and the periodic boundary conditions

$$u(-\ell) = u(\ell), \ \frac{du}{dx}(-\ell) = \frac{du}{dx}(\ell)$$

(for the interval $[-\ell, \ell]$). Generally speaking, given a symmetric differential operator, a sequence of orthogonal eigenfunctions exists, and it can be used to represent functions in a Fourier series. However, it may not be easy to find the eigenfunctions in some cases (see Exercise 4).

Exercises

1. Use the trigonometric identity

$$\sin\alpha\sin\beta = \frac{1}{2}\left(\cos\left(\alpha-\beta\right) - \cos\left(\alpha+\beta\right)\right)$$

to verify that, if $n \neq m$, then $(\psi_n, \psi_m) = 0$, where $\psi_1, \psi_2, \psi_3, \ldots$ are given in (5.13).

2. Define K as in Exercise 5.1.8. We know (from the earlier exercise) that K has only real, positive eigenvalues. Find all of the eigenvalues and eigenvectors of K.

3. Repeat Exercise 2 for the differential operator $L_{\tilde{m}}$ defined in Exercise 5.1.7. The resulting eigenfunctions are the quarter-wave cosine functions.

4. **(Hard)** Define L_R as in Exercise 5.1.11.
   (a) Show that L_R has only positive eigenvalues.
   (b) Show that L_R has an infinite sequence of positive eigenvalues. Note: The equation that determines the positive eigenvalues cannot be solved analytically, but a simple graphical analysis can be used to show that they exist and to estimate their values.
   (c) For $\alpha = \kappa = 1$, find the first two eigenpairs by finding accurate estimates of the two smallest eigenvalues.

5. **(Hard)** Consider the differential operator $M : C^2_m[0,1] \to C[0,1]$ defined by

$$Mu = -\frac{d^2u}{dx^2} + \frac{du}{dx} + 5u$$

(recall that $C^2_m[0,1] = \left\{u \in C^2[0,1] \ : \ u(0) = \frac{du}{dx}(1) = 0\right\}$). Analyze the eigenpairs of M as follows:

   (a) Write down the characteristic polynomial of the ODE

$$\frac{d^2u}{dx^2} - \frac{du}{dx} + (\lambda - 5)u = 0. \qquad (5.18)$$

   Using the quadratic formula, find the characteristic roots.

(b) There are three cases to consider, depending on whether the discriminant in the quadratic formula is negative, zero, or positive (that is, depending on whether the characteristic roots are complex conjugate, or real and repeated, or real and distinct). In each case, write down the general solution of (5.18).

(c) Show that in the case of real roots (either repeated or distinct), there is no nonzero solution.

(d) Show that there is an infinite sequence of values of λ, leading to complex conjugate roots, each yielding a nonzero solution of (5.18). Note: The equation that determines the eigenvalues cannot be solved analytically. However, a simple graphical analysis is sufficient to show the existence of the sequence of eigenvalues.

(e) Find the first two eigenpairs (that is, those corresponding to the two smallest eigenvalues), using some numerical method to compute the first two eigenvalues accurately.

(f) Show that the first two eigenfunctions are *not* orthogonal.

6. Define

$$C_b^2[0,\ell] = \left\{ u \in C^2[0,\ell] \,:\, u(\ell) = a_{11}u(0) + a_{12}\frac{du}{dx}(0), \right.$$
$$\left. \frac{du}{dx}(\ell) = a_{21}u(0) + a_{22}\frac{du}{dx}(0) \right\},$$

and define $L_b : C_b^2[0,\ell] \to C[0,\ell]$ by

$$L_b u = -\frac{d^2u}{dx^2}.$$

Find necessary and sufficient conditions on the coefficients $a_{11}, a_{12}, a_{21}, a_{22}$ for the operator L_b to be symmetric.

7. Compute the Fourier sine series, on the interval $[0, 1]$, for each of the following functions:[29]

(a) $g(x) = x$

(b) $h(x) = \frac{1}{2} - \left|x - \frac{1}{2}\right|$

(c) $m(x) = x - x^3$

(d) $k(x) = 7x - 10x^3 + 3x^5$

Graph the error in approximating each function by 10 terms of the Fourier sine series.

[29] The use of *Mathematica* or some other program to compute the necessary integrals is recommended.

8. Explain why a Fourier sine series, if it converges to a function on $[0,\ell]$, defines an odd function of $x \in \mathbf{R}$. (A function $f : \mathbf{R} \to \mathbf{R}$ is *odd* if $f(-x) = -f(x)$ for all $x \in \mathbf{R}$.)

9. What is the Fourier sine series of $f(x) = \sin(3\pi x)$ on the interval $[0,1]$?

10. Repeat Exercise 7 using the quarter-wave sine series (see Section 5.2.3).[29]

11. Repeat Exercise 7 using the quarter-wave cosine series (see Exercise 3).[29]

5.3 Solving the BVP using Fourier series

We now return to the problem of solving the BVP

$$\begin{aligned} -T\frac{d^2u}{dx^2} &= f(x),\ 0 < x < \ell, \\ u(0) &= 0, \\ u(\ell) &= 0, \end{aligned} \tag{5.19}$$

which can be expressed simply as $L_D u = f$.

5.3.1 A special case

We begin with a special case that is easy to solve. Suppose f is of the form

$$f(x) = \sum_{n=1}^{N} c_n \sin\left(\frac{n\pi x}{\ell}\right), \tag{5.20}$$

where the coefficients $c_1, c_2, \ldots, c_N$ are known. We then look for the solution in the form

$$u(x) = \sum_{n=1}^{N} b_n \sin\left(\frac{n\pi x}{\ell}\right).$$

By construction, $u \in C_D^2[0,\ell]$, and so u satisfies the boundary conditions (regardless of the values of $b_1, b_2, \ldots, b_N$). We therefore choose the coefficients $b_1, b_2, \ldots, b_N$ so that the differential equation is satisfied.

Since u is expressed in terms of the eigenfunctions, the expression for $L_D u$ is very simple:

$$(L_D u)(x) = -T\frac{d^2u}{dx^2}(x) = \sum_{n=1}^{N} \frac{Tn^2\pi^2}{\ell^2} b_n \sin\left(\frac{n\pi x}{\ell}\right).$$

The equation $L_D u = f$ then becomes

$$\sum_{n=1}^{N} \frac{Tn^2\pi^2}{\ell^2} b_n \sin\left(\frac{n\pi x}{\ell}\right) = \sum_{n=1}^{N} c_n \sin\left(\frac{n\pi x}{\ell}\right).$$