rightarrow \quad x_2 = x_1 - f(x_1)\underbrace{\left[\frac{x_1-x_0}{f(x_1)-f(x_0)}\right]}_{\text{reciprocal of slope of secant line of } f}$$

General form: $x_{n+1} = x_n - f(x_n)\left[\frac{x_n-x_{n-1}}{f(x_n)-f(x_{n-1})}\right]$