o Math 4334 Midterm Exam 3/1/17
Dr. Minkoff

Instructions: You may not use notes or books on this exam. Don't spend too much time on any one problem. Show your work!

NAME:

1 /9	2 /15	3 /10	4 /15
5 /12	6 /14	EC /5	T /80

[9 pts](1) Imagine you were given the following table showing convergence of three root finding methods (bisection, Newton, Secant). Indicate which column most likely corresponds to which method and give a brief justification for your answer.

iteration number (n)	f(x_n)	g(x_n)	h(x_n)
4	-1.68 $\times 10^{-1}$	5.50 $\times 10^{-4}$	-0.6167
5	-2.24 $\times 10^{-2}$	7.11 $\times 10^{-8}$	0.2333
6	9.54 $\times 10^{-4}$	1.55 $\times 10^{-15}$	0.0616
7	-5.07 $\times 10^{-6}$	–	-0.0196

(1) g(xn) is Newton (looks quadratically convergent)

(2) f(xn) is secant (between Newton & bisection

(3) h(xn) is bisection (slowest of 3 methods)

[15 pts] (2a) Determine an interpolating polynomial of degree 2 for the function $f(x) = \cos x$ which is exact at the points $(0, 1)$, $(\pi/2, 0)$ and $(\pi, -1)$.

x	$f(x)$	$f[\,,\,]$	$f[\,,\,,\,]$
0	1		
		$-2/\pi$	
$\pi/2$	0		0
		$-2/\pi$	
π	-1		

$$\frac{1-0}{0-\pi/2} = -2/\pi$$

$$\frac{-1-0}{\pi-\pi/2} = \frac{-1}{\pi/2} = -2/\pi$$

So $\quad P_2(x) = 1 + (-2/\pi)(x-0) + 0(x-0)(x-\pi/2)$

$$\boxed{P_2(x) = 1 - \frac{2x}{\pi}}$$

check:
$P_2(0) = 1$ ✓
$P_2(\pi/2) = 1 - \frac{2(\pi/2)}{\pi} = 0$ ✓
$P_2(\pi) = 1 - 2\,\pi/\pi = -1$ ✓

(b) Using one of the formulas given in class for polynomial interpolation error, give an upper bound on the error in the approximation you found in Part (a).

Note: node points are equally spaced so

$$|f(x) - p(x)| \le \frac{1}{4(n+1)} M h^{n+1}$$

$$n = 2, \quad h = \frac{\pi - 0}{2} = \pi/2$$

So error $|f(x) - p(x)| \le \frac{1}{4(3)} M h^3$

need $\quad |f^{(2+1)}(x)| \le M$

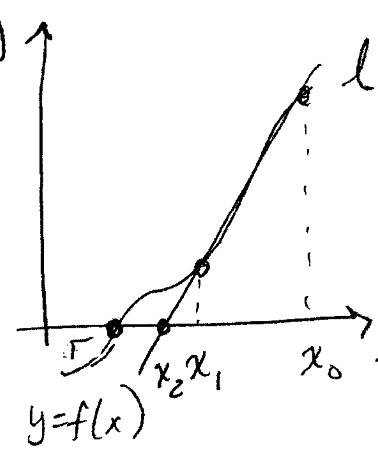
$f(x) = \cos x$
$f'(x) = -\sin x$
$f''(x) = -\cos x$
$f^{(3)}(x) = \sin x$

$|f^{(3)}(x)| = |\sin x| \le 1$

So $\quad |f(x) - p(x)| \le \frac{1}{12}(1)\left(\frac{\pi}{2}\right)^3 = \boxed{\frac{\pi^3}{96} \rightarrow \text{upper bound on error}}$

[10 pts](3) Give a very careful graphical derivation of the secant method. (Note: to receive credit you must give a well labeled graph and a corresponding algebraic derivation.)

Note that 2 pts determine a line
If we have 2 pts x_0, x_1 which we think are good guesses as roots of f, we can draw a line between points $(x_0, f(x_0))$ and $(x_1, f(x_1))$.
If x_2 is new root guess we get after an iteration of secant we know 3 points on secant line l: $(x_0, f(x_0))$, $(x_1, f(x_1))$, $(x_2, 0)$.



Line has one slope so

$$\frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{0-f(x_1)}{x_2-x_1}$$

Solving for x_2 gives

$$(x_2-x_1)\left[\frac{f(x_1)-f(x_0)}{x_1-x_0}\right] = -f(x_1)$$

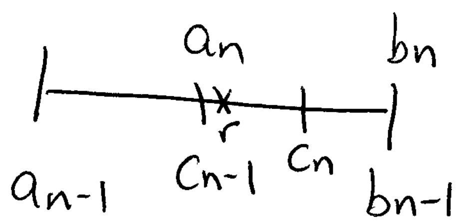
$$\Rightarrow \quad x_2 = x_1 - f(x_1)\left[\frac{x_1-x_0}{f(x_1)-f(x_0)}\right]$$

General form: $x_{n+1} = x_n - f(x_n)\underbrace{\left[\frac{x_n-x_{n-1}}{f(x_n)-f(x_{n-1})}\right]}$ reciprocal of slope of secant line of f

[15 pts](4) Using the notation developed in class for the bisection method, determine which of these assertions are true and which are false. If the assertion is false, give a counter example. If it is true, prove it.

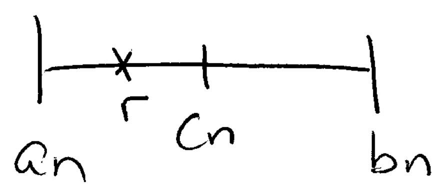
1. $|r - c_n| < |r - c_{n-1}|$

Can be false



2. $c_n \le r \le b_n$

false if



3. $0 \le r - a_n \le 2^{-n}(b_0 - a_0)$

True because

$0 \le r - a_n$ as a_n is left endpt of interval root is contained in.

and $r - a_n \le b_n - a_n = 2^{-n}(b_0 - a_0)$

[12 pts](5) Determine the decimal number that has the machine representation $[C705A700]_{16}$ on the Marc-32 machine.

C705 A700

$$1\underbrace{100\ 0111\ 0}_{\text{expt}}\underbrace{000\ 0101\ 1010\ 0111\ 0000\ 0000}_{\text{mantissa}}$$

Sign ⇒ negative number

$$(-1)^s = (-1)^1 < 0$$

$$\text{Expt} = 10001110 = 2+4+8+128 = 142$$

$$\text{so } 142 - 127 = 15$$

mantissa represents number

$$1.0000\ 1011\ 0100\ 1110\ 000\ 0000 \times 2^{15}$$

$$= 1000010110100111.00000000$$

$$= 2^0 + 2^1 + 2^2 + 2^5 + 2^7 + 2^8 + 2^{10} + 2^{15}$$

$$= 1+2+4+32+128+256+1024+32768$$

$$= \boxed{-34215}$$

[14 pts](6) Consider the function

$$g(x) = \frac{x - \sin(x)}{\tan(x)}.$$

(a) For what values of x might catastrophic cancellation occur?

for $x \approx 0$

(b) Find an alternate formula for g(x) that eliminates the catastrophic cancellation seen in Part (a) above.

Using a Taylor series for $\sin(x)$

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

on computer you would truncate series. Let's stop at x^5 term.

Plugging into $g(x)$ we get

$$h(x) \approx \frac{x - \left[x - \frac{x^3}{3!} + \frac{x^5}{5!}\right]}{\tan(x)}$$

$$h(x) \approx \frac{\frac{x^3}{3!} - \frac{x^5}{5!}}{\tan(x)}$$

note that for small x, $\frac{x^3}{6} \neq \frac{x^5}{120}$

so we eliminated catastrophic cancellation

(c) Give a formula for the error in using the approximation in Part (b) above.

Error would come from Taylor series remainder which in this case is next term in T.S. or

$$|R_7(x)| \le \left|-\frac{x^7}{7!}\right|$$ so

formula for error is

$$|\text{Error}| \le \left|\frac{x^7}{7!\tan(x)}\right|$$

[5 pts](EXTRA CREDIT) Show one iteration of Newton's method applied to finding $\sqrt[3]{2}$. (Hint: reformulate this problem as a root finding problem and use an initial guess of $x_0 = 1$.)

$$x = \sqrt[3]{2} \Rightarrow 2 = x^3 \Rightarrow x^3 - 2 = 0$$

so $f(x) = x^3 - 2$

$f'(x) = 3x^2$

Newton $\quad x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$f(x_0) = 1^3 - 2$
$= -1$

$f'(x_0) = 3(1)^2$
$= 3$

$$x_1 = 1 - \frac{-1}{3}$$

$$= 4/3$$